Homological invariants of Legendrian submanifolds Lecture 1

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Contact structures

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Contact structure ξ *on* Y *= hyperplane distribution maximally non integrable:*

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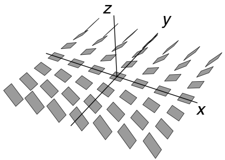

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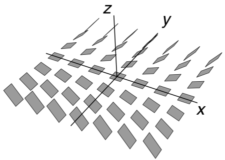

Theorem (Darboux)

This is the unique local model for contact structures.

Legendrian submanifolds

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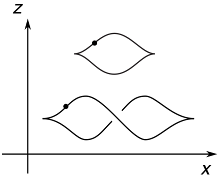

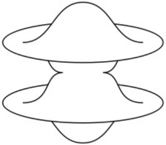

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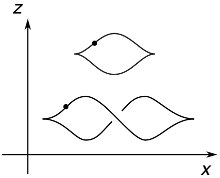

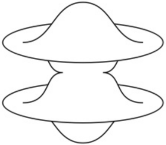

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It is natural to study Λ modulo Legendrian isotopy
$\rightsquigarrow$ generalization of knot theory

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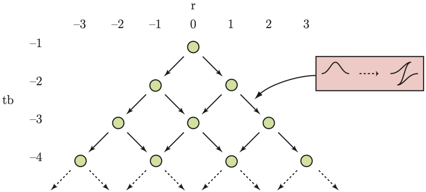

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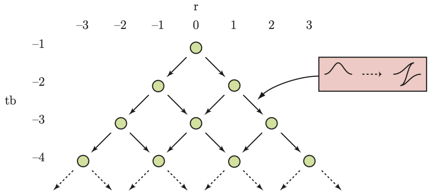

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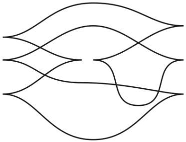

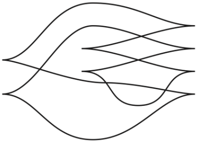

Chekanov $\overline{5_2}$ knots: classical invariants do not always suffice.

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Gromov (1985): use almost complex structure J on (X, ω)
$\rightsquigarrow$ Gromov-Witten invariants = count of J-holomorphic curves.

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Symplectization $(\mathbb{R} \times Y, \omega = d(e^t \lambda))$, with $t \in \mathbb{R}$.
For $(Y, \xi) = (\mathbb{R}^{2n-1}, \xi_0)$, extend J to $\mathbb{R} \times Y \ni (t, p)$ by $J\frac{\partial}{\partial t} = \frac{\partial}{\partial z}$.
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BUT all such F are constant: $\int_{\mathbb{D}^2} F^*\omega = \int_{\partial\mathbb{D}^2} F^*(e^t\lambda) = 0$.

Holomorphic disks in symplectizations

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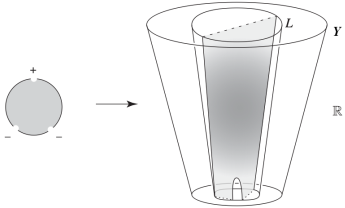

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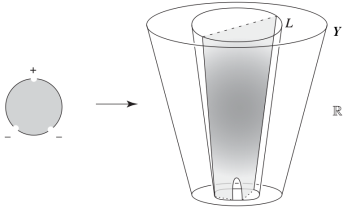

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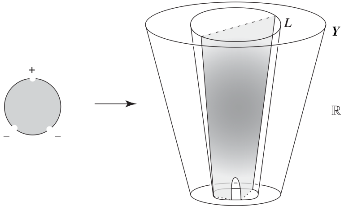

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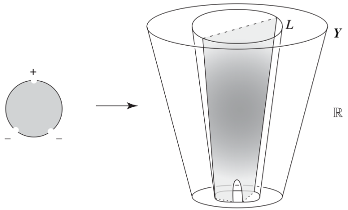

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$\int_{\mathbb{D}^2} F^* d\omega = +\infty$ and $0 \le \int_{\mathbb{D}^2} F^* d\lambda = \sum_{i=1}^k T_i^+ - \sum_{i=1}^\ell T_i^-$.

Constructing Legendrian invariants

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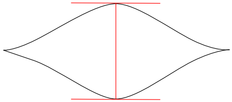

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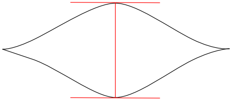

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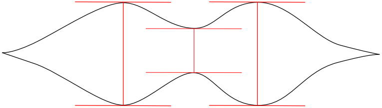

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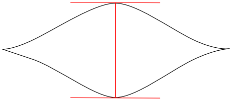

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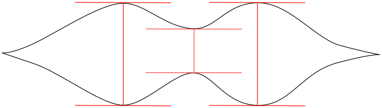

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Instead, count of disks $\rightsquigarrow$ coefficients in a differential:
$\rightsquigarrow$ chain complex generated by Reeb chords,
$\rightsquigarrow$ homological invariants of Legendrian submanifolds.

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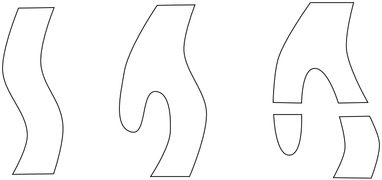
Limits of analogy: compactification of moduli spaces.



More negative punctures can appear.

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	$k = 1$ and genus 0		$k \geq 1$ and $\ell \geq 0$	
	"$\ell = 1$"	$\ell \geq 0$	genus $= 0$	genus ≥ 0
no Λ	cylindrical or linearized CH	contact homol. (CH)	rational SFT	SFT
Λ	linearized LCH	Leg. CH (LCH)	version due to Ekholm	not defined

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Very general framework (Eliashberg, Givental, Hofer) for holomorphic curves in symplectic cobordisms (with or without Λ).

Hol. curves with k positive and ℓ negative punctures.
Remember that $\sum_{i=1}^{k} T_i^+ - \sum_{i=1}^{\ell} T_i^- \geq 0$ so $k \geq 1$.

	$k = 1$ and genus 0		$k \geq 1$ and $\ell \geq 0$	
	"$\ell = 1$"	$\ell \geq 0$	genus $= 0$	genus ≥ 0
no Λ	cylindrical or linearized CH	contact homol. (CH)	rational SFT	SFT
Λ	linearized LCH	Leg. CH (LCH)	version due to Ekholm	not defined

Next lectures:

focus on linearized LCH, its variants and properties.