Diffusion in almost-Riemannian geometry 3/3

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We have seen the definition of 2-ARS as the generalized Riemannian structure obtained locally by declaring that a pair of vector fields satisfying the Hormander condition is an orthonormal frame

→if the orthonormal frame is global we say that it is **free**

We have seen

- properties of the singular set
- type of points (Riemannian Grushin and Tangency) and normal forms
- Geodesics: they are smooth and can cross the singular set, but
  - spheres of finite diameter can have infinite (Riemannian) volume and infinite (Riemannian) perimeter
  - a conjugate locus could be present even if the curvature is always negative where it is defined

A Gauss Bonnet Theorem

The Riemannian Gauss-Bonnet theorem

Theorem (Gauss-Bonnet)

Let M be a 2D, compact, orientable Riemannian manifold. Then $\int_M K dA = 2\pi\ \chi(M) = 2\pi(2 - 2\mathbf{g})$

can we extend such a result in the AR context?

Of course this cannot be done directly. For instance for 2-step structures we have that $K \to -\infty$ on the singular set.

Definition

A 2-ARS is orientable if $\mathbf{U}$ is orientable as vector bundle.

Roughly this is equivalent to ask that we can take all matrices of changes of orthonormal frames in $SO(2)$.

$$Y_1(q) = \cos(\theta)X_1(q) + \sin(\theta)X_2(q),$$
$$Y_2(q) = -\sin(\theta)X_1(q) + \cos(\theta)X_2(q).$$

→This concept is unrelated from the orientability of M.
→There exists non orientable structures on orientable manifolds and viceversa.

Definition

A 2-ARS is **fully orientable** if

- it is orientable
- and M is orientable.

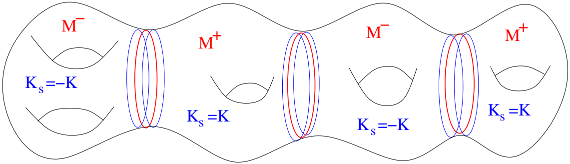
Definition

Consider a **fully-orientable, 2-step** 2-ARS. Let $\omega \in \Lambda^2(M)$ be a never vanishing two-form, defining an orientation on M. Define

$$M^{\pm} = \{p \in \Omega_i \setminus \mathcal{Z} | \quad i \in I, \pm\omega(X_i, Y_i)(p) > 0\} \tag{1}$$

We call signed curvature the function (defined on $M \setminus \mathcal{Z}$)

$$K_s(p) = \begin{cases} K(p), & \text{if } p \in M^+, \\ -K(p), & \text{if } p \in M^-, \end{cases} \tag{2}$$



For every $\varepsilon > 0$, let $M_\varepsilon = \{p \in M | \ d(p, \mathcal{Z}) > \varepsilon\}$ We say that K_s is integrable on M if

$$\lim_{\varepsilon \to 0} \int_{M_\varepsilon} K_s \ dA \tag{3}$$

exists and is finite. In this case we denote such limit by $\int \mathcal{K}_s dA$.

Theorem

Consider a **fully-orientable, 2-step** *2-ARS. Then*

$$\int \mathcal{K}_s dA = 2\pi(\chi(M^+) - \chi(M^-)), \tag{4}$$

where χ *denotes the Euler characteristic.*

Free structures

Lemma

If M is orientable and the 2-step 2-ARS is free ($\Rightarrow$ fully orientable) then $(\chi(M^+) - \chi(M^-)) = 0$.

Corollary

Let M be a compact oriented two-dimensional manifold and consider a free 2-step 2-ARS on it. Then

$$\int \mathcal{K}_s dA = 0 \tag{5}$$

This is a deep fact: If the metric is defined globally by a couple of vector fields, then $\int \mathcal{K}_s dA = 0$. This is what happens in Riemannian geometry: on the torus!

$\rightarrow$we “force the manifold to be parallelizable by accepting singularities”

Manifold M (compact orientable)
topolgy implies zeros of vector fields
(Σ index of zeros=χ(M))

Riemannian construction

A Riemannian structure without singularities is defined locally on charts by vector fields

$$\int_M K \, dA = 2\pi\chi(M)$$

(topological information)

Almost Riemannian (free) construction

A Riemannian structure defined globally by the vector fields

$$\int_M K_s \, dA = 0$$

has singularities
(topological information)

(only) on the torus the two constructions coincide

Non free structures

If the structure is not free one can prove that

$$\chi(M^+) - \chi(M^-) = e(\mathbf{U})$$

Where $e(\mathbf{U})$ is the Euler number of U.
and then

Theorem

Consider a **fully-orientable, 2-step** *2-ARS. Then*

$$\int \mathcal{K}_s dA = 2\pi e(\mathbf{U}) \tag{6}$$

Extensions in presence of tangency points are possible, but less natural.

Diffusion and Schroedinger evolution on 2-ARS

How to define the heat and the Schroedinger equation in a 2-ARS?

$$\partial_t \phi = \frac{1}{2}\Delta\phi, \qquad i\partial_t \phi = -\frac{\hbar^2}{2}\Delta\phi$$

What is the right operator Δ?

$\Delta = X_1^2 + X_2^2$ is not a good choice because

- it is not invariant by change of orthonormal frame
- it is not global if the structure cannot be defined with one frame only

To have an invariant operator the simplest thing is to look for an operator of the form

$$\Delta = \mathrm{div}_\omega \circ \mathrm{grad}$$

(the definition of grad extends with no difficulties to ARSs)

The problem is the choice of ω. We have $\infty + 1$ choices

- either we choose from outside a regular volume $\omega \Rightarrow$ we get a well defined hypoelliptic operator but then the diffusion will depend on ω
- or we take the Riemannian area dA. In this case we obtain the **Laplace-Beltrami** operator which diverges on $\mathcal{Z}$.

in 2-ARS an intrinsic volume which is not diverging is not known (in a sense it does not exist)

Let us start to study the (diverging) **Laplace Beltrami** operator

$$\Delta = \mathrm{div}_{dA} \circ \mathrm{grad} = X_1^2(\phi) + X_2^2(\phi) + \mathrm{div}(X_1)X_1(\phi) + \mathrm{div}(X_2)X_2(\phi)$$

The first order term are diverging

For the Grushin metric $X_1 = \partial_x$, $X_2 = x\partial_y$, $dA = \frac{1}{|x|}$ and we obtain

$$\Delta\phi := \mathrm{div}_{dA}(\mathrm{grad}(\phi)) = \left(\partial_x^2 + x^2\partial_y^2 - \frac{1}{x}\partial_x\right)\phi,$$

If we take $\omega = dx\,dy$ we obtain

$$\tilde{\Delta}\phi := \mathrm{div}_\omega(\mathrm{grad}(\phi)) = \left(\partial_x^2 + x^2\partial_y^2\right)\phi,$$

Is there a propagation of the heat through the singular set? (same question for the Schroedinger equation)

This question is essentially equivalent to

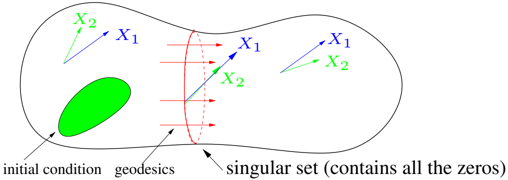
Q: Let M be a 2D manifold endowed with a 2-ARS. Let Ω be a connected component of $M \setminus \mathcal{Z}$. Let Δ be the corresponding Laplace-Beltrami defined on $C_0^\infty(\Omega)$. Is Δ essentially self-adjoint on $L^2(\Omega, dA)$?

→**If Δ is essentially self-adjoint** on $L^2(\Omega, dA)$ then the Cauchy problems for the heat and Schoredinger equations are well defined in $L^2(\Omega, dA)$ without the need of boundary conditions $\Rightarrow$ **no propagation.**

→If it is not essentially self-adjoint then one needs boundary conditions. For instance

- Dirichlet-like conditions (killing condition)
- Neumann-like (reflecting condition)
- other conditions permitting to connect the two sides (**bridging extensions** see [B. Prandi 2016])

A *a priori* one expects a negative answer to this question (i.e. Δ is not essentially self adjoint) since a positive answer would imply that neither the heat flow, neither a quantum particle can pass through $\mathcal{Z}$, while classical geodesics cross it with no singularities.



A positive answer

Theorem (U.B, Camille Laurent, 2013)

Let M be a 2D compact orientable manifold endowed with a 2-ARS. Assume that

$$(H) \quad \text{for every } q \in M,\ \blacktriangle(q) + [\blacktriangle, \blacktriangle](q) = T_qM.$$

Let Ω be a connected component of $M \setminus \mathcal{Z}$. Then Δ with domain $C_0^\infty(\Omega)$ is essentially self-adjoint on $L^2(\Omega, dA)$.

Remark (H) is an open condition implying that
→$\mathcal{Z}$ is a finite union of non-intersecting circles
→only Riemannian and Grushin points are present
→the compactness hypothesis is not necessary (but simplify the statement)

Consequences

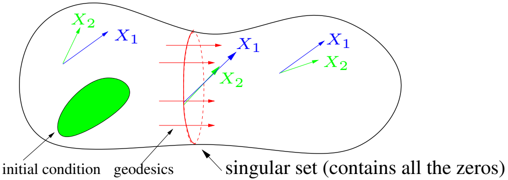
Corollary

Consider the unique solution ϕ of the Schroedinger equation,

$$i\partial_t \phi = -\frac{\hbar^2}{2}\Delta\phi \quad (7)$$

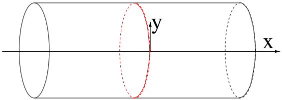
$$\phi(0) = \phi_0 \in L^2(M, dA) \quad (8)$$

with ϕ_0 supported in a connected component Ω of $M \setminus \mathcal{Z}$. Then, $\phi(t)$ is supported in Ω for any $t \geq 0$. The same holds for the solution of the heat or for the solution of the wave equation.



Idea of the proof of the self-adjointness: the Grushin cylinder

- The Grushin Cylinder is the Grushin plane in which we compactifies the y variable



- By setting $f = \sqrt{|x|}g$

$$\begin{gathered}\partial_x^2 + x^2\partial_y^2 - \frac{1}{x}\partial_x \qquad \text{on } L^2(\tfrac{1}{|x|}dx\,dy) \\ \Downarrow \\ \partial_x^2 + x^2\partial_y^2 - \frac{3}{4}\frac{1}{x^2} \qquad \text{on } L^2(dx\,dy)\end{gathered} \tag{9}$$

- By making Fourier transform in y, we are reduced to study the selfadjointness of:

$$\partial_x^2 - \frac{3}{4}\frac{1}{x^2} - k^2x^2 \quad \text{on }]0,\infty[$$

$$\text{i.e. of } -\partial_x^2 + \frac{3}{4}\frac{1}{x^2} + k^2x^2 \quad \text{on }]0,\infty[$$

Proposition (Reed-Simon)

The operator $-\partial_x^2 + \frac{c}{|x|^2}$ defined on $L^2(]0,+\infty[)$ with domain $C_0^\infty(]0,+\infty[)$ is essentially self-adjoint if and only $c \geq \frac{3}{4}$.

(uses the limit-point limit circle theory Weyl's Theorem)

→The rest of the proof for an almost Riemannian structure consists in generalizing this result for a normal form around a connected component of the singular set and to treat it as a perturbation of the Grushin case.

Remarks

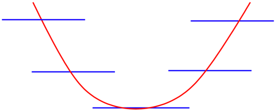
- the explosion of the area naturally acts as a barrier, which prevents the particles from crossing the degeneracy zone.
- One cannot relate the heat kernel $p_t(q_1, q_2)$ to the distance as in the Riemannian case

$$-4t \log p_t(q_1, q_2) \to d(q_1, q_2)^2 \text{ (Varadhan '67)}$$ is not true on ARSs

- No semiclassical theory for the Schroedinger equation

→such a phenomenon is today called Quantum Confinement (and there are today many further results on rank-varying Sub-Riemannian structures, see Rizzi, Prandi, Seri, Franceschi)

→The theorem is not proved in presence of tangency points (hard open question)

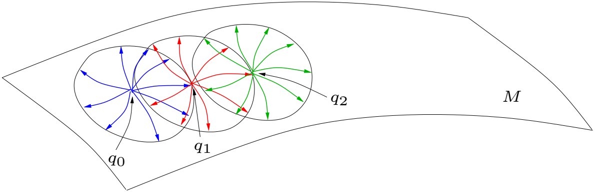


Tangency

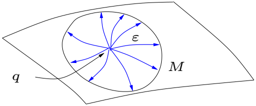
heat blocking by the Grushin singularity: a Random walk interpretation

In Riemannian geometry the Laplacian can be constructed as the generator of a limit random walk.
We consider a particle that

- at time zero is in q_0;
- at time δt jumps on a point q_1 of the sphere of radius ε centered in q_0, uniformly on the sphere, by following a geodesic;
- at time $2\delta t$ jumps on a point q_2 of the sphere of radius ε centered in q_1 uniformly on the sphere, by following a geodesic;
- ....



If ϕ is the density of probability of finding the particle in q we have that:



how much ϕ is increasing at a point q in time δt is proportional to the difference between the average of ϕ in a sphere of radius ε centered in q and the value of $\phi(q,t)$.

$$\phi(q,t+\delta t)-\phi(q,t)=\int_{S^{n-1}}\Big(\phi\big(\exp_q(\varepsilon,\theta),t\big)-\phi(q,t)\Big)d\theta.$$

Dividing by δt we obtain

$$\frac{\phi(q,t+\delta t)-\phi(q,t)}{\delta t}=\frac{1}{\delta t}\int_{S^{n-1}}\Big(\phi\big(\exp_q(\varepsilon,\theta),t\big)-\phi(q,t)\Big)d\theta.$$

Taking the parabolic scaling ($\leftrightarrow$ infinite velocity) $\delta t=\varepsilon^2$, and for $\delta t\to 0$,

$$\partial_t\phi=L\phi,\qquad \text{where } L\phi(q,t)=\lim_{\varepsilon\to 0}\underbrace{\frac{1}{\varepsilon^2}\int_{S^{n-1}}\Big(\phi\big(\exp_q(\varepsilon,\theta),t\big)-\phi(q,t)\Big)d\theta}_{\text{generator of the random walk}}. \quad (10)$$

in Riemannian geometry Δ is proportional to L

Up to constants we have

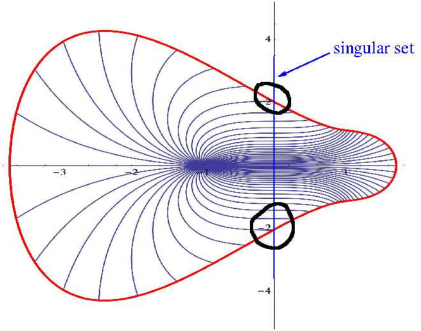
$$L = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{S^{n-1}} \Big(\phi\big(\exp_q(\varepsilon, \theta), t\big) - \phi(q,t) \Big) d\theta = \Delta$$

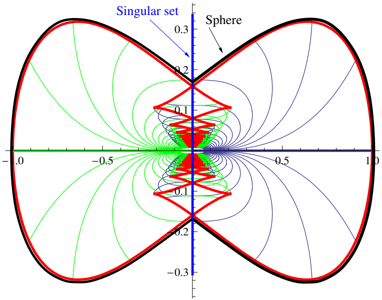
there is a way of passing from convergence of the operator to convergence of the process.

in 2D almost-Riemannian geometry.......

in 2D almost-Riemannian geometry the **front and the sphere are not admissible curves when they intersect the singular set.** Hence their length is infinite and the heat is trapped in the singularity!

e.g. for the Grushin plane





→We do not have a so intuitive interpretation for the Schroedinger equation

→For the Grushin plane we proved that we are stochastically incomplete i.e. that

$$\int_M \phi(t,q)dA(q) \text{ is decreasing}$$

(the singular set is eating the heat)

→the quantum sphere is an example of compact manifolds ($\Rightarrow$ geodesically complete) that is stochastically incomplete.

(in Riemannian geometry on compact manifold we are always geodesically and stochastically complete)

Schroedinger evolution

Some new results

Schroedinger: semiclassical interpretation

In quantum mechanics semiclassical analysis says roughly:

- if ψ is wave packet solution of the Schroedinger equation, then for $\hbar \to 0$, its maximum satisfies the classical equation of the motion.

However the theory is delicate and on Riemannian manifolds even **how to construct the quantum operator starting from the classical Hamiltonian** is not obvious (the quantization procedure is not unique).

$$H(p,q) = \frac{1}{2}\left(\langle p, X_1(q)\rangle^2 + \langle p, X_2(q)\rangle^2\right) = \frac{1}{2} g^{ij} p_i p_j$$

$$\Downarrow$$

$$??? \quad -\frac{\hbar^2}{2m}\Delta \quad ???$$

→there are many possible quantizations (i.e., ways of writing the Schroedinger equation starting from the classical Hamiltonian) that can be divided in two categories:

- intrinsic quantizations (usually called “geometric quantizations”)
- extrinsic quantizations (using the embedding in $\mathbf{R}^n$)

the intrinsic approach: Geometric quantization

Literature on geometric quantization, path integrals etc. suggests that in a **Riemannian manifold** in dimension 2

$$H(p,q) = \frac{1}{2}\left(\langle p, X_1(q)\rangle^2 + \langle p, X_2(q)\rangle^2\right) = \frac{1}{2} g^{ij} p_i p_j$$

$$\Downarrow$$

$$-\frac{\hbar^2}{2m}(\Delta - cK(q))$$

where

- $K(q)$ is the Gaussian curvature. In dimension ≥ 2 one should use the scalar curvature R (notice that $R = 2K$ in dimension two).
- $c \geq 0$ is a constant that depends on the quantization procedure. Most used values are $c = 0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$. See [Andersson-Driver1999]
  - formal expansion methods: $c = \frac{1}{3}$, $c = \frac{1}{2}$, $c = \frac{2}{3}$;
  - geometric quantization: $c \in [0, 2/3]$. Weyl quantization $c = 0$;
  - path integral: there is an ambiguity reflecting in the ambiguity of c.

for $\hbar \to 0$ for a Riemannian manifolds the curvature term becomes irrelevant, but for 2-ARS this is not the case since it is diverging.

For the Grushin cylinder
(after the unitary transformation $L^2(\frac{1}{|x|}dx\,dy) \to L^2(dx\,dy)$)

$$\begin{aligned} i\hbar\partial_t\psi &= -\frac{\hbar^2}{2m}(\Delta - cK)\psi \\ &= -\frac{\hbar^2}{2m}\Big((\partial_x^2 + x^2\partial_y^2 - \frac{3}{4}\frac{1}{x^2}) - c(-\frac{2}{x^2})\Big)\psi \\ &= \frac{\hbar^2}{2m}\Big(-(\partial_x^2 + x^2\partial_y^2) + \frac{\frac{3}{4} - 2c}{x^2}\Big)\psi \end{aligned}$$

→Clearly for $c = 3/8$ the divergence term disappear and we are not essentially self-adjoint in $\mathbf{R}^+ \times \mathbf{R}$.

→Actually all $c > 0$ destroy the self-adjointness because $\frac{3}{4} - 2c > \frac{3}{4}$ (remember the Reed-Simon theorem).

Theorem

Let M be a 2D compact orientable manifold endowed with a 2-ARS. Assume that

$$(H) \quad \text{for every } q \in M,\ \blacktriangle(q) + [\blacktriangle, \blacktriangle](q) = T_q M.$$

Let Ω be a connected component of $M \setminus \mathcal{Z}$. Then $\Delta - cK(q)$ with domain $C_0^\infty(\Omega)$ is essentially self-adjoint on $L^2(\Omega, dA)$ iff $c = 0$.

→For $c > 0$ one should be able to recover the classical limit for certain self-adjoint extensions. **“How” is an interesting open question**.

→again the compactness hypothesis is not necessary

→For the proof see:

- $c \in [0, 1/2)$ I. Beschastnyi, E. Pozzoli, U. B. Quantum Confinement for the Curvature Laplacian $-\Delta + cK$ on 2D-Almost-Riemannian Manifolds. Potential Theory 2022.
- $c \geq 1/2$, I. Beschastnyi, Closure of the Laplace-Beltrami operator on 2D almost-Riemannian manifolds and semi-Fredholm properties of differential operators on Lie manifolds. arXiv:2104.07745

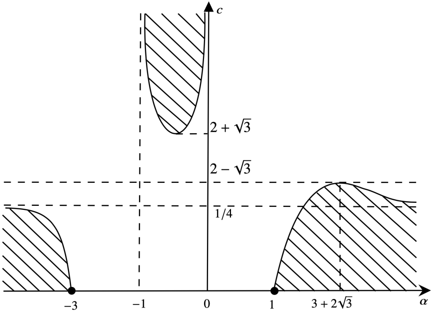
The second case is harder because $-\Delta + cK$ is not “non-negative modulo a Kato small perturbation”.

If we remove the 2-step hypotheses the situation could be very different. For instance for the structure

$$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ x^\alpha \end{pmatrix}, \quad \alpha \in \mathbf{R}$$

(for α integer this structure is almost-Riemannian of step $1+\alpha$)

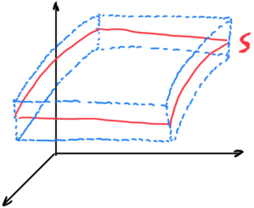
the description of the self-adjointness for $\Delta - cK$ is given by:



the extrinsic approach

Beside geometric quantization, there is a different approach making appearing curvature terms in the operator.

Consider a surface S in $\mathbf{R}^3$ and let us consider an ε-neighborhood of it.



Let us put boundary conditions (Dirichlet-Dirichlet or Neumann-Neumann)

Let us consider the corresponding Laplacians and after a suitable blow up let us send ε to zero. We have:

$L^{\varepsilon}_{NN} = \Delta + o(\varepsilon)$

$L^{\varepsilon}_{DD} = -\left(\frac{\pi}{2\varepsilon}\right)^2 + \Delta + (2K - 4H^2) + o(\varepsilon)$

(these convergences are weak limits on special test functions)

→here K is the Gaussian curvature and H is the mean curvature

→See Duclos, Exner [1995], Lampart, Teufel, Wachsmuth [2010] for quantum waveguides and Krejcirik [2014].

- For the heat equation only the NN (or periodic) boundary conditions make sense since Dirichlet conditions implies a killing of the process.
- For the Schroedinger equation both boundary conditions make sense (DD after renormalization)
- for the Schroedinger equation other mixed conditions make sense.

Comparison between intrinsic and extrinsic quantization

Geometric Quantization	extrinsic quantization
$\Delta - cK, \quad c \geq 0$	Δ (Neumann-Neumann) $\Delta + 2K - 4H^2$ (Dirichlet-Dirichlet)

Claim Geometric quantization coincides with extrinsic quantization only for $c = 0$ and for Neumann-Neumann boundary conditions.

can we embed the Grushin Cylinder in $\mathbf{R}^3$ and compute $\Delta + 2K - 4H^2$?

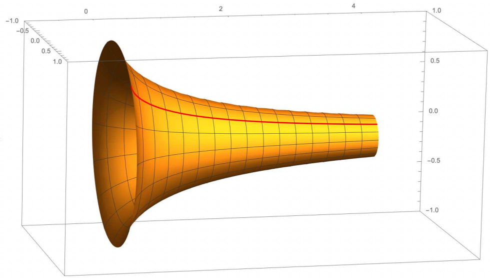
→the embedding of the Grushin cylinder can have other applications

can we compute $\Delta + 2K - 4H^2$ for the Grushin cylinder?

The Grushin cylinder is too singular to be embedded globally in $\mathbf{R}^3$.
However we can embed part of it.

For $x \in [1, \infty[$ and $y \in [0, 2\pi]$ it can be embedded isometrically in $\mathbf{R}^3$ as

$$z_1 = const + \int_1^x \sqrt{1 - \frac{1}{s^4}}\, ds, \quad z_2 = \frac{\cos(y)}{x}, \quad z_3 = \frac{\sin(y)}{x}, \qquad \text{here } (z_1, z_2, z_3) \in \mathbf{R}^3$$



here the coordinate x in $\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{x^2} \end{pmatrix}$ is the curvilinear lenght of the red curve

(Grushin's trumpet bell)

→In this case
$$K = -\frac{2}{x}$$
$$H = \frac{x^4 - 3}{2x\sqrt{x^4 - 1}}$$
$$\Delta + 2K - 4H^2 = (\partial_x^2 + x^2\partial_y^2 - \frac{3}{4}\frac{1}{x^2}) + 2\Big(-\frac{2}{x^2}\Big) - 4\Big(\frac{x^4 - 3}{2x\sqrt{x^4 - 1}}\Big)^2$$
$$= \partial_x^2 + x^2\partial_y^2 - \frac{1}{x-1} + O(1) \text{ for } x \to 1$$

Claim (expected result): On $[1, \infty[\times S^1$, this operator is not essentially self-adjoint.

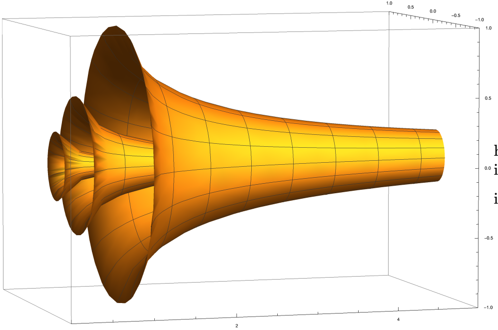
Can we go closer to zero?

If one would like to embed the Grushin cylinder closer to zero up to $1/n$ one should wind it n^2 times.

More precisely for $x \in [\frac{1}{n}, \infty[$ and $y \in [0, 2\pi]$ we can be embedded isometrically in $\mathbf{R}^3$ as the surface:

$$z_1 = const + \frac{1}{n}\int_1^{nx} \sqrt{1 - \frac{1}{s^4}}\, ds, \quad z_2 = \frac{\cos(n^2 y)}{n^2 x}, \quad z_3 = \frac{\sin(n^2 y)}{n^2 x}$$

This embedding is only local since the surface is winded n^2 times.



here $const$ is fixed in such a way that $z_1(x = 1) = 1$ independently of c

In this case $H = \frac{n^4 x^4 - 3}{2x\sqrt{n^4 x^4 - 1}}$ diverges at $x = \frac{1}{n}$ and $\Delta + 2K - 4H^2$ is not essentially self adjoint in $[\frac{1}{n}, \infty[$.

$\rightarrow$it is hard (and not very meaninful) to to say something for $n \to \infty$.

Concluding

while for the heat equation the self-adjointness of Δ on $M \setminus \mathcal{Z}$ (in the 2-step hypothesis) can be well understood in terms of random walks, for the Schroedinger equations this is harder to be interpreted.

$\rightarrow$intrinsic quantizations provides $\Delta - cK$ which is not self-adjoint except for $c = 0$.
$\rightarrow$extrinsic quantization provides Δ (NN) or $\Delta + 2K - 4H^2$ (DD).

- the first choice coincide with $c = 0$ of the intrinsic quantization.
- the second choice cannot be studied at the singularity since the embedding become singular before.

Definition

The *Euler number* a fully orientable 2D almost-Riemannian structure $(\mathbf{U}, f)$ on a compact manifold M is the Euler number $e(\mathbf{U})$ of $\mathbf{U}$. It is the self-intersection number of M in $\mathbf{U}$, where M is identified with the zero section.

self-adjointness

we have to specify what we mean by Δ at the singularity
$\updownarrow$
study the self-adjointness of Δ.

Working with self-adjoint operators is crucial because:

Stone's theorem

On any Hilbert space there is a one-to-one correspondence,

A self-adjoint operator $\longleftrightarrow$ e^{-itA} strongly continuous unitary group

Theorem

On any Hilbert space there is a one-to-one correspondence,

A nonpositive definite self-adjoint operator $\longleftrightarrow$ e^{tA} strongly continuous semigroup

Self-adjointness of the Laplacian

Let Ω be an open subset of M (possibly coinciding with M).

we recall that an operator $L : D(L) \subset \mathcal{H} \to \mathcal{H}$ is (we assume $D(L)$ dense):

- symmetric if $\langle Lu, v\rangle = \langle u, Lv\rangle$
- self-adjoint if it is symmetric and $D(L^*) = D(L)$

As for the standard Laplacian, Δ is never self-adjoint on $L^2(\Omega, \omega)$ since the domain of Δ^* (that is H^2) is larger than the domain of Δ (e.g., C_c^∞, or C_c^2 or $\{\phi \in C^2 \cap L^2 \text{ with } \Delta\phi \in L^2\}$) .

Then one defines Δ on $\mathcal{C}_c^\infty(\Omega)$. We say A is a self-adjoint extension of Δ if

$$D(\overline{\Delta|_{C_c^\infty(N)}}) \subset D(A) = D(A^*) \subset D(\Delta^*)$$

$$A^*\phi = \Delta^*\phi \quad \text{for any } \phi \in D(A).$$

We have two cases:

→**There are several self-adjoint extensions.** Each self-adjoint extension corresponds to a different choice of conditions at $\partial\Omega$.

→**there exists only one such extension.** In this case the operator is called **essentially self-adjoint** and the extension the **Friedrichs** extension. Conditions at $\partial\Omega$ are automatically satisfied.

what is important for us is that

if Δ is essentially self-adjoint then the Cauchy problems for the heat and Schroedinger equations are well posed without the need of boundary conditions

- for the Schroedinger equation the essential self-adjointness is a necessay and sufficient condition for the well-posedness of the problem
- for the heat equation it is a sufficient condition