UPPSALA UNIVERSITET

DG-Algebraic Aspects of Contact Invariants

42th Winter School in Geometry and Physics, Srní

Georgios Dimitroglou Rizell

Uppsala University

15 Jan – 22 Jan 2022

Plan

Lecture I:

- DGAs and DG-modules
- Derived category & **R**Hom
- Bar resolution, short resolution

Lecture II

- Koszul resolution (commutative case)
- Functoriality and invariance of **R**Hom.
- A_∞-category of bounding cochains dual to the DGA

Lecture III:

- A_∞-category of bounding cochains dual to the DGA
- Equivalence with DG-category of modules
- Augmentation variety

Recap

Last lecture we saw:

- How to define a "DG category" denoted by $\mathcal{C}_{dg}(\mathcal{A})$ associated to any DGA, whose objects are DG-modules, endomorphisms are themselves DGAs denoted by $\mathcal{A}^*_{dg}(M, M)$, and morphisms are DG-modules $\mathcal{A}^*_{dg}(M, N)$ over the latter DGAs.
- The right-derived Hom complex can then be constructed as follows:
  - Replace M and N with semi-free modules S_M and S_N; then
  - $\mathbf{R}\mathrm{Hom}^*_{\mathcal{A}}(M, N) = \mathcal{A}^*_{dg}(S_M, S_N)$.

UPPSALA UNIVERSITET

Lecture II

UPPSALA UNIVERSITET

Section 2

The Koszul resolution ($\mathcal{A}$ commutative)

The Koszul resolution

When $\mathcal{A}$ is commutative algebra, e.g. with trivial differential. When $\mathcal{A}$ is the regular function ring of a smooth n-dimensional affine algebraic variety, then the *Koszul resolution* is a well-behaved resolution of $\mathcal{A}$ as an $\mathcal{A}$-bimodule. It is given by:

$$S_{\mathcal{A}}^{Koszul} := \{\Lambda^n(E) \xrightarrow{\partial_E^{(n)}} \ldots \to \Lambda^2(E) \xrightarrow{\partial_E^{(2)}} \Lambda^1(E) \xrightarrow{\partial_E^{(1)}} \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A}\}.$$

- $E = \mathcal{A} \otimes_{\mathbf{k}} \Omega^1(\mathcal{A})$ is a free $\mathcal{A}$-bimodule,
- $\Omega^k(\mathcal{A})$ is the Kähler differential k-forms,
- $\Lambda^k(E) = \mathcal{A} \otimes_{\mathbf{k}} \Omega^k(\mathcal{A})$.

The differentials are given by

$$\partial_E^{(1)}(a \otimes b\, dx) = ax \otimes b - a \otimes xb.$$

The Koszul resolution

The higher differentials are given by

$$\partial_E^{(k)}(a \otimes b\, dx_1 \wedge \ldots \wedge dx_k) =$$
$$= \sum_{i=1}^{k}(-1)^{i+1}\left(ax_i \otimes b\, dx_1 \wedge \ldots dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_k + \right.$$
$$\left. -a \otimes bx_i\, dx_1 \wedge \ldots dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_k\right).$$

- The canonical bimodule morphism $m\colon \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \to \mathcal{A}$ trivially extends to a quasi isomorphism of $\mathcal{A}$-bimodules

$$S_{\mathcal{A}}^{Koszul} \xrightarrow{q.is} \mathcal{A}.$$

- For a left $\mathcal{A}$-module M_* we then obtain a semi-free resolution by the tensor product

$$S_M = S_{\mathcal{A}}^{Koszul} \otimes_{\mathcal{A}} M_* \xrightarrow{q.is} M_*.$$

Augmentations

We recall the following basic but important definition from last time

Definition 2.1

An *augmentation* is a degree zero DGA-morphism $\varepsilon \colon \mathcal{A} \to \mathbf{k}$.

- An augmentation is equivalent to the choice of structure of $\mathcal{A}$-module on the one-dimensional vector space $\mathbf{k}$, we write $\mathbf{k}_\varepsilon$ for the corresponding module.
- When $\mathcal{A}$ is an affine $\mathbb{C}$-algebra, augmentations are in bijection with *maximal ideals* $\ker \epsilon \subset \mathcal{A}$.
- Maximal ideals, in turn, correspond to *points* on the variety $\mathbf{Sp}(\mathcal{A})$ by Hilbert's Nullstellensatz.

Computation of $\mathbf{R}$Hom

Exercise 2.2

Consider the commutative polynomial ring

$$\mathcal{A} = \mathbf{k}[x_1, \ldots, x_n]$$

supported in degree zero with trivial differential (or Laurent polynomial ring).
Compute

$$H(\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(\mathbf{k}_{\varepsilon_0}, \mathbf{k}_{\varepsilon_1})) = 0$$

whenever $\varepsilon_0 \neq \varepsilon_1$, while

$$H(\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(\mathbf{k}_{\varepsilon}, \mathbf{k}_{\varepsilon})) = H^*(\mathbb{T}^n, \mathbf{k}).$$

(This holds on the level of algebras!)

Section 3

Invariance Properties

Functorial properties

- A unital DG-morphism $\Phi\colon \mathcal{A} \to \mathcal{B}$ between DGAs induces a natural functor

$$\begin{gathered} \mathcal{C}_{dg}(\mathcal{B}) \to \mathcal{C}_{dg}(\mathcal{A}), \\ (M_*, l) \to (M_*, l \circ (\Phi \otimes \mathrm{Id}_{\mathcal{M}})), \end{gathered}$$

  where $l\colon \mathcal{B} \otimes_{\mathbf{k}} M \to M$ denotes $\mathcal{B}$-module multiplication.
- Since this identification also sends $\mathcal{B}$-module morphisms to $\mathcal{A}$-module morphisms, we indeed have an chain map $\mathcal{B}^*_{dg}(M, N) \hookrightarrow \mathcal{A}^*_{dg}(M, N)$ on the level of morphisms.

Functorial properties

- To obtain well-behaved functorial properties with respect to the derived morphisms $\mathbf{R}\mathrm{Hom}_{\mathcal{A}}$ one needs to systematically choose which semi-free resolution to work with.
- The bar resolution is functorial: there is a natural morphism

$$S^{bar}_{\mathcal{A}} = \left\{ \ldots \xrightarrow{\partial^{(4)}_E} \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \xrightarrow{\partial^{(3)}_E} \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \right\}$$

$$\downarrow$$

$$S^{bar}_{\mathcal{B}} = \left\{ \ldots \xrightarrow{\partial^{(4)}_E} \mathcal{B} \otimes_{\mathbf{k}} \mathcal{B} \otimes_{\mathbf{k}} \mathcal{B} \xrightarrow{\partial^{(3)}_E} \mathcal{B} \otimes_{\mathbf{k}} \mathcal{B} \right\}$$

Functorial properties

- To obtain well-behaved functorial properties with respect to the derived morphisms $\mathbf{R}\mathrm{Hom}_{\mathcal{A}}$ one needs to systematically choose which semi-free resolution to work with.
- The bar resolution is functorial: there is a natural morphism

$$S^{bar}_{\mathcal{A}} \otimes_{\mathcal{A}} M = \left\{ \ldots \xrightarrow{\partial^{(4)}_E} \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \otimes_{\mathbf{k}} M \xrightarrow{\partial^{(3)}_E} \mathcal{A} \otimes_{\mathbf{k}} M \right\}$$

$$\downarrow$$

$$S^{bar}_{\mathcal{B}} \otimes_{\mathcal{B}} M = \left\{ \ldots \xrightarrow{\partial^{(4)}_E} \mathcal{B} \otimes_{\mathbf{k}} \mathcal{B} \otimes_{\mathbf{k}} M \xrightarrow{\partial^{(3)}_E} \mathcal{B} \otimes_{\mathbf{k}} M \right\}$$

Functorial properties

- To obtain well-behaved functorial properties with respect to the derived morphisms $\mathbf{R}\mathrm{Hom}_{\mathcal{A}}$ one needs to systematically choose which semi-free resolution to work with.
- The bar resolution is functorial: there is a natural morphism

$$\begin{array}{c} \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(M,N) = \mathcal{A}^*_{dg}(S^{bar}_{\mathcal{A}} \otimes_{\mathcal{A}} M, S^{bar}_{\mathcal{A}} \otimes_{\mathcal{A}} N) \\ \uparrow \\ \mathbf{R}\mathrm{Hom}_{\mathcal{B}}(M,N) = \mathcal{A}^*_{dg}(S^{bar}_{\mathcal{B}} \otimes_{\mathcal{B}} M, S^{bar}_{\mathcal{B}} \otimes_{\mathcal{B}} N) \end{array}$$

Functorial properties

Theorem 3.1

A quasi-isomorphism $\Phi\colon \mathcal{A} \to \mathcal{B}$ of DGAs induces a quasi-isomorphism of right-derived homomorphism complexes

$$\Phi^*\colon\ \mathbf{R}\mathrm{Hom}_{\mathcal{B}}(M, N) \to \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(M, N).$$

Remark 3.2

This functor might not be surjective on modules, but one can show that it is surjective on quasi-isomorphism classes of modules; the reason is that every $\mathcal{A}$-module has a semi-free resolution S, and $S \otimes_{\mathcal{A}} \mathcal{B}$ is a semi-free $\mathcal{B}$-module.

Functorial properties

(1/2).

- First, the semi-free modules $S^{bar}_{\mathcal{A}} \otimes_{\mathcal{A}} N$ and $S^{bar}_{\mathcal{B}} \otimes_{\mathcal{B}} N$ are quasi-isomorphic as complexes, where the quasi-isomorphism is induced by $\mathcal{B}$.
- It then follows that there are quasi-isomorphisms

$$\begin{aligned} \mathcal{B}^*_{dg}(\mathcal{B}, S^{bar}_{\mathcal{B}} \otimes_{\mathcal{B}} N) &\cong_{q.i.s} S^{bar}_{\mathcal{B}} \otimes_{\mathcal{B}} N \\ \cong_{q.i.s} \quad S^{bar}_{\mathcal{A}} \otimes_{\mathcal{A}} N &\cong_{q.i.s} \mathcal{A}^*_{dg}(\mathcal{A}, S^{bar}_{\mathcal{A}} \otimes_{\mathcal{A}} N). \end{aligned}$$

(First and last follow since the domain is free, middle q.is. follows from the previous point.)

Functorial properties

(2/2).

- The sought quasi-isomorphism

$$\mathcal{B}^*_{dg}(S^{bar}_{\mathcal{B}} \otimes_{\mathcal{B}} M, S^{bar}_{\mathcal{B}} \otimes_{\mathcal{B}} N) \to \mathcal{A}^*_{dg}(S^{bar}_{\mathcal{A}} \otimes_{\mathcal{A}} M, S^{bar}_{\mathcal{A}} \otimes_{\mathcal{A}} N)$$

can ultimately be seen to follow from the fact that both complexes are iterated cones built from direct sums of the quasi-isomorphic complexes $S^{bar}_{\mathcal{B}} \otimes_{\mathcal{B}} N \cong S^{bar}_{\mathcal{A}} \otimes_{\mathcal{A}} N$.

□

Section 4

A_∞-category of bounding cochains

Augmentations

We begin by recalling the connections between one-dimensional representations and augmentations:

Definition 4.1

An *augmentation* is a degree zero DGA-morphism $\varepsilon\colon \mathcal{A} \to \mathbf{k}$

We let $\mathbf{k}_\varepsilon$ denote the one-dimensional DG-module with module multiplication defined by $l(a \otimes m) = \varepsilon(a) \cdot m$.

Consequences:

- $\varepsilon \circ \partial_{\mathcal{A}} = 0$ (the chain map property, together with the fact that $\mathbf{k}$ has a trivial differential)
- $\varepsilon(a) = 0$ for any $|a| \neq 0$
- $\varepsilon(1) = 1$ and $\varepsilon(a_1 \cdot a_2) = \varepsilon(a_1) \cdot \varepsilon(a_2)$.

A_∞-algebras from DGAs

- Since Chekanov first introduced the Legendrian invariant in form of the DGA now called the Chekanov–Eliashberg algebra $\mathcal{A}(\Lambda)$ in [Che02], it was clear that studying the *linearized complex induced by an augmentation* was an important tool for distinguishing Legendrians.
- Civan–Koprowki–Etnyre–Sabloff [Civ+11] it was shown that an augmentation induces an invariant in the form of an A_∞-algebra.
- In [BC14] Bourgeois and Chantraine finally upgraded this invariant to the *augmentation category* denoted by $\mathrm{Aug}_-(\Lambda)$.

A_∞-algebras from DGAs

The point with this section is to

- Recall the construction of the A_∞-category $\mathrm{Aug}_-(\mathcal{A})$;
- Show its equivalence to the DG-algebra category (as observed by Ekholm-Lekili [EL17]);
- Combining these perspectives and Theorem 3.1 we learn something that is not completely obvious from the original construction:

A quasi-isomorphism $\Phi\colon \mathcal{A} \to \mathcal{B}$ of DGAs induces a quasi-isomorphic embedding of the augmentation category $\mathrm{Aug}_-(\mathcal{B})$ into $\mathrm{Aug}_-(\mathcal{A})$.

A_∞-algebras from DGAs

We start with a *noncommutative finitely generated semi-free DGA*

$$\mathcal{A} = \mathbf{k}\langle a_1, \ldots, a_k \rangle,$$

i.e. a noncommutative polynomial ring in graded variables a_i where:

- The differential is determined by a sum

  $$\partial a_i = \sum_{b_1 \ldots b_d} c^{a_i}_{b_1 \ldots b_d} \cdot b_1 \cdot \ldots \cdot b_d, \;\; c^{a_i}_{b_1 \ldots b_d} \in \mathbf{k},$$

  over words of length $d = 0, 1, 2, 3, \ldots$ in the letters $b_i \in \{a_1, \ldots, a_k\}$.
- There is an action filtration, and $\ell(b_i) < \ell(a_i)$ (this is why it called semi-free); In particular: the above sum is finite!
- Recall the graded Leibniz rule

  $$\partial(\mathbf{a} \cdot \mathbf{b}) = \partial(\mathbf{a}) \cdot \mathbf{b} + (-1)^{|\mathbf{a}|} \mathbf{a} \cdot \partial(\mathbf{b})$$

  which determines the differential on general elements.

An induced CoDGA

The algebra $\mathcal{A}$ can be written as the tensor algebra

$$\mathcal{A} = \mathcal{T}(A) = \mathbf{k} \oplus A \oplus A^{\otimes 2} \oplus A^{\otimes 3} \oplus \ldots$$

for a finite-dimensional graded $\mathbf{k}$-vector space

$$A = \mathbf{k} \cdot a_1 \oplus \ldots \oplus \mathbf{k} \cdot a_k.$$

We then pass to its component-wise dual

$$\mathcal{A}^{\#} = \mathbf{k} \oplus A^* \oplus (A^*)^{\otimes 2} \oplus (A^*)^{\otimes 3} \oplus \ldots$$

$\mathcal{A}^{\#}$ is a CoDGA with differential $\partial^{\#}$ given as the adjoint of ∂.

Remark 4.2

This makes sense since $\partial(A^{\otimes k}) \subset A^{\otimes(k-1)} \oplus A^{\otimes k} \oplus A^{\otimes(k+1)} \oplus \ldots$

An induced CoDGA

- Leibniz rule implies that $\partial^\#$ is determined by operations $\mu^l \colon (A^*)^{\otimes l} \to A^*$ as follows

$$\partial^\#(b_1 \otimes \ldots \otimes b_m) = \sum_{i,l} (-1)^\star b_1 \otimes \ldots \otimes \mu^l(b_{1+i} \otimes \ldots \otimes b_{1+l}) \otimes \ldots \otimes b_m$$

- These operations are easily determined from the structure constants of ∂:

$$\mu^d(b_1, \ldots, b_d) = \sum_i c^{a_i}_{b_1 \ldots b_d} \cdot a_i.$$

- In the following we will consider μ^d as operations on the suspension $(\Sigma A^*)^{\otimes d}$.

An induced CoDGA

- $\partial^2 = 0 \Rightarrow (\partial^\#)^2 = 0$
- $(\partial^\#)^2 = 0$ implies that $\{\mu^d\}_{d=0,1,2\ldots,D}$ satisfy the *curved* A_∞*-relations*:

For each $d = 1, 2, 3, \ldots$:

$$\sum_{m,n} (-1)^{\maltese_n} \mu^{d-m+1}(b_1, \ldots, b_n, \mu^m(b_{n+1}, \ldots, b_{n+m}), b_{n+m+1}, \ldots, b_d) = 0$$

where $\maltese_n = |b_1| + \ldots + |b_n| - n$

- When $\mu^d = 0$ i.e. ∂ has no constant term $c_\emptyset^{a_i} = 0$, the above give the *strict* A_∞*-relations*.

A_∞-relations

Examples of curved A_∞-relations

- $d = 1$: ("chain complex")

$$0 = \mu^1 \circ \mu^1(a) + \mu^2(\mu^1(a), \mu^0) - \mu^2(\mu^0, \mu^1(a))$$

(μ^1 is a differential when μ^0 vanishes)

- $d = 2$: ("graded Leibniz rule")

$$0 = \mu^1 \circ \mu^2(a, b) + \mu^2(\mu^1(a), b) + (-1)^{|a|-1}\mu^2(a, \mu^1(b)) + \dots$$

(+ additional terms involving μ^0)

- $d = 3$: ("graded associativity")

$$0 = \mu^1 \circ \mu^3(a, b, c) + \mu^2(\mu^2(a, b), c) + (-1)^{|a|-1}\mu^2(a, \mu^2(b, c)) + \dots$$

(+ additional terms involving μ^0 and μ^1).

A_∞-relations and sign convention

Remark 4.3

We follow the conventions of [Sei08] by which a strict A_∞-algebra with vanishing μ^d for $d \geq 3$ becomes a DGA with differential "∂" and multiplication "$\cdot$" after introducing the sign changes $\partial(a) := (-1)^{|a|}\mu^1(a)$ and $a_1 \cdot a_2 := (-1)^{|a_1|}\mu^2(a_1, a_2)$.

Maurer–Cartan equation

The strictly unital A_∞-category constructed by Bourgeois–Chantraine in [BC14], [Cha19] can now be recovered as follows:

- Add a “strict unit” in degree zero denoted by e, i.e. extend μ^d to the vector space

$$B_{\mathcal{A}} := \mathbf{k} \cdot e \oplus \Sigma A^*$$

  subject to
  - $\mu^2(e, a) = a = (-1)^{|a|}\mu^2(a, e)$,
  - $\mu^1(e) = 0$ and $\mu^d(\ldots, e, \ldots) = 0$ for $d \geq 3$.
- The objects of this category are solutions $b \in \mathcal{MC}(B_{\mathcal{A}})$ to the generalised Maurer–Cartan equation

$$0 = \sum_{d \geq 0} \mu^d(\underbrace{b, \ldots, b}_{d}) = \mu^0 + \mu^1(b) + \mu^2(b, b) + \ldots$$

  where $b \in \Sigma A^*$ is an element of degree one.

Maurer–Cartan equation

Next time we will continue the construction of this A_∞-category. We end this lecture with the following exercise:

Exercise 4.4

Show that solutions b of the Maurer–Cartan equation are in natural bijective correspondence with augmentations $\varepsilon_b \colon \mathcal{A} \to \mathbf{k}$.

UPPSALA UNIVERSITET

References I

[BC14] Frédéric Bourgeois and B. Chantraine. "Bilinearized Legendrian contact homology and the augmentation category". In: *J. Symplectic Geom.* 12.3 (2014), pp. 553–583. ISSN: 1527-5256. URL: http://projecteuclid.org/euclid.jsg/1409319460.

[Cha19] B. Chantraine. "Augmentations des sous-variétés legendriennes: invariants associés et applications". Habilitation à diriger des recherches. Université de Nantes - Faculté des Sciences et Techniques, Dec. 2019. URL: https://hal.archives-ouvertes.fr/tel-02415052.

[Che02] Y. Chekanov. "Differential algebra of Legendrian links". In: *Invent. Math.* 150.3 (2002), pp. 441–483. ISSN: 0020-9910. DOI: 10.1007/s002220200212. URL: http://dx.doi.org/10.1007/s002220200212.

[Civ+11] Gokhan Civan et al. "Product structures for Legendrian contact homology". In: *Math. Proc. Cambridge Philos. Soc.* 150.2 (2011), pp. 291–311. ISSN: 0305-0041. DOI: 10.1017/S0305004110000460. URL: http://dx.doi.org/10.1017/S0305004110000460.

[EL17] T. Ekholm and Y. Lekili. "Duality between Lagrangian and Legendrian invariants". Preprint, https://arxiv.org/abs/1701.01284 [math.SG]. 2017. eprint: 1701.01284.

UPPSALA UNIVERSITET

References II

[Sei08] P. Seidel. *Fukaya categories and Picard-Lefschetz theory*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008, pp. viii+326. ISBN: 978-3-03719-063-0. DOI: 10.4171/063. URL: https://doi.org/10.4171/063.