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DG-Algebraic Aspects of Contact Invariants

42th Winter School in Geometry and Physics, Srní

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Plan

Lecture I:

- DGAs and DG-modules
- Derived category & $\mathbf{R}$Hom
- Bar resolution, short resolution

Lecture II

- Koszul resolution (commutative case)
- Functoriality and invariance of $\mathbf{R}$Hom.
- A_∞-category of bounding cochains dual to the DGA

Lecture III:

- A_∞-category of bounding cochains dual to the DGA
- Equivalence with DG-category of modules
- Augmentation variety

Recap

Last lecture we saw:

- DG-modules (e.g. one dimensional modules induced by augmentations) pull back under a morphism $\Phi \colon \mathcal{A} \to \mathcal{B}$ of DGAs, inducing natural chain maps

$$\Phi^* \colon \ \mathbf{R}\mathrm{Hom}^*_{\mathcal{B}}(M, N) \to \mathbf{R}\mathrm{Hom}^*_{\mathcal{A}}(M, N);$$

  If Φ is a quasi-isomorphism, then Φ^* is as well.

- A finitely generated semi-free non-commutative DG-algebra $\mathcal{A} = \mathcal{T}A$ induces a *strictly unital curved A_∞-algebra*

$$\left(B_{\mathcal{A}} = \mathbf{k} \cdot e \oplus \Sigma A^*, \{\mu^d\}_{d=0,1,2,\ldots}\right),$$

  where

$$A^* = \mathbf{k} \cdot a_1 \oplus \ldots \oplus \mathbf{k} \cdot a_k$$

  (in fact: only finitely many nonzero operations μ^d).

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Lecture II

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Section 2

The A_∞-category $\mathcal{MC}(B_\mathcal{A})$ (a.k.a. $\mathrm{Aug}_-(\mathcal{A})$)

The category of bounding cochains

A strictly unital A_∞-algebra B with finitely many non-zero operations gives rise to a category $\mathcal{MC}(B)$ of bounding cochains in the following manner, going back to work by Fukaya–Ohta–Ono–Oh [Fuk+09a], [Fuk+09b]:

- **Objects:** Solutions $b \in \mathcal{MC}(B)$ to the generalised Maurer–Cartan equation

$$\sum_{d=0}^{\infty} \mu^d(\underbrace{b, \ldots, b}_{d}) = \mu^0 + \mu^1(b) + \mu^2(b, b) + \ldots = 0$$

  also called *bounding cochains*.
- **Morphisms:** $(\mathrm{Hom}^*_B(b_0, b_1) = B, \mu^1_{(b_0,b_1)})$ with differential

$$\mu^1_{(b_0,b_1)}(a) := \sum_{d=1+d_0+d_1 \geq 1}^{\infty} \mu^d(\underbrace{b_1, \ldots, b_1}_{d_1}, a, \underbrace{b_0, \ldots, b_0}_{d_0}).$$

The strict A_∞-category of bounding cochains

The curved A_∞-relations together with the generalised Maurer–Cartan equation for b_1 and b_0 implies that $(B, \mu^1_{(b_0,b_1)})$ is indeed a chain complex. (Unlike (B, μ^1)!)

$$
\begin{aligned}
&(\mu^1_{(b_0,b_1)})^2(a) = \\
&\overset{\text{def}}{=} \sum_{d'',d'>0}^{\infty} \mu^{d''}(b_1,\ldots,\mu^{d'}(b_1,\ldots,b_1,a,b_0,\ldots,b_0),\ldots,b_0) \\
&\overset{A_\infty\text{-rel}}{=} -\sum_{d'\geq 0}^{\infty} \mu^{d''}(b_1,\ldots,\mu^{d'}(b_1,\ldots,b_1),\ldots,b_1,a,b_0,\ldots,b_0) + \\
&\quad - \sum_{d'\geq 0}^{\infty} \mu^{d''}(b_1,\ldots,b_1,a,b_0,\ldots,\mu^{d'}(b_0,\ldots,b_0),\ldots,b_0) \overset{\text{MC-eqn}}{=} 0.
\end{aligned}
$$

The strict A_∞-category of bounding cochains

A_∞-**Operations:** For each $d \geq 1$ and sequence of $d+1$ number of objects

$$(b_0, b_1, \ldots, b_d), \;\; b_i \in \mathcal{MC}(B),$$

we have the operations defined for $a_i \in \mathrm{Hom}_B(b_{i-1}, b_i)$:

$$\begin{aligned} &\mu^d_{(b_0,b_1,\ldots,b_d)}(a_d, \ldots, a_1) := \\ &= \sum_{d'>0} \mu^{d'}(b_d, \ldots, b_d, a_d, \ldots, b_{i-1}, a_i, b_i, \\ &\qquad \ldots, b_1, a_1, b_0, \ldots, b_0) \end{aligned}$$

The strict A_∞-category of bounding cochains

- A homotopy-associative *composition* is given by

$$\mu^2_{(b_0,b_1,b_2)}\colon \operatorname{Hom}_B(b_1,b_2)\otimes_{\mathbf{k}}\operatorname{Hom}_B(b_0,b_1)\to\operatorname{Hom}_B(b_0,b_2)$$

- In general there are higher compositions:

$$\mu^d_{(b_0,b_1,\ldots,b_d)}\colon \operatorname{Hom}_B(b_{d-1},b_d)\otimes_{\mathbf{k}}\ldots\otimes_{\mathbf{k}}\operatorname{Hom}_B(b_0,b_1)\to\operatorname{Hom}_B(b_0,b_d)$$

- The above operations satisfy the strict A_∞-relations for an A_∞-category.

The A_∞-relations

For each $d = 1, 2, 3, \ldots$, $b_i \in \mathcal{MC}(B)$, and $a_i \in \mathrm{Hom}_B(b_{i-1}, b_i)$:

$$0 = \sum_{m>0,n} (-1)^{\maltese_n} \mu^{d-m+1}_{(b_0,\ldots,b_d)}(a_d, \ldots, a_{n+m+2},\\ \mu^m_{(b_n,\ldots,b_{n+m+1})}(a_{n+m}, \ldots, a_{n+1}), a_n, \ldots, a_1)$$

where $\maltese_n = |a_{n+m+2}| + \ldots + |a_d| - (d - n - m - 1)$

Remark 2.1

Since b_i satisfy the Maurer–Cartan equation, the obtained A_∞-relations are strict (no curvature terms).

This makes $\mathcal{MC}(B)$ into an A_∞-category with strict units as defined e.g. in [Sei08].

The strict A_∞-category of bounding cochains

Exercise 2.2

1. Show that $\mu^1_{(b_0,b_1)}(e) = b_1 - b_0$.
2. Show that the endomorphisms $(\mathrm{Hom}_B(b,b), \{\mu^d_{(b,\ldots,b)}\}_{d=1,2,\ldots})$ form strict (uncurved) A_∞-algebras with e a strict unit.

Remark 2.3

The element $e \in \mathrm{Hom}_B(b,b')$ is a strict unit only if $b = b'$ by the above exercise. (Otherwise it is not closed.)

In the following we will typically write $e_{ij}, a_{ij} \in \mathrm{Hom}_B(b_i, b_j)$ to designate in which Hom-space a given morphism lives (all these Hom-spaces are equal to B *as a* $\mathbf{k}$*-vector space*).

The category of bounding cochains

It is immediate that the category $\mathcal{MC}(B)$ induced by the curved A_∞-algebra B that corresponds to the DGA $\mathcal{A}$ is the same as the version of the augmentation category by Bourgeois–Chantraine described in [Cha19].

In particular, when $\mathcal{A}$ is the Chekanov–Eliashberg algebra of a Legendrian Λ, and the bounding cochain b_i corresponds to an augmentation $\varepsilon_i \colon \mathcal{A} \to \mathbf{k}$, there is a canonical isomorphism

$$\mathrm{Hom}_B(b_0, b_1) = \mathrm{Cone}(\mathbf{k} \cdot e \xrightarrow{f} LCC^*_{\varepsilon_1,\varepsilon_0}(\Lambda))$$

where

$$f(e) = \sum_i (\varepsilon_1(a_i) - \varepsilon_0(a_i))a_i.$$

Section 3

Isomorphism

Definition of isomorphism

The closed morphisms $a \in \mathrm{Hom}_B(b_0, b_1)$ descend to the "homology category" with the same objects as $\mathcal{MC}(B)$ but with $\mathrm{Mor}(b_0, b_1) = H(\mathrm{Hom}_B(b_0, b_1))$; this is a strict category in the classical sense.

Definition 3.1

Two objects b_0 and b_1 in an A_∞-category are *isomorphic* if there exists closed morphisms $a_{ij} \in \mathrm{Hom}^*_B(b_i, b_j)$ so that

$$[\mu^2_{(b_0,b_1,b_0)}(a_{10}, a_{01})] = [e_{00}] \text{ and } [\mu^2_{(b_1,b_0,b_1)}(a_{01}, a_{10})] = [e_{11}]$$

holds in homology; i.e. the morphism $[a_{ij}]$ in the homology category is an isomorphism in the standard sense.

Reformulation of isomorphism property via "Yoneda"

The isomorphism property in a category can be characterised in terms of the "Yoneda embedding" i.e. the concrete action on by morphisms on Hom-spaces induced by (pre)composition.
More precisely,

Lemma 3.2

Assume that a_{01} is a closed element, for which the induced chain maps

1. $\mu^2_{(b_0,b_1,b)}(\cdot, a_{01})\colon \operatorname{Hom}^*_B(b_1, b) \to \operatorname{Hom}^*_B(b_0, b)$
2. $\mu^2_{(b,b_0,b_1)}(a_{01}, \cdot)\colon \operatorname{Hom}^*_B(b, b_0) \to \operatorname{Hom}^*_B(b, b_1)$

are quasi-isomorphisms for all objects b. (a_{01} acts by both pre-composition and composition.) Then a_{01} is an isomorphism in the A_∞-category.

Reformulation of isomorphism property via "Yoneda"

Proof.

- Setting $b = b_0$ in (1) we find $a_{10} \in \mathrm{Hom}^*_B(b_1, b_0)$ for which $[\mu^2_{(b_0,b_1,b_0)}(a_{10}, a_{01})] = [e_{00}]$.
- Setting $b = b_1$ in (1) we find $a'_{10} \in \mathrm{Hom}^*_B(b_1, b_0)$ for which $[\mu^2_{(b_1,b_0,b_1)}(a_{01}, a'_{10})] = [e_{11}]$.
- Since morphisms which are left and right invertible coinciding right and left inverses in classical unital categories, a_{01} is an isomorphism.

□

Condition for isomorphism

Proposition 3.3

Two objects $b_0, b_1 \in \mathcal{MC}(B_{\mathcal{A}})$, where $B_{\mathcal{A}}$ is the curved A_∞-algebra that corresponds to the semi-free DGA $\mathcal{A}$, are isomorphic if and only if $e_{01} \in Hom^*_B(b_0, b_1)$ can be extended to a cycle of the form

$$e_{01} + c_1 a_1 + \ldots + c_n a_n, \;\; c_i \in \mathbf{k}, a_i \in \Sigma A^*.$$

(Here $a_i \in \Sigma A^*$ is of degree zero, i.e. $a_i \in A$ is of degree -1.)

Remark 3.4

See work by Bourgeois–Galant for the connection to the notion of DG-homotopy between augmentations [BG20]; the above is equivalent to $\varepsilon_{b_i} \colon \mathcal{A} \to \mathbf{k}$, $i = 0, 1$, being DG-homotopic.

Condition for isomorphism

Proof.

We show the "if" part; the "only if" part is left as an exercise.

- By Lemma 3.2 it suffices to verify that the element
$$a_{01} \coloneqq e_{01} + c_1 a_1 + \ldots + c_n a_n,$$
which is closed by assumption, induces quasi-isomorphisms
$$\mu^2_{(b,b_0,b_1)}(a_{01}, \cdot) \text{ and } \mu^2_{(b_0,b_1,b)}(\cdot, a_{01}).$$
- The element e_{01} is the strict unit of the weak A_∞-algebra B;
- Combined with the fact that μ^k are *strictly action increasing*, we conclude that the above maps even are *chain isomorphisms*.

□

Section 4

Functorial properties

Functoriality

A unital DG-morphism

$$\Phi\colon \mathcal{A}_+ \to \mathcal{A}_-$$

between finitely generated semi-free non-commutative DGAs with underlying tensor algebras

$$\mathcal{A}_\pm = \mathbf{k} \oplus A_\pm \oplus A_\pm^{\otimes 2} \oplus A_\pm^{\otimes 3} \oplus \ldots$$

can be described by its values $\Phi(a_i^+)$ on the generators of $\mathcal{A}_+$; in this case we can write

$$\Phi(a_i^+) = \sum_{b_1^- \cdots b_d^-} d^{a_i^+}_{b_1^- \cdot \ldots \cdot b_d^-} b_1^- \cdot \ldots \cdot b_d^-, \;\; d^{a_i^+}_{b_1^- \cdot \ldots \cdot b_d^-} \in \mathbf{k},$$

where $b_i^- \in \{a_i^-\}$, and where $\{a_i^\pm\}$ is a choice of basis for the $\mathbf{k}$-vector space $A_\pm$.

Functoriality

- Again we consider the CoDGA induced by the graded dual with respect to the word length-grading:

$$\mathcal{A}_{\pm}^{\#} = \mathbf{k} \oplus (A_{\pm}^{*}) \oplus (A_{\pm}^{*})^{\otimes 2} \oplus (A_{\pm}^{*})^{\otimes 3} \oplus \ldots$$

- As for the differential, Φ is not homogeneous with respect to the word length filtration; Since Φ can decrease the word-length by at most one, we still get a well-defined adjoint

$$\Phi^{\#} \colon \mathcal{A}_{-}^{\#} \to \mathcal{A}_{+}^{\#}$$

which is a morphism of CoDGAs.

Functoriality

- There is an induced curved A_∞-morphism

$$B_{\mathcal{A}_-} \to B_{\mathcal{A}_+}$$

  of curved A_∞-algebras.
- Recall that a curved A_∞-morphism is a collection of (in this case a finite number of) morphisms

$$\{f^d \colon (B_{\mathcal{A}_-})^{\otimes d} \to B_{\mathcal{A}_+}\}_{d=0,1,2,\ldots}$$

- If $b \in \mathcal{MC}(B_{\mathcal{A}_-})$ then

$$\sum_{d \geq 0} f^d(\underbrace{b, \ldots, b}_{d}) \in \mathcal{MC}(B_{\mathcal{A}_+}),$$

  i.e. there is a well-defined map

$$\Phi^\# \colon \mathcal{MC}(B_{\mathcal{A}_-}) \to \mathcal{MC}(B_{\mathcal{A}_+}).$$

Functoriality

- There is an induced A_∞-functor

$$\mathcal{MC}(B_{\mathcal{A}_-}) \to \mathcal{MC}(B_{\mathcal{A}_+})$$

between strict (uncurved) A_∞-categories with strict units.

- In particular, there are chain maps

$$f^1_{b_0^-,b_1^-} \colon \operatorname{Hom}_{B_-}(b_0^-, b_1^-) \to \operatorname{Hom}_{B_+}(b_0^+, b_1^+)$$

that satisfy

$$f^1_{b_0,b_1}(e_{01}^-) = e_{01}^+ + \text{“terms of higher action”}$$

Section 5

A quasi-equivalence between categories

From A_∞ to DG

- Recall the "DG-category" $\mathcal{C}_{dg}(\mathcal{A})$ associated to $\mathcal{A}$ from Lecture I, whose objects are the DG-modules of $\mathcal{A}$.
- It has a sub-category $\mathcal{C}_{dg}^{Aug}(\mathcal{A}) \subset \mathcal{C}_{dg}(\mathcal{A})$ that consists of one-dimensional modules (one dimensional modules are in bijection with augmentations).
- We replace the Hom's of this sub-category with the right-derived Hom's

$$\mathrm{Hom}(\varepsilon_0, \varepsilon_1) = \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(\mathbf{k}_{\varepsilon_0}, \mathbf{k}_{\varepsilon_1}).$$

- A DG-category is a particular case of an A_∞-category, for which all higher morphisms $\mu^d = 0$, $d > 2$, vanish.

From A_∞ to DG

The Bar–Cobar adjunction from homological algebra [LV12] translates to:

Lemma 5.1

There is a quasi-isomorphism

$$(B_{\mathcal{A}}, \mu^d_{b,b}) \cong_{q.is} \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(\mathbf{k}_{\varepsilon_b}, \mathbf{k}_{\varepsilon_b})$$

of A_∞-algebras (the left-hand side is a DGA, i.e. a very special A_∞-algebra).

From A_∞ to DG

Proof (1/2).

For simplicity, we assume that $(\mathcal{A}, \partial)$ is a DGA for which

- ∂ has no constant term (this is possible after conjugating ∂ to $\Psi^{-1} \circ \partial \circ \Psi$ via an automorphism of the underlying graded algebra which is defined on the generators by $\Psi(a) = a - \epsilon(a)$);
- $\varepsilon = \varepsilon_0$ is the trivial augmentation (which sends all words of positive length to zero);
- $b = b_0$ is the corresponding trivial bounding cochain on the uncurved strictly unital A_∞-algebra $(B_\mathcal{A}, \{\mu^d = \mu^d_{(b_0,b_0)}\})$.

From A_∞ to DG

Proof (2/2).

The Bar–Cobar adjunction provides a canonical A_∞-morphism

$$\mathrm{Hom}(b_0, b_0) \xrightarrow{q.is} \Omega(\overline{B}(\mathrm{Hom}(b_0, b_0))) =$$
$$= \Omega(\mathcal{A}^{\#}) = \mathcal{A}^{*}_{dg}(\overline{\mathcal{S}}^{bar} \otimes_{\mathcal{A}} \mathbf{k}, \mathbf{k}) = \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(\mathbf{k}_{\varepsilon_0}, \mathbf{k}_{\varepsilon_0}).$$

Where

- $\overline{B}(...)$ denotes the reduced bar construction on an augmented A_∞-algebra (or DGA);
- $\Omega(...)$ denotes the reduced cobar construction on a co-augmented co-DGA, and
- $\overline{\mathcal{S}}^{bar}$ is the reduced bar resolution the bimodule $\mathcal{A}$.

□

Remark 5.2

In the case when the DGA $\mathcal{A}$ has generators in non-positive degree, one needs to take additional care and consider the word-length filtration to get the equivalences

$$\Omega(\mathcal{A}^{\#}) = \mathcal{A}^{*}_{dg}(\overline{\mathcal{S}}^{bar} \otimes_{\mathcal{A}} \mathbf{k}, \mathbf{k})^{finite} \cong_{q.is.} \mathcal{A}^{*}_{dg}(\overline{\mathcal{S}}^{bar} \otimes_{\mathcal{A}} \mathbf{k}, \mathbf{k})$$

(otherwise the left-hand side must be completed).

From A_∞ to DG

Similarly, one can show the categorical statement

Theorem 5.3

The A_∞-category $\mathcal{MC}(B_{\mathcal{A}})$ of bounding cochains is quasi-equivalent to the augmentation DG-category $\mathcal{C}_{dg}^{Aug}(\mathcal{A})$.

Section 6

The Augmentation Variety

Parametrising augmentations

Ng [Ng03; Ng08] introduced the following canonical algebras and varieties associated to a semifree DGA:

- The *characteristic algebra*

$$\mathcal{C}_{\mathcal{A}} := \mathcal{A}/\mathcal{A} \cdot \partial\mathcal{A} \cdot \mathcal{A}$$

  which admits a unital DG-morphism from $\mathcal{A}$;
- All algebra maps to an algebra with trivial differential factorises through the characteristic algebra;
- Any augmentation $\varepsilon \colon \mathcal{A} \to \mathbf{k}$ factorises through a the quotient of a polynomial algebra

$$\mathcal{A}_{AugVar} := \mathbf{k}[a_1^0, \ldots, a_k^0]/\langle \pi_0 \partial(a_i^1) \rangle$$

  where a_i^j enumerates the generators in degree j, and π_0 projects to words consisting of letters of degree zero only.

Parametrising augmentations

- The algebra $\mathcal{A}_{AugVar}$ can be considered as the regular function ring of a variety (possibly non-reduced) called the *augmentation* variety
- To get a quasi-isomorphism invariant, one must also consider the quotient by an action induced by homotopy of augmentations.
- In other words, the DG-category of augmentations or, equivalently, the A_∞-category of bounding cochains of $B_{\mathcal{A}}$, can be parametrized by the points of a d-dimensional variety with function ring $\mathcal{A}_{AugVar}$.
- There is a unital DG-morphism $\Phi\colon \mathcal{A} \to \mathcal{A}_{AugVar}$: Hence there is a unital algebra map from $H^*(\mathbb{T}^d, \mathbf{k})$ into $H(\mathrm{Hom}^*_{B_{\mathcal{A}}}(b, b))$.

When is the DGA affine?

We seem to lack tools for answering this problem in general:

Does the semi-free DGA $\mathcal{A}$ admit a quasi-isomorphism to a (smooth) affine algebra concentrated in degree zero?

- A necessary condition when $\mathbf{k} = \mathbf{C}$: the DG-category of *finite-dimensional* DG-modules consists of twisted complexes built from one-dimensional modueles (i.e. induced by augmentations); Put differently: augmentations generate the category of proper DG-modules.

Idea of proof.

Consider the classification of finite-dimensional commutative $\mathbf{C}$-algebras together with the Jordan normal form. □

When is the DGA affine?

We seem to lack tools for answering this problem in general:

Does the semi-free DGA $\mathcal{A}$ admit a quasi-isomorphism to a (smooth) affine algebra concentrated in degree zero?

Example 6.1

The reduced bar construction gives rise to a semi-free model

$$B(H^*(\mathbf{T}^d,\mathbf{k}))^\# = (\mathcal{T}(\tilde{H}^*(\mathbf{T}^d,\mathbf{k})),\partial) \cong_{q.is.} \mathbf{k}[b_1,\ldots,b_d]$$

of the commutative polynomial algebra.

When is the DGA affine?

Example 6.2

In joint upcoming work with Ghiggini, we give some sporadic examples, such as the *DGA of the Legendrian trefoil*:

$$\mathcal{A} = \mathbf{k}\langle a_1, a_2, b_1, b_2, b_3\rangle, \ \ |a_i| = 1, |b_i| = 0,$$
$$\partial(a_1) = 1 + b_1 + b_3 + b_3 b_2 b_1,$$
$$\partial(a_2) = 1 + b_1 + b_3 + b_1 b_2 b_3$$

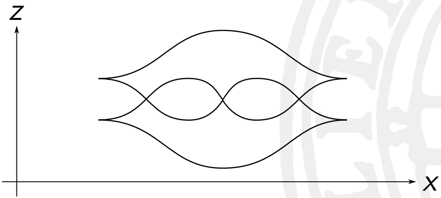

Figure: Front projection of the standard Legendrian trefoil.

When is the DGA affine?

Exercise 6.3

Show that a Legendrian knot with Thurston–Bennequin invariant different from $+1$ has a DGA which is not affine. (Hint: compute the Euler-characteristic of $\mathbf{R}\mathrm{Hom}$'s)

This is no surprise; The mirror of a Weinstein manifold is typically not an affine variety.

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References I

[BG20] F. Bourgeois and D. Galant. "Geography of bilinearized Legendrian contact homology". Preprint, https://arxiv.org/abs/1905.12037 [math.SG]. 2020. arXiv: 1905.12037 [math.SG].

[Cha19] B. Chantraine. "Augmentations des sous-variétés legendriennes: invariants associés et applications". Habilitation à diriger des recherches. Université de Nantes - Faculté des Sciences et Techniques, Dec. 2019. URL: https://hal.archives-ouvertes.fr/tel-02415052.

[Fuk+09a] K. Fukaya et al. *Lagrangian intersection Floer theory: anomaly and obstruction. Part I*. Vol. 46. AMS/IP Studies in Advanced Mathematics. Providence, RI: American Mathematical Society, 2009, pp. xii+396. ISBN: 978-0-8218-4836-4.

[Fuk+09b] K. Fukaya et al. *Lagrangian intersection Floer theory: anomaly and obstruction. Part II*. Vol. 46. AMS/IP Studies in Advanced Mathematics. Providence, RI: American Mathematical Society, 2009, i–xii and 397–805. ISBN: 978-0-8218-4837-1.

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References II

[LV12] J.-L. Loday and B. Vallette. *Algebraic operads*. Vol. 346. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Heidelberg, 2012, pp. xxiv+634. ISBN: 978-3-642-30361-6. DOI: 10.1007/978-3-642-30362-3. URL: https://doi.org/10.1007/978-3-642-30362-3.

[Ng03] Lenhard Ng. "Computable Legendrian invariants". In: *Topology* 42.1 (2003), pp. 55–82. ISSN: 0040-9383. DOI: 10.1016/S0040-9383(02)00010-1. URL: http://dx.doi.org/10.1016/S0040-9383(02)00010-1.

[Ng08] Lenhard Ng. "Framed knot contact homology". In: *Duke Math. J.* 141.2 (2008), pp. 365–406. ISSN: 0012-7094. DOI: 10.1215/S0012-7094-08-14125-0. URL: http://dx.doi.org/10.1215/S0012-7094-08-14125-0.

[Sei08] P. Seidel. *Fukaya categories and Picard-Lefschetz theory*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008, pp. viii+326. ISBN: 978-3-03719-063-0. DOI: 10.4171/063. URL: https://doi.org/10.4171/063.