Homological invariants of Legendrian submanifolds Lecture 3

Frédéric Bourgeois

Laboratoire de Mathématiques d'Orsay

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Examples of closed contact manifolds

Example 1: $Y = S^{2n-1} \subset \mathbb{C}^n$ with coordinates $z_j = x_j + iy_j$,
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$\rightsquigarrow$ Reeb flow is the geodesic flow for (M, g).

Graph trick in symplectic geometry

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Definition

A contactomorphism ψ of $(Y, \xi = \ker \lambda)$ is a diffeomorphism of Y such that $\psi^*\lambda = e^g\lambda$ for some function g on Y.

Notion of contact product

If (Y_1, ξ_1) and (Y_2, ξ_2) are contact manifolds,
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The lifted graph of ψ, defined by

$$\hat{\Gamma}_\psi = \{(x, \psi(x), g(x)) \mid x \in Y, \psi^* \lambda = e^g \lambda\}$$

is a Legendrian submanifold of $(\hat{Y}, \hat{\xi})$.

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- True if (Y, ξ) is hypertight, i.e. there exists λ such that R_λ has no contractible periodic orbit (Albers, Fuchs, Merry).

Legendrian approach to translated points

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$\rightsquigarrow$ LCH-type theories can be used to prove this conjecture.

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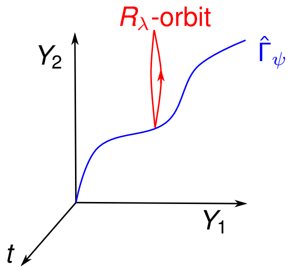

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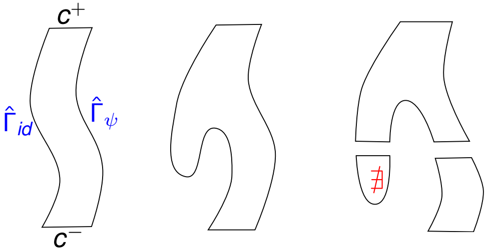

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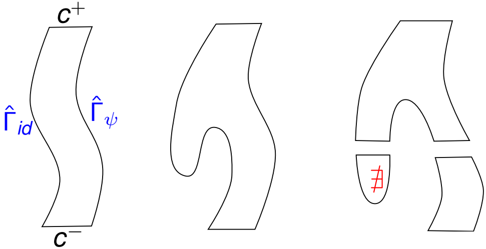

Define $\partial : C_* \to C_{*-1}$ by $\partial c = \sum_{c^-} \#\mathcal{M}(c^+; c^-)/\mathbb{R}\ c^-$,
so that $\partial \circ \partial = 0$.

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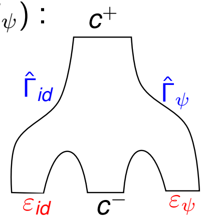

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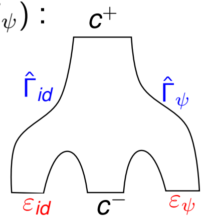

ε_{id} and ε_{ψ} can be induced from an augmentation of (Y, ξ).

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$\rightsquigarrow \hat{\Lambda} = \Lambda_1 \times \Lambda_2 \times \mathbb{R} \subset (Y_1 \times Y_2 \times \mathbb{R}, \hat{\xi} = \ker(e^t \lambda_1 + e^{-t} \lambda_2))$
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$\rightsquigarrow \hat{\Lambda} = \Lambda_1 \times \Lambda_2 \times \mathbb{R} \subset (Y_1 \times Y_2 \times \mathbb{R}, \hat{\xi} = \ker(e^t \lambda_1 + e^{-t} \lambda_2))$
is Legendrian.

$R_{\hat{\lambda}} = \frac{1}{2}(e^{-t} R_{\lambda_1} + e^t R_{\lambda_2})$.
$\rightsquigarrow$ Chords of $\hat{\Lambda} \overset{1:1}{\longleftrightarrow}$ (chords of Λ_1) $\times$ (chords of Λ_2).
$\rightsquigarrow \hat{\mathscr{A}} = \mathscr{A}_1 \otimes \mathscr{A}_2$.

Theorem (Zénaïdi)

If (Y_i, ξ_i) are contactizations of exact sympl. mfds with $c_1 = 0$, if $H_1(\Lambda_i) = 0$ and ε_i are augmentations for Λ_i, $i = 1, 2$, then there exists $\hat{\varepsilon}$ augmentation for $\hat{\Lambda}$ such that

$$LCH^{\varepsilon_1}(\Lambda_1) \otimes LCH^{\varepsilon_2}(\Lambda_2) \simeq LCH^{\hat{\varepsilon}}(\hat{\Lambda}).$$

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This is a step towards an axiomatic definition of LCH^{ε}.

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*Legendrian $\Lambda, \Lambda' \subset (Y^3, \xi)$ with λ_0 hypertight, adapted to Λ, Λ'.
If $LCH^{strip}(\Lambda \to \Lambda')$ has exp. homotopical growth rate $a > 0$ (*),
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(*) means:
the number of homotopy classes ρ of paths from Λ to Λ'
containing only Reeb chords of length $< C$
and such that $LCH^{strip,\rho}(\Lambda \to \Lambda') \neq 0$
grows faster than e^{aC+b} for $a > 0$.

Complexity of Reeb flows (2)

Corollary (Alves)

We have $h_{top} > 0$ for all Reeb flows on:

- $(ST^*\Sigma_g, \xi_{can})$ *for $g \geq 2$ (Schlenk, Macarini),*
- *a class of toroidal contact 3-manifolds constructed by Colin,*
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And using similar techniques based on LCH^{strip}:

Theorem (Alves, Colin, Honda)

We have $h_{top} > 0$ for all Reeb flows on (Y^3, ξ)
having a supporting open book decomposition
with connected binding and pseudo-Anosov monodromy
with fractional Dehn twist coefficient $\frac{k}{n}$ such that $k \geq 5$.

Thank you for your attention!