DG-Algebraic Aspects of Contact Invariants

42th Winter School in Geometry and Physics, Srní

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15 Jan – 22 Jan 2022

Plan

Lecture I:

- DGAs and DG-modules
- Derived category & **R**Hom
- Bar resolution, short resolution

Lecture II

- Koszul resolution (commutative case)
- Functoriality and invariance of **R**Hom.
- A_∞-category of bounding cochains dual to the DGA

Lecture III:

- A_∞-category of bounding cochains dual to the DGA
- Equivalence with DG-category of modules
- Augmentation variety

Motivation

- The DGAs that we consider here arise as algebraic invariants in symplectic topology, with the differentials defined by counts of pseudoholomorphic curves.
- The main example is the Chekanov–Eliashberg algebra of a Legendrian $\mathcal{A}_{C_*\Omega(\Lambda)}(\Lambda)$ which is a Legendrian isotopy invariant of the Legendrian submanifold Λ of a contact manifold.

Motivation

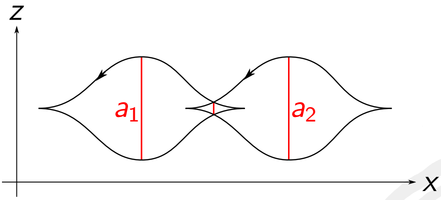

Figure: The front projection of the Hopf link Λ_{Ho} in a Darboux chart.

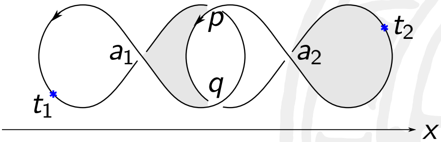

Figure: The Lagrangian projection (to the xy-plane) of the Hopf link in a Darboux chart. The shaded disc on the right contributes to $\partial a_1 = qp$, and the shaded disk on the right gives $\partial a_2 = t_2$.

Motivation

- The DGAs that we consider here arise as algebraic invariants in symplectic topology, with the differentials defined by counts of pseudoholomorphic curves.
- The main example is the Chekanov–Eliashberg algebra of a Legendrian $\mathcal{A}_{C_*\Omega(\Lambda)}(\Lambda)$ which is a Legendrian isotopy invariant of the Legendrian submanifold Λ of a contact manifold.
- Since the DGA is an infinite-dimensional free algebra, it is rather complicated to distinguish two DGAs up to quasi-isomorphism. One of the goals with these lectures is to introduce the **Augmentation Variety** by Ng [Ng08] and the **Augmentation Category** due to Bourgeois–Chantraine [BC14].
- An important point: These constructions are purely algebraic; i.e. once geometry has given us the DGA, we can proceed to study algebra.

Motivation

The DGA $\mathcal{A}$ associated to a Legendrian is a Legendrian isotopy invariant. There are related invariants associated to related geometric objects.

- Passing to its derived category $D^b(\mathcal{A})$, or the subcategories $Perf(\mathcal{A})$ of perfect complexes, or $\mathcal{T}w(\mathcal{A}) \subset Perf(\mathcal{A})$ of twisted complexes, one obtains an invariant of the symplectic manifold obtained by a Weinstein-handle attachment on the Legendrian. (See the surgery formula by Bourgeois–Ekholm–Eliashberg [BEE12].)
- In many cases this derived category has been shown to be equivalent to the derived category $D^b(Coh(X^\vee))$ of coherent sheaves on a mirror algebraic variety $X^\vee$. (Kontsevich's homological mirror symmetry.)

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Lecture I

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Section 2

DG algebras and modules

Preliminaries

- All algebras here will be associative and unital **k**-algebras for some field **k**; when $\operatorname{char} \mathbf{k} \neq 2$, we need all gradings to be in an abelian group G for which $G/2G \neq 0$ (e.g. $G = \mathbb{Z}_2$ or $\mathbb{Z}$).
- Chain complexes (C_*, ∂) will be endowed with a differential of degree -1, i.e. $\partial \colon C_* \to C_{*-1}$.
- Co-complexes (C^*, d) will be endowed with a differential of degree 1, i.e. $d \colon C^* \to C^{*+1}$.
- Suspension $(\Sigma C)_* = C[1]_* := C_{*-1}$ induces an auto-equivalence on the category of complexes, we require that $\Sigma\partial = (-1)\partial$ and $\Sigma d = (-1)d$.

DG algebras

Definition 2.1

A *differential graded algebra (DGA)* is a graded chain complex $(\mathcal{A}_*, \partial)$ endowed with a multiplication $m\colon \mathcal{A}_* \otimes_{\mathbf{k}} \mathcal{A}_* \to \mathcal{A}_*$, written $m(a \otimes b) = a \cdot b$, which is a chain map of degree zero, and which makes $\mathcal{A}$ into a unital associative $\mathbf{k}$-algebra.

Consequences:

- The grading satisfies $|a \cdot b| = |a| + |b|$;
- $1 \in \mathcal{A}_0$;
- The *graded Leibniz rule*: $\partial(a \cdot b) = \partial(a) \cdot b + (-1)^{|a|} a \cdot \partial(b)$. (Recall that the differential on the tensor product is $\partial^{\otimes}(a \otimes b) = (\partial a) \otimes b + (-1)^{|a|} a \otimes \partial b$).

DG modules

Definition 2.2

A *(left) differential graded module (DGM)* over a DGA $\mathcal{A}$ is a chain complex (M_*, ∂_M) which is a left $\mathcal{A}_*$-module in the classical sense, and for which left multiplication $l\colon \mathcal{A}_* \otimes_{\mathbf{k}} M_* \to M_*$, written $l(a \otimes m) = a \cdot m$, is a chain map of degree zero.

We write $M \in \mathcal{A}$-Mod; The definitions for right modules Mod-$\mathcal{A}$ and bimodules $\mathcal{A}$-Mod-$\mathcal{A}$ are analogous.

Consequences:

- Grading satisfies $|a \cdot m| = |a| + |m|$;
- $\partial_M(a \cdot m) = \partial(a) \cdot m + (-1)^{|a|} a \cdot \partial_M(m)$;

DG modules

Remark 2.3

- Recall that a $\mathcal{A}$ DG-bimodule can be equivalently characterised as a left $\mathcal{A} \otimes_{\mathbf{k}} \mathcal{A}^{op}$-DG module.
- It is immediate that $\mathcal{A}$ itself is both a left, right DG $\mathcal{A}$-module, and hence a DG-bimodule; we call it the *diagonal bimodule*.
- It is important to note that, while $\mathcal{A} \otimes_{\mathbf{k}} \mathcal{A}$ is free as a bimodule, $\mathcal{A}$ is not.

DG category of morphisms

For an $M, N \in \mathcal{A}$-Mod, the classical (non-DG) $\mathcal{A}$-module morphisms φ^i of homogeneous degree i are denoted by

$$\mathcal{A}^i_{dg}(M, N)$$

(i.e. φ^i might not be a chain map). We will call the elements of $\mathcal{A}_{dg}(M, N)$ pre-DG module morphisms.

- $\mathcal{A}^*_{dg}(M, N)$ a chain complex when endowed with the differential

  $$D(\varphi^i) = (-1)^i \varphi^i \circ \partial_M - \partial_N \circ \varphi^i$$

  of degree -1.
- The endomorphisms

  $$\mathcal{A}^*_{dg}(M, M)$$

  naturally becomes a DGA when endowed with the above differential, and multiplication given by composition.

DG category of morphisms

The above makes the DG-modules form the objects of a *DG-category* that we denote by $\mathcal{C}_{dg}(\mathcal{A})$.
We do not give more details here about DG-categories here, but direct the reader to [Kel94] for the definition. Later we will relate this category to an A_∞-category associated to the same DGA, which is a generalisation of the notion of a DG-category.

DG category of morphisms

Exercise 2.4

Verify that

1. $\mathcal{A}^*_{dg}(N, N)$ is a DGA;
2. $\mathcal{A}^*_{dg}(M, N)$ is a left (resp. right) DG-module over $\mathcal{A}^*_{dg}(N, N)$ (resp. $\mathcal{A}^*_{dg}(M, M)$).
3. $\mathcal{A}$ is naturally a left $\mathcal{A}$-DG module and there is a natural isomorphism
$$\mathcal{A}^*_{dg}(\mathcal{A}, \mathcal{A}) = \mathcal{A}^{op}_*$$
of DGAs.
4. There is a natural isomorphism $\mathcal{A}^*_{dg}(\mathcal{A}, N) = N_*$ of right $\mathcal{A}^{op}$-DG modules (equivalently left $\mathcal{A}$-DG modules).

DG-module morphisms

Definition 2.5

The DG-morphisms are the cycles in degree zero

$$\mathrm{Hom}_{\mathcal{A}\text{-Mod}}(M,N) := Z_0(\mathcal{A}^*_{dg}(M,N)) \subset \mathcal{C}^0_{dg}(M,N)$$

I.e. pre DG-module morphisms that moreover are degree zero chain maps.

This subset of morphisms gives rise to a sub-category

$$\mathcal{C}(\mathcal{A}) \subset \mathcal{C}_{dg}(\mathcal{A})$$

whose objects are the same.

DG modules

Remark 2.6

- The complex $\mathcal{A} \otimes_{\mathbf{k}} M$ is a left DG $\mathcal{A}$-module and that $l \colon \mathcal{A} \otimes_{\mathbf{k}} M \to M$ is a DG-module morphism by the associativity of module and algebra multiplication;
- Similarly, the map $m \colon \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \to \mathcal{A}$ is moreover a DG-bimodule morphism.

Section 3

Derived DG-category

Quasi-isomorphism

Definition 3.1

A morphism of DGAs or DG-modules is said to be a *quasi-isomorphism* if it induces isomorphism in homology.

- The DG-category $\mathcal{C}_{dg}(M, N)$ itself is not the correct object here; we need a category that behaves well with respect to e.g. quasi-isomorphism class of $\mathcal{A}$;
- Grothendieck and Verdier introduced the derived category. There are now several alternative invariants that are based upon the same basic idea: We need to a pass to a category in which quasi-isomorphic modules are isomorphic.
- Main problem: If $\varphi \colon M \to N$ is a quasi-isomorphism, the homology inverse is not necessarily representable by an chain map $\psi \colon N \to M$.

Semi-free modules

- A good feature of free modules: it is easy to construct maps from them.
- On the other hand, it is difficult to construct maps *to* them (unless the domain is free).
- Following Keller [Kel94] we construct the full DG-subcategory

$$\mathcal{SF}(\mathcal{A}) \subset \mathcal{C}_{dg}(\mathcal{A})$$

consisting of *semi-free DG-modules*; the objects are isomorphic to "iterated cones of free modules"; The finitely generated semi-free DG-modules will play an important role, and are denoted by

$$\mathcal{T}w(\mathcal{A}) \subset \mathcal{SF}(\mathcal{A})$$

(The notation "$\mathcal{T}w$" alludes to *twisted complexes*.)

Semi-free modules

Definition 3.2

- A DG-module S_* is said to be *free* if

$$S_* \cong \bigoplus_{\iota \in \mathcal{I}} \mathcal{A}[i_\iota]$$

- A DG-module S_* is said to be *semi-free* if there is a filtration

$$F_1 M \subset F_2 M \subset \ldots\ldots F_j M \subset \ldots S_*$$

  by DG-submodules in which $F_{j+1}S/F_jS$ are free modules. In particular; $F_1 S$ is free.

Semi-free modules

Up to isomorphism, we can identify a semi-free DG-module S_* with

$$S_* \cong \mathcal{A} \otimes_{\mathbf{k}} V$$

where V is a graded vector space with an additional filtration $F_i V$, so that for $v \in F_i V \setminus F_{i-1} V$ we have

$$\partial(a \otimes v) = \partial(a) \otimes v + \partial_E(a \otimes v)$$

where $\partial_E(a \otimes v) \subset \mathcal{A} \otimes_{\mathbf{k}} F_{i-1} V$. (The filtration on S_* is thus given b

$$\mathcal{A} \otimes_{\mathbf{k}} F_1 V \subset \ldots \mathcal{A} \otimes_{\mathbf{k}} F_2 V \subset \ldots \subset S_*.$$

In particular, the differential consists of the internal differential induced by the differential on $\mathcal{A}$, and a filtration-decreasing "external" differential ∂_E.

Semi-free modules

Exercise 3.3

Show that for a semi-free module S_*, the external differential

$$\partial_E|_{F_{j+1}M/F_jM} \colon F_{j+1}M/F_jM \to F_jM$$

of the differential is itself a DG-module morphism.

It follows that

Lemma 3.4

Any semi-free DG-module can be built by starting with a free DG-module F_0, and subsequently taking iterated mappings cones

$$\mathrm{Cone}(F_k \to \mathrm{Cone}(F_{k-1} \to \mathrm{Cone}(\ldots \to \mathrm{Cone}(F_1 \to F_0))))$$

from free DG-modules F_i.

Semi-free replacement

An important feature of semi-free modules is the following:

Lemma 3.5

1. *For any $M \in \mathcal{C}_{dg}(\mathcal{A})$ we can find a semi-free module S and a quasi isomorphism $S \to M$ of DG $\mathcal{A}$-modules, which moreover can be taken to be surjective.*
2. *If M_* is acyclic, then $\mathcal{A}^*_{dg}(S, M)$ and $N_* \otimes_{\mathcal{A}} S_*$ are also acyclic for any semifree S_*, and right DG $\mathcal{A}$-module N_*.*

Remark 3.6

Assumptions must be made on M in order to ensure that the complex is bounded from below; if the grading is not $\mathbb{Z}$-valued and bounded from below, we need to equip M with e.g. an action filtration that is bounded from below.

Semi-free replacement

Proof.

(1): Later we will construct the bar complex, which is a resolution which always work. In general, the idea is as follows: Start by covering the cycles in M. Then add relations by taking cones Continue if necessary. (In general we might need infinitely many iterations.)

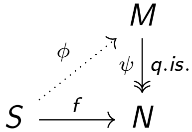
(2): The statement is clear if we replace S by the free module $F_1 S$. Then the 5-lemma inductively to show acyclicity for S replaced with $F_i S$ for any $i \geq 0$. □

Semi-free replacement

The next important feature is "cofibrancy":

Lemma 3.7

For a surjective quasi-isomorphism ψ and morphism $f\colon S \to N$ from a semi-free module S, there exists a lift ϕ that makes the following diagram of DG-module morphisms commute.



(Here all morphisms are DG-module morphisms.)

Semi-free replacement

Corollary 3.8

Let M and N be DG-modules, and let $S_M \to M$ and $S_N \to N$ be surjective quasi-isomorphisms from semi-free DG-modules. Then the complex

$$\mathbf{R}\mathrm{Hom}^*_{\mathcal{A}}(M,N) := \mathcal{A}^*_{dg}(S_M, S_N) \cong \mathcal{A}^*_{dg}(S_M, N)$$

is well-defined up to quasi-isomorphism, independently on the choice of semi-free modules S_M and S_N.

Definition 3.9

$$\mathrm{Ext}_{\mathcal{A}}(M,N) := H(\mathbf{R}\mathrm{Hom}^*_{\mathcal{A}}(M,N))$$

Semi-free replacement

In particular,

- $\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(M, M)$ is a unital DGA, and
- $\mathrm{Ext}_{\mathcal{A}}(M, M)$ is a unital algebra.

Remark 3.10

These algebras are typically not commutative, even in the case when $\mathcal{A}$ is commutative; they capture endomorphisms of certain high rank modules.

Semi-free replacement

- The quasi-isomorphism

$$\mathcal{A}^*_{dg}(S_M, S_N) \cong_{q.is.} \mathcal{A}^*_{dg}(S_M, N)$$

  is a consequence of Part (2) of Lemma 3.5, since $g\colon S_N \to N$ is a quasi-isomorphism. Consider e.g. $\mathcal{A}^*_{dg}(S_M, \mathrm{Cone}(g))$ for the acyclic mapping cone $\mathrm{Cone}(g)$.
- The surjection $f\colon S_M \to M$ induces a natural inclusion of complexes

$$\mathcal{A}^*_{dg}(M, N) \hookrightarrow \mathcal{A}^*_{dg}(S_M, N) \cong_{q.is.} \mathbf{R}\mathrm{Hom}^*_{\mathcal{A}}(M, N).$$

  Studying a concrete semi-free replacement such as the bar-complex in the next section, one can see that the induced map in homology is an inclusion as well.

Section 4

Examples of DGAs and resolutions

Bar resolution

The *unreduced bar resolution* or *standard resolution* of a unital associative DGA is the infinitely generated semi-free $\mathcal{A}$-bimodule

$$S_{\mathcal{A}}^{bar} := \left\{ \ldots \xrightarrow{\partial_E^{(5)}} \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \xrightarrow{\partial_E^{(4)}} \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \xrightarrow{\partial_E^{(3)}} \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \right\}$$

where the external differential is given by

$$\partial_E^{(n+2)}(a_0 \otimes \ldots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \ldots \otimes m(a_i \otimes a_{i+1}) \otimes \ldots \otimes a_{n+1}.$$

(In fact, it is a bimodule morphism, so it would be enough to describe it on a free generating set.)

Lemma 4.1

The morphism $m = \partial_E^{(2)} \colon S_{\mathcal{A}} \to \mathcal{A}$ to the non-free diagonal bimodule $\mathcal{A}$ is a quasi-isomorphism.

Bar resolution

Lemma 4.2

The morphism $\partial_E^{(2)} \colon S_{\mathcal{A}}^{bar} \to \mathcal{A}$ *to the non-free diagonal bimodule* $\mathcal{A}$ *is a quasi-isomorphism.*

Proof.

Consider the mapping cone $\mathrm{Cone}(\partial_E^{(2)})$. The identity of this complex is homotopic to zero via the e.g. the chain homotopy

$$h(a_0 \otimes \ldots \otimes a_n) = \{1\} \otimes a_0 \otimes \ldots \otimes a_n \in \mathcal{A}^{\otimes_{\mathbf{k}} n+1}$$

of $\mathbf{k}$-complexes (this map is not $\mathcal{A}$-linear!). □

Bar resolution

- Since $\mathcal{S}_{\mathcal{A}}^{bar}$ is free as a right DG $\mathcal{A}$-module, and since $\mathcal{A} \otimes_{\mathcal{A}} M = M$ for any left DG $\mathcal{A}$-module, Part (2) of Lemma 3.5 implies that
$$S_M := \mathcal{S}_{\mathcal{A}}^{bar} \otimes_{\mathcal{A}} M$$
admits a surjective quasi-isomorphism onto M induced by $\partial_E^{(2)}$.
- We can thus always use this (unfortunately very big) resolution when computing $\mathbf{R}\mathrm{Hom}_{\mathcal{A}}^{*}(M, \cdot)$.

Bar resolution

To conclude, the *unreduced bar-resolution* is given

$$S^{bar}_{\mathcal{A}} := \left\{ \ldots \xrightarrow{\partial^{(5)}_E} \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \xrightarrow{\partial^{(4)}_E} \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \xrightarrow{\partial^{(3)}_E} \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \right\}$$

where the external differential is given by

$$\partial^{(n+2)}_E(a_0 \otimes \ldots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \ldots \otimes m(a_i \otimes a_{i+1}) \otimes \ldots \otimes a_{n+1},$$

with a canonical map

$$m \colon S^{bar}_{\mathcal{A}} \to \mathcal{A}.$$

Bar resolution

Taking the tensor product with M we get the left $\mathcal{A}$ DG-module

$$S_{\mathcal{A}}^{bar} \otimes_{\mathcal{A}} M :=$$
$$\left\{ \ldots \xrightarrow{\partial_E^{(5)}} \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \otimes_{\mathbf{k}} M \xrightarrow{\partial_E^{(4)}} \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \otimes_{\mathbf{k}} M \xrightarrow{\partial_E^{(3)}} \mathcal{A} \otimes_{\mathbf{k}} M \right\}$$

where the external differential is given by

$$\begin{aligned} &\partial_E^{(n+2)}(a_0 \otimes \ldots a_n \otimes m) = \\ &\quad = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes m(a_i \otimes a_{i+1}) \otimes \ldots \otimes m \\ &\quad + (-1)^{n+1} a_0 \otimes \ldots \otimes a_{n-1} \otimes l(a_n \otimes m) \end{aligned}$$

with a canonical quasi-isomorphism

$$l \colon S_{\mathcal{A}}^{bar} \to M.$$

A short resolution

The geometry gives us DGAs of a very particular type. In particular, the Chekanov–Eliashberg DGA (a Legendrian isotopy invariant) is typically a DGA of the form of a fully non-commutative polynomial algebra

$$\mathcal{A} = \mathbf{k}\langle a_1, \ldots a_n \rangle$$

of a finite number of variables a_i in homogeneous degree. (There is moreover an action-filtration, and the differential is strictly action-decreasing.)
Again there is a surjective morphism from a free bimodule

$$m \colon \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \to \mathcal{A}$$

to the diagonal bimodule given by the algebra multiplication.

A short resolution

Lemma 4.3

If the differential of $\mathcal{A}$ is action-decreasing (or grading is in $\mathbb{Z}$ and bounded from below), then the kernel

$$\ker m = \langle (a_i \otimes 1 - 1 \otimes a_i) \rangle \subset \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A}$$

is itself a semi-free bimodule generated by $\hat{a}_i = a_i \otimes 1 - 1 \otimes a_i$

Proof.

Forgetting the differential we obtain a free bimodule. The action filtration gives us the sought iterated cone structure. □

A short resolution

Lemma 4.4

If the differential of $\mathcal{A}$ is action-decreasing (or grading is in $\mathbb{Z}$ and bounded from below), then the kernel

$$\ker m = \langle (a_i \otimes 1 - 1 \otimes a_i) \rangle \subset \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A}$$

is itself a semi-free bi-submodule generated by $\hat{a}_i = a_i \otimes 1 - 1 \otimes a_i$

Hence there is a resolution

$$S_{\mathcal{A}}^{short} := \left(\bigoplus_{i=1}^{n} (\mathcal{A} \otimes_{\mathbf{k}} \mathcal{A}) \hat{a}_i, \hat{\partial} \right) \xrightarrow{\partial_E} \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A},$$

$$\partial_E(\hat{a}_i) = a_i \otimes 1 - 1 \otimes a_i,$$

of the diagonal bimodule, which has the good property of having *finite rank*.

A short resolution

The short resolution

$$S^{short}_{\mathcal{A}} := \left(\bigoplus_{i=1}^{n} (\mathcal{A} \otimes_{\mathbf{k}} \mathcal{A})\hat{a}_i, \hat{\partial} \right) \xrightarrow{\partial_E} \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A},$$
$$\partial_E(\hat{a}_i) = a_i \otimes 1 - 1 \otimes a_i,$$

has a differential $\hat{\partial}$ obtained as follows.
Write the differential of $\mathcal{A}$ of a generator a_i as

$$\partial(a_i) = c_0 + \sum_j c_j b_{j_1} \cdot \ldots \cdot b_{j_{k_j}}, \;\; c_j \in \mathbf{k},$$

then the bimodule $\oplus_{i=1}^{n}(\mathcal{A} \otimes_{\mathbf{k}} \mathcal{A})\hat{a}_i$ is endowed with the differential

$$\hat{\partial}(w_1 \hat{a}_i w_2) = \sum_j \sum_k (-1)^k c_j w_1 b_{j_1} \cdot \ldots b_{j_{k-1}} \hat{b}_{j_k} b_{j_{k+1}} \ldots \cdot b_{j_{k_j}} w_2.$$

Augmentations

Definition 4.5

An *augmentation* is a degree zero DGA-morphism $\varepsilon\colon \mathcal{A} \to \mathbf{k}$

Since $\mathbf{k} = \mathrm{End}_{\mathbf{k}}(\mathbf{k})$ any choice of augmentation endows $\mathbf{k}$ with the structure of a left DG $\mathcal{A}$-module structure with multiplication

$$l(a \otimes m) = \varepsilon(a) \cdot m;$$

we denote this module by $\mathbf{k}_\varepsilon$.

Computation of **R**Hom

Exercise 4.6

Consider the noncommutative free graded algebra

$$\mathcal{A} = \mathbf{k}\langle a_1, \ldots, a_n \rangle$$

with trivial differential.
Compute

$$H(\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(\mathbf{k}_{\varepsilon_0}, \mathbf{k}_{\varepsilon_1})) = H\left(\mathbf{k} \cdot \mathrm{Id} \oplus \bigoplus_{i=1}^{n} \mathbf{k}[|a_i| + 1] \cdot a_i, d\right)$$

where the differential is determined by

$$d(\mathrm{Id}) = \sum_i (\varepsilon_1(a_i) - \varepsilon_0(a_i)) a_i.$$

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