UNIVERSITY OF BERGEN

Curvature in sub-Riemannian geometry

Lecture 1

42nd Winter School: Geometry and Physics

Erlend Grong, erlend.grong@uib.no

Srní, January 19, 2022

University of Bergen, Norway

Some advertisement

Geometry and Probability with constraints

`https://sites.google.com/view/codysmageoproco/geoproco`

To get slides and notes

`https://sites.google.com/site/erlendgrong/home`

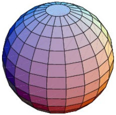
or just google ‘Erlend Grong’ and pick the Google Sites one

Scrool down to ‘Recent events’.

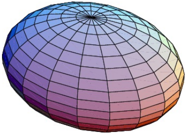
Introduction

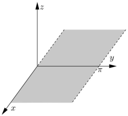
What is differential geometry?

From differential topology to differential geometry of manifolds. From “clay shapes” to “paper shapes”:

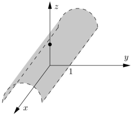


$\ncong$





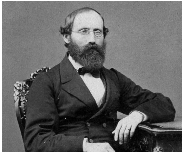
$\cong$



.

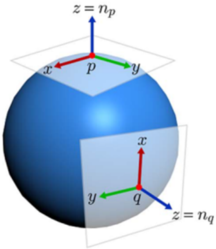
Two objects considered equivalent if you can transform one to the other by a map that preserves all lengths (an isometry).

What is differential geometry?



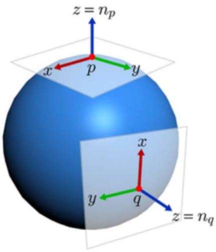
Bernhard Riemann
1826 - 1866

Riemannian manifold M: We have a smoothly varying inner product $g = \langle \cdot, \cdot \rangle_g$ to measure length and angles of vectors tangent to M.



What is differential geometry?

Riemannian manifold M: We have a smoothly varying inner product $g = \langle \cdot, \cdot \rangle_g$ to measure length and angles of vectors tangent to M.



Gives us a way to measure the length $L(\gamma)$ of a curve γ in M. This gives us a distance on M

$$d_g(x, y) = \inf\{L(\gamma) \,:\, \gamma \text{ is a curve in } M \text{ from } x \text{ to } y\}.$$

What is differential geometry?

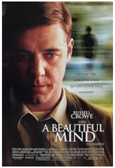
Riemannian manifolds: Is it really necessary to deal with all of this abstract manifold terminology?

Can't you just consider these spaces as subsets of $\mathbb{R}^N$ instead, just like the sphere S^n?

What is differential geometry?

Riemannian manifolds: Is it really necessary to deal with all of this abstract manifold terminology?

Can't you just consider these spaces as subsets of $\mathbb{R}^N$ instead, just like the sphere S^n?



John Forbes Nash
1928 - 2015

Nash: Yes, all such spaces can be viewed as subspaces of $\mathbb{R}^N$ with the euclidean structure for some $N > 0$.

What is differential geometry?

Nash: Yes, all such spaces can be viewed as subspaces of $\mathbb{R}^N$ with the euclidean structure for some $N > 0$.
However, viewing a Riemannian manifold M like this is often unhelpful.

(a) You will often have do deal with a very complicated isometric immersion of M into $\mathbb{R}^N$ where N is very large.
Ex: The hyperbolic plane $\mathbb{H}^2$ is topologically just $\mathbb{R}^2$. It cannot be isometrically immersed into $\mathbb{R}^3$, but it is possible for $\mathbb{R}^5$. It is still an open question if it is possible in $\mathbb{R}^4$.
(b) Usually, we are looking for properties of the geometry that are intrinsic, where this embedding should not matter.

When then are two shapes different?

Consider $\mathbb{R}^2$ with coordinates (x, y) with corresponding vector field ∂_x, ∂_y. The eulidean metric g_{Eucl} this is given by

$$\langle \partial_x, \partial_x \rangle_{g_{Eucl}} = 1, \qquad \langle \partial_y, \partial_y \rangle_{g_{Eucl}} = 1, \qquad \langle \partial_x, \partial_y \rangle_{g_{Eucl}} = 0.$$

This can be written as either

$$g_{Eucl} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{or} \qquad g_{Eucl} = dx^2 + dy^2.$$

When then are two shapes different?

Consider $\mathbb{R}^2$ with coordinates (x, y) with corresponding vector field ∂_x, ∂_y. The eulidean metric g_{Eucl} this is given by

$$\langle \partial_x, \partial_x \rangle_{g_{Eucl}} = 1, \qquad \langle \partial_y, \partial_y \rangle_{g_{Eucl}} = 1, \qquad \langle \partial_x, \partial_y \rangle_{g_{Eucl}} = 0.$$

This can be written as either

$$g_{Eucl} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{or} \qquad g_{Eucl} = dx^2 + dy^2.$$

A Riemannian metric $g = \langle \cdot, \cdot \rangle_g$ varies from point to point.

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} \qquad \text{or} \qquad g = g_{11} dx^2 + 2 g_{21} dx dy + g_{22} dy^2,$$

where g_{ij} are functions.

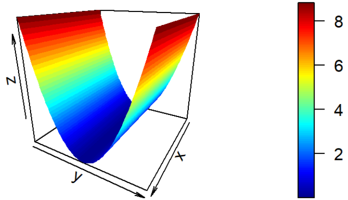
When then are two shapes different?

Example

$$M = \left\{(x, y, y^2) \; : \; (x, y) \in \mathbb{R}^2\right\}.$$

We can then view this as $\mathbb{R}^2$ with the inner product

$$g = dx^2 + (1 + 4y^2)dy^2.$$

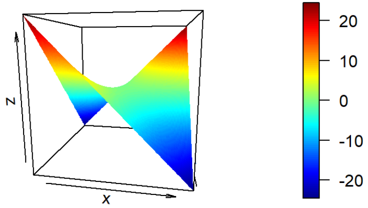


Example

$$M = \left\{(x, y, xy) \,:\, (x, y) \in \mathbb{R}^2\right\}.$$

We can then view this as $\mathbb{R}^2$ with the inner product

$$g = (1 + y^2)dx^2 + (1 + x^2)dy^2 + 2xydxdy.$$



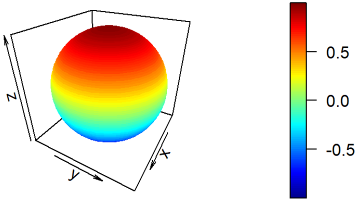
When are two shapes different?

Example

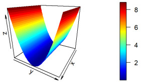
$$M = S^2 \setminus \{(0,0,1)\}.$$

Stereographic projection: Let (p^0, p^1, p^2) be the coordinates of $\mathbb{R}^3$. Define $x = \frac{p^1}{1-p^0}, y = \frac{p^2}{1-p^0}$.

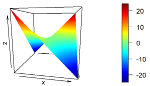
$$g = \frac{4}{(1+x^2+y^2)^2}(dx^2 + dy^2).$$



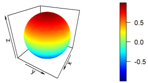
When are two shapes different?



$$g = dx^2 + (1 + 4y^2)dy^2.$$



$$g = (1 + y^2)dx^2 + (1 + x^2)dy^2 + 2xydxdy.$$



$$g = \frac{4}{(1+x^2+y^2)^2}(dx^2 + dy^2).$$

Are any of these shapes just the same as the Euclidean space?

When are two shapes different?

Look at the following example:

$$g = (1+x^2)dx^2 + \frac{2x}{1+y^2}dxdy + \frac{1+x^2+x^2y^2}{(1+y^2)^2}.$$

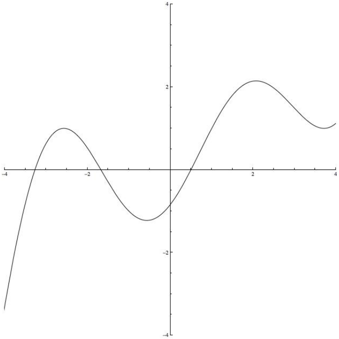
It is very difficult explicitely show that this is a change of variable from the standard Euclidean space (which is it). On the other hand, it is even more difficult to show that such a change is impossible.

Solution: Gaussian curvature.

Geometry of curves: Calculus I

What shape does this graph have?

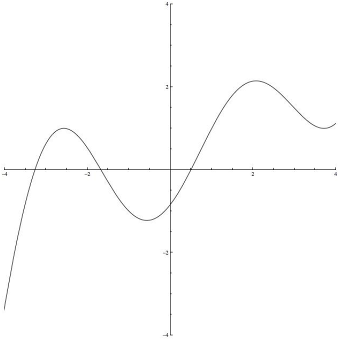
$$f(x) = (x-1)\sin(1+x) + x$$



Geometry of curves: Calculus I

What shape does this graph have?

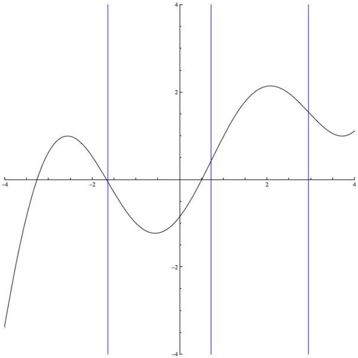
$$f(x) = (x-1)\sin(1+x) + x$$



	-4	-3	-2	-1	0	1	2	3	4
f'	$+$	$+$	$-$	$-$	$+$	$+$	$+$	$-$	$+$
f''	$-$	$-$	$-$	$+$	$+$	$-$	$-$	$+$	$+$

Geometry of curves: Calculus I

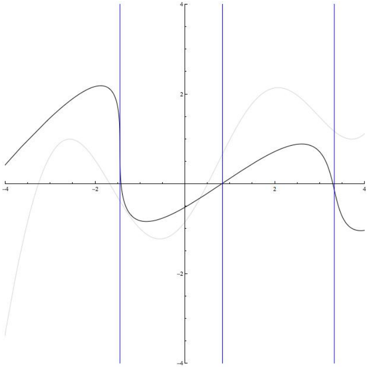
$$f(x) = (x-1)\sin(1+x) + x$$



- The derivative says nothing about the shape.

Geometry of curves: Calculus I

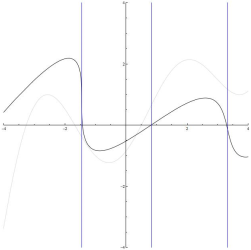
$g(x) =$ rotation of $f(x)$ by 30° clockwise   Same shape



- Local maximum and minimum not preserved.
- Inflection points and concavity/convexity preserved.

Geometry of curves: Calculus I

$g(x) =$ rotation of $f(x)$ by 30° clockwise



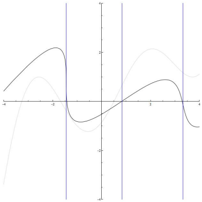
Curvature of the oriented curve

$$\kappa(x) = \frac{f''(x)}{(1 + f'(x))^{3/2}}.$$

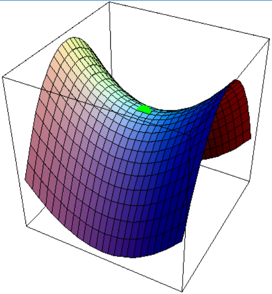
Preserved under rotations and translation.

Geometry of curves: Calculus I

$$\kappa(x) = \frac{f''(x)}{(1+f'(x))^{3/2}}.$$



However, if we consider $M = \{(x, f(x)) \,:\, x \in \mathbb{R}\}$ as a Riemannian manifold, we have no way of discovering κ from its internal geometry. This is an extrinsic property, i.e. is says something about how we have embedded M as a submanifold of $\mathbb{R}^2$.

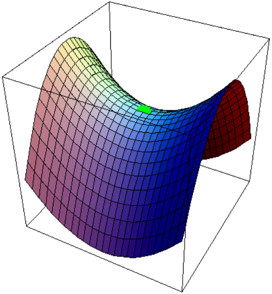


∇f does not tell you anything about the shape, but the Hessian

$$\operatorname{Hess}(f) = \begin{pmatrix} \partial_x^2 f & \partial_x \partial_y f \\ \partial_x \partial_y f & \partial_y^2 f \end{pmatrix}$$

does. Sign of $\det \operatorname{Hess}(f)$ invariant under rotation.

Geometry of surfaces: Calculus II



More precisely, the Gaussian curtvature

$$\kappa = \frac{\det \operatorname{Hess}(f)}{(1 + \partial_x f + \partial_y f)^2}$$

is invariant.

Geometry of surfaces: Calculus II

$$\kappa = \frac{\det \mathrm{Hess}(f)}{(1 + \partial_x f + \partial_y f)^2}.$$

Carl Friedrich Gauss
1777–1854

Theorema Egregium (1828)

κ is an invariant of the Riemannian geometry of

$$M = \{(x, y, f(x, y)) \, : \, (x, y) \in \mathbb{R}^2\},$$

This is an intrinsic property, i.e any isometric imbedding of M in euclidean space will give the same answer.

Geometry of surfaces: Calculus II

Theorema Egregium (1828)

This is an intrinsic property, i.e any isometric imbedding of M in euclidean space will give the same answer.

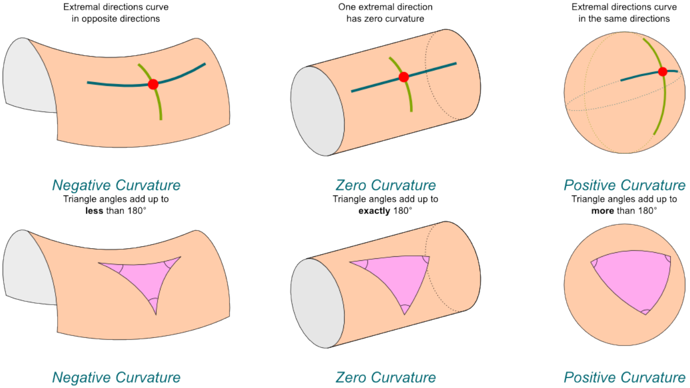
Theorem

A 2-d Riemannian manifold is locally isometric to the Euclidean space if and only if the Gaussian curvature vanishes.

Click here later if you want to see an application to eating pizza.

Gaussian curvature

Two different points of view of curvature. A 2-dimensional math student will not be able to see the curvature of the extremal directions, but will be able to measure angles.



Used to measure the shape of our universe in the BOOMERanG experiment.

This lecture series will take these ideas to other types of spaces ...

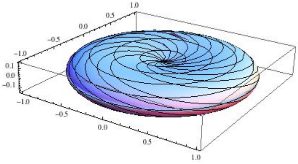
The Heisenberg group: Consider a curve $(x(t), y(t))$ in $\mathbb{R}^2$ with the usual Euclidean metric. Let $z(t)$ be, up to a constant, half the oriented area formed by $x(t)$ and $y(t)$, that is

$$z(t) = z_0 + \frac{1}{2}\int_0^t (x(s)dy(s) - y(s)dx(s)).$$

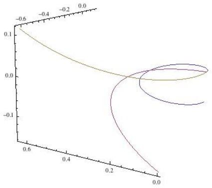
Define the length of $(x(t), y(t), z(t))$ as the length of $(x(t), y(t))$ and define a distance $d_{cc}(x, y)$ on $\mathbb{R}^3$ as the infimum of all such curves connecting x and y.

The Heisenberg group

$$z(t) = z_0 + \frac{1}{2}\int_0^t (x(s)dy(s) - y(s)dx(s)).$$



Unit ball



Geodesics

Relative to this distance, for the coordinate functions, we have

$$x, y \in O(d_{cc}(0,\cdot)), \qquad z \in O(d_{cc}(0,\cdot)^2).$$

We have that $d_{cc}(0,(x,y,x))$ is comparable with $\sqrt{x^2 + y^2 + |z|}$.

Such spaces are called sub-Riemannian spaces. Can we see the difference between these spaces?

The Heisenberg group

The Heisenberg group can be described having this orthonormal basis

$$X_1 = \partial_x - \frac{1}{2} y \partial_z, \qquad X_2 = \partial_x + \frac{1}{2} x \partial_z.$$

How can we see that the orthonormal basis

$$Y_1 = (\frac{\pi}{2} + \tan^{-1} z)\partial_x,$$

$$Y_2 = \frac{x^2}{2(1+z^2)(\frac{\pi}{2} + \tan^{-1} z)}\partial_x + \frac{\frac{\pi}{2} + \tan^{-1} z}{x}\partial_y + \frac{1}{2}\frac{x}{\frac{\pi}{2} + \tan^{-1} z}\partial_z$$

gives an isometric space?

Outline for these lectures

- Lecture 1: Introduction + What is a sub-Riemannian manifold and why should you care about them?
- Lecture 2: “Cartan geometry light” on Riemannian manifolds. Carnot groups and symbols.
- Lecture 3: The sub-Riemannian frame bundle and a canonical connection. The sub-Riemannian equivalence problem: Determining when sub-Riemannian spaces are isometric.

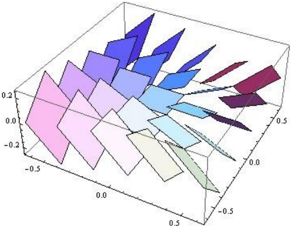
Definition: Sub-Riemannian manifolds

What is a sub-Riemannian manifold?

Definition

A sub-Riemannian manifold is a triple (M, E, g) where

1. M is a connected manifold with (dimension at least 3).
2. E is a subbundle of the tangent bundle TM.
3. $g = \langle \cdot, \cdot \rangle_g$ is a smoothly varying inner product only defined on E.



The metric on such manifold

A sub-Riemannian manifold is a triple (M, E, g).

1. An absolutely continuous curve $\gamma(t)$ in M is called horizontal if $\dot{\gamma}(t) \in E_{\gamma(t)}$ for almost every t.
2. We can define the length of a horizontal curve $\gamma : [a, b] \to M$ by

$$L(\gamma) = \int_a^b \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_g^{1/2} \, dt.$$

3. We define the sub-Riemannian distance or the Carnot-Carathéodory distance $d_g(x, y)$ as the infiumum of the lengths of all of the horizontal curves connecting x and y.

The metric on such manifold

A sub-Riemannian manifold is a triple (M, E, g).

1. An absolutely continuous curve $\gamma(t)$ in M is called horizontal if $\dot{\gamma}(t) \in E_{\gamma(t)}$ for almost every t.
2. We can define the length of a horizontal curve $\gamma : [a, b] \to M$ by

$$L(\gamma) = \int_a^b \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_g^{1/2} \, dt.$$

3. We define the sub-Riemannian distance or the Carnot-Carathéodory distance $d_g(x, y)$ as the infiumum of the lengths of all of the horizontal curves connecting x and y.

.. This looks like it can go wrong. And indeed it can go wrong.

.. so let us talk about subbundles

- Define $\underline{E}^1 = \Gamma(E)$, the sections of E and let

$$\underline{E}^{k+1} = \underline{E}^k + [\underline{E}^1, \underline{E}^k].$$

- Write $E^k_x = \underline{E}^k|_x$.
- E is called bracket-generating if for any x, there is a (minimal) $s(x)$ such that $E^{s(x)} = T_xM$. $s(x)$ is called the step at x.
- $\mathfrak{G}(x) = (\operatorname{rank} E^1, \operatorname{rank} E^2, \ldots, E^{s(x)})$ is called the growth vector at x.

.. so let us talk about subbundles

- E is called bracket-generating if for any x, there is a (minimal) $s(x)$ such that $E^{s(x)} = T_xM$. $s(x)$ is called the step at x.

The Chow-Rashevskiĭ theorem ++

If E is bracket-generating, then every pair of points (x, y) in M can be connected by a horizontal curve. It follows that d_g is a well-defined metric on M. Furthermore, it induces the same topology as the manifold topology.

So as long as the vector fields with values in E and their iterated Lie brackets span all of TM, we are good. We will always assume this from now on.

W.L. Chow.,
Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung.
Math. Ann., 117:98–105, 1939.

P. K. Rashevskiĭ.
On the connectability of two arbitrary points of a totally nonholonomic space by an admissible curve..
Uchen. Zap. Mosk. Ped. Inst. Ser. Fiz.Mat. Nauk, 3(2):83–94, 1938.,

A second look at the Heisenberg group

Let $M = \mathbb{R}^3$ and define

$$X = \partial_x - \frac{1}{2} y \partial_z, \qquad Y = \partial_y + \frac{1}{2} x \partial_z.$$

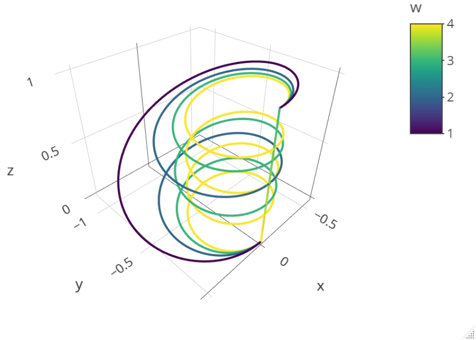
Write $E = \mathsf{span}\{X, Y\}$ and $\langle X, X \rangle_g = \langle Y, Y \rangle_g = 1$, $\langle X, Y \rangle_g = 0$.

$$[X, Y] = \partial_z, \quad \text{so} \quad E^2 = TM.$$

The growth vector is $\mathfrak{G} = (2, 3)$.

A second look at the Heisenberg group

Examples of local length minimizers from 0 to $(0, 0, 1)$.



and so on. The curve with only one rotation is the shortest.

The Hopf fibration

We can consider the Hopf fibration as the surjective map from $S^3 \subseteq \mathbb{C}^2$ to the corresponding complex line in $\mathbb{C}P^1 \cong S^2$, given by

$$\pi : (z, w) \mapsto [z, w] \text{ (equivalence class).}$$

Let S^3 have its usual metric and define $E = (\ker \pi_*)^\perp$ with g the restriction of this metric to E. We then have a sub-Riemannian manifold (S^3, E, g)

The Hopf fibration

$$\pi : (z, w) \mapsto [z, w] \text{ (equivalence class).}$$

Identify S^3 with the Lie group $\mathrm{SU}(2)$ whose Lie algebra $\mathfrak{su}(2)$ is spanned by three elements X, Y, Z, having cyclic bracket relations

$$[X, Y] = Z, \qquad [Y, Z] = X, \qquad [Z, X] = Y.$$

Define $K = \exp(\mathbb{R}Z)$. Then we can see π as the map $\mathrm{SU}(2) \mapsto \mathrm{SU}(2)/K$, $E = \mathrm{span}\{X, Y\}$ and $\langle X, X \rangle_g = \langle Y, Y \rangle_g = 1$ and $\langle X, Y \rangle_g = 0$.

The Martinet distribution

Let $M = \mathbb{R}^3$ and define

$$X = \partial_x, \qquad Y = \partial_y + \frac{1}{2}x^2\partial_z.$$

Write $E = \mathrm{span}\{X, Y\}$ and $\langle X, X\rangle_g = \langle Y, Y\rangle_g = 1$, $\langle X, Y\rangle_g = 0$.

$$[X, Y] = x\partial_z, \quad [X, [X, Y]] = \partial_z.$$

$$E^2_{x,y,z} = \begin{cases} T_{x,y,z} & x \neq 0 \\ \mathrm{span}\{\partial_x, \partial_y\} & x = 0 \end{cases} \qquad E^3 = TM.$$

$$\mathfrak{G}(x, y, z) = \begin{cases} (2, 3) & x \neq 0 \\ (2, 2, 3) & x = 0 \end{cases}$$

The Martinet distribution

Let $M = \mathbb{R}^3$ and define

$$X = \partial_x, \qquad Y = \partial_y + \frac{1}{2}x^2\partial_z.$$

Write $E = \mathrm{span}\{X, Y\}$ and $\langle X, X\rangle_g = \langle Y, Y\rangle_g = 1$, $\langle X, Y\rangle_g = 0$.

$$[X, Y] = x\partial_z, \quad [X, [X, Y]] = \partial_z.$$

$$E^2_{x,y,z} = \begin{cases} T_{x,y,z} & x \neq 0 \\ \mathrm{span}\{\partial_x, \partial_y\} & x = 0 \end{cases} \qquad E^3 = TM.$$

$$\mathfrak{G}(x, y, z) = \begin{cases} (2, 3) & x \neq 0 \\ (2, 2, 3) & x = 0 \end{cases}$$

If the growth vector is constant, E is called equiregular.

Why we cannot have nice things

The key tool for getting invariants on Riemannian geometry:
The Levi-Civita connection.

Why we cannot have nice things

The key tool for getting invariants on Riemannian geometry: The Levi-Civita connection.

We try to define the same thing in sub-Riemannian geometry. We want a connection that takes orthonormal basis of E to an orthonormal basis of E under parallel transport. This is equivalent to say that for any $x \in M$, $Y, Y_2 \in \Gamma(E)$ and $X \in \Gamma(TM)$, we have

$$\nabla_X Y|_x \in \Gamma(E), \qquad X\langle Y, Y_2\rangle_g = \langle \nabla_X Y, Y_2\rangle + \langle Y, \nabla_X Y_2\rangle.$$

Why we cannot have nice things

The key tool for getting invariants on Riemannian geometry: The Levi-Civita connection.

We try to define the same thing in sub-Riemannian geometry. We want a connection that takes orthonormal basis of E to an orthonormal basis of E under parallel transport. This is equivalent to say that for any $x \in M$, $Y, Y_2 \in \Gamma(E)$ and $X \in \Gamma(TM)$, we have

$$\nabla_X Y|_x \in \Gamma(E), \qquad X\langle Y, Y_2\rangle_g = \langle \nabla_X Y, Y_2\rangle + \langle Y, \nabla_X Y_2\rangle.$$

Exercise

Prove this is equivalent to taking orthonormal basese of E through orthonormal bases of E though parallel transport.

Why we cannot have nice things

$$\nabla_X Y|_x \in \Gamma(E), \qquad X\langle Y, Y_2\rangle_g = \langle \nabla_X Y, Y_2\rangle_g + \langle Y, \nabla_X Y_2\rangle_g.$$

Proposition

If E is bracket-generating and properly contained in TM, then there are no compatible connections of (E, g) that are torsion-free.

Why we cannot have nice things

$$\nabla_X Y|_x \in \Gamma(E), \qquad X\langle Y, Y_2\rangle_g = \langle \nabla_X Y, Y_2\rangle_g + \langle Y, \nabla_X Y_2\rangle_g.$$

Proposition

If E is bracket-generating and properly contained in TM, then there are no compatible connections of (E, g) that are torsion-free.

Proof from contradiction: If $X, Y \in \Gamma(E)$ and ∇ is compatible and torsion-free then

$$[X, Y] = \nabla_X Y - \nabla_Y X, \text{ is also in } E,$$

contradicting that E was bracket-generating.

Why we cannot have nice things

Proposition

If E is bracket-generating and properly contained in TM, then there are no compatible connections of (E, g) that are torsion-free.

We therefore need a different point of view to look at a good choice of connection.

Why care about sub-Riemannian geometry?

Where does sub-Riemannian geometry appear?

- Control theory, when we have fewer number of controls than compared to our full space.
- Theory of metric spaces.
- Limits of Riemannian metrics, as the metric outside a certain subbundle go to infinity. Can be applied to fibrations, foliations, etc.
- Gauge theory (Montgomery, 1984).
- Image processing (ask Ugo Boscain).
- Rough paths.
- Statistics on Riemannian manifolds. (Stefan Sommer)

Second order hypoelliptic operators

If we have a second order elliptic operator $L = \sum a_{ij}\partial_i\partial_j + \sum b_k\partial_k$, where (a_{ij}) positive definite, then there is a corresponding Riemannian metric g such that

$$\frac{1}{2}(L(f_1 f_2) - f_1 L f_2 - f_2 L f_1) = \langle \nabla f_1, \nabla f_2 \rangle_g.$$

Curvature of g control properties of L and $P_t = e^{tL}$.

If (a_{ij}) is only positive semidefinite, we have a sub-Riemannian metric (E, g) such that

$$\frac{1}{2}(L(f_1 f_2) - f_1 L f_2 - f_2 L f_1) = \langle \nabla^E f_1, \nabla^E f_2 \rangle_g.$$

Curvature-like quantities of g gives control properties of L and $P_t = e^{tL}$.

F. Baudoin and N. Garofalo.
Curvaturedimension inequalities and Ricci lower bounds for sub-Riemannian manifolds with transverse symmetries.
J. Eur. Math. Soc. (JEMS), 19(1):151–219, 2017.

E. Grong and A. Thalmaier.
Curvature dimension inequalities on sub-Riemannian manifolds obtained from Riemannian foliations: part I and II
Math. Z., 282(12), 2012.

E. Grong and A. Thalmaier.
Stochastic completeness and gradient representations for sub-Riemannian manifolds.
Potential Anal., 51(2):219–254, 2019.

Geodesics and topology

For a Riemannian manifolds, positive curvature makes geodesics move slower appart, negative curvature faster. Some results: Postively curved manifolds are compact with finite fundamental groups, non-positively curved manifolds are diffeomorphic to $\mathbb{R}^n$.

There are similar relations appearing in sub-Riemannian geometry by considering variations of geodesics.

Geodesics and topology

There are similar relations appearing in sub-Riemannian geometry by considering variations of geodesics.

I. Zelenko and C. Li.
Differential geometry of curves in Lagrange Grassmannians with given Young diagram.
Differential Geom. Appl., 27(6):723–742, 2009.,

D. Barilari and L. Rizzi.
Comparison theorems for conjugate points in sub-Riemannian geometry.
ESAIM Control Optim. Calc. Var., 22(2):439–472, 2016.

Carnot groups: The flat sub-Riemannain spaces

The flat spaces: Carnot groups

Let $\mathfrak{g}$ be a nilpotent Lie algebra. A stratification of a nilpotent Lie algebra is a decomposition

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_s,$$

such that

$$[\mathfrak{g}_1, \mathfrak{g}_k] = \mathfrak{g}_{k+1}, \qquad [\mathfrak{g}_1, \mathfrak{g}_s] = 0.$$

A Carnot algebra is a Lie algebra $\mathfrak{g}$ with a stratification and an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}_1$.

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_s,$$

$$[\mathfrak{g}_1, \mathfrak{g}_k] = \mathfrak{g}_{k+1}, \qquad [\mathfrak{g}_1, \mathfrak{g}_s] = 0.$$

Let G be the corresponding simply connected Lie group. We can then define a subbundle E by left translation of $\mathfrak{g}_1$. In other words, $E_a = a \cdot \mathfrak{g}_1$. We can define a sub-Riemannian metric g on E by

$$\langle v, w \rangle_g = \langle a^{-1} \cdot v, a^{-1} \cdot v \rangle.$$

Example: The n-th Heisenberg group

Consider the Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where

$$\mathfrak{g}_1 = \mathrm{span}\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}, \qquad \mathfrak{g}_2 = \mathrm{span}\{Z\}.$$

$$[X_i, X_j] = 0, \qquad [Y_i, Y_j] = 0, \qquad [X_i, Y_j] = \delta_{ij} Z.$$

$$X_1, \ldots, X_n, Y_1, \ldots, Y_n \text{ form an orthonormal basis.}$$

Corresponding group $G = \{(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, with

$$(x, y, z) \cdot (\tilde{x}, \tilde{y}, \tilde{z}) = \left(x + \tilde{x}, y + \tilde{y}, z + \tilde{z} + \frac{1}{2}(\langle x, \tilde{y} \rangle - \langle y, \tilde{x} \rangle \right).$$

We have that the corresponding left invariant vector fields are given by

$$X_j = \partial_{x^i} - \frac{1}{2} y^j \partial_z, \qquad Y_j = \partial_{y^i} + \frac{1}{2} x^j \partial_z, \qquad Z = \partial_z.$$

Example: Free nilpotent group of step 2

Consider the Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where

$$\mathfrak{g}_1 = \mathbb{R}^n, \qquad \mathfrak{g}_2 = \wedge^2 \mathbb{R}^n.$$

$$[p, q] = p \wedge q, \qquad [A, q] = [A, B] = 0, \qquad p, q \in \mathfrak{g}_1, A, B \in \mathfrak{g}_2.$$

$\mathbb{R}^n$ has the standard metric. Corresponding group $G = \mathbb{R}^n \times \wedge^2 \mathbb{R}^n$, with

$$(p, A) \cdot (q, B) = \left(p + q, A + B + \frac{1}{2} p \wedge q \right).$$

We see that we can always find the group operation using the Baker-Campbell-Hausdorff formula.

The flat spaces: Carnot groups

A special thing for Carnot group is that they have dilations. If $\mathfrak{g}$ is a Carnot algebra, for any $r > 0$, we define a linear map $\mathrm{dil}_r : \mathfrak{g} \to \mathfrak{g}$ by

$$\mathrm{dil}_r(A) = r^k A \qquad \text{for any } A \in \mathfrak{g}_k.$$

Exercise

(a) Show that for a stratified Lie algebra, $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$, where we interpret $\mathfrak{g}_{i+j} = 0$ if $i + j > s$.

(b) Explain why dil_r is a Lie algebra isomorphism.

Since G is simply connected, there is a Lie group isomorphism $\mathrm{Dil}_r : G \to G$ such that

$$\mathrm{Dil}_r(\exp(A)) = \exp(\mathrm{dil}_r A).$$

Since G is simply connected, there is a Lie group isomorphism $\mathsf{Dil}_r : G \to G$ such that

$$\mathsf{Dil}_r(\exp(A)) = \exp(\mathsf{dil}_r A).$$

Exercise

(a) Show that if γ is a horizontal curve in G, then $L(\mathsf{Dil}_r \gamma) = rL(\gamma)$.

(b) Explain why $d_g(\mathsf{Dil}_r(x), \mathsf{Dil}_r(y)) = rd_g(x, y)$.

The significance of the dilations of Carnot groups

- $\mathbb{R}^n$ with the Euclidean metric and Carnot groups, look the same when we “zoom in”.
- Riemannian manifolds all look like the Euclidean space when we “zoom in”.
- Sub-Riemannian manifolds (M, E, g) with E equiregular will all look like Carnot groups when we “zoom in”. But not necessarily the same at every point.
- This zooming in can be made formal using Gromov-Haussdorff convergence of pointed metric spaces.

The significance of the dilations of Carnot groups

A. Bellaïche.
The tangent space in sub-Riemannian geometry.
In Sub-Riemannian geometry, volume 144 of Progr. Math., pages 1–78. Birkhäuser, Basel, 1996.

M. Gromov.
Carnot-Carathéodory spaces seen from within.
In Sub-Riemannian geometry, volume 144 of Progr. Math., pages 79–323. Birkhäuser, Basel, 1996.

Isometries of Carnot groups

- Let $l_a : G \to G$ be the left translation $l_a(b) = a \cdot b$, $a, b \in G$. Since the sub-Riemannian structure is left invariant, each such map will be an isometry.
- Consider a Lie group isomorphism $\Phi : G \to G$ with induced $\varphi = \Phi_{*,1} : \mathfrak{g} \to \mathfrak{g}$. Then Φ is an isometry, if and only if φ maps $\mathfrak{g}_1$ to itself isometrically. We will call this an Carnot algebra isometry.

Isometries of Carnot groups

Theorem

Assume that $\Phi : G \to G$ is an isometry of a Carnot group with $\Phi(1) = a$. Then $\Phi \circ l_{a^{-1}}$ is both an isometry and a Lie algebra isomorphism. In other words, any isometry of a Carnot group is the composition of a left translation and a Lie group isomorphism.

E. Le Donne and A. Ottazzi.
Isometries of Carnot groups and sub-Finsler homogeneous manifolds.
J. Geom. Anal., 26(1):330–345, 2016,

Example: Free nilpotent group of step 2

Consider the Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where

$$\mathfrak{g}_1 = \mathbb{R}^n, \qquad \mathfrak{g}_2 = \wedge^2 \mathbb{R}^n.$$

$$[p, q] = p \wedge q, \qquad [A, q] = [A, B] = 0, \qquad p, q \in \mathfrak{g}_1, A, B \in \mathfrak{g}_2.$$

$\mathbb{R}^n$ has the standard metric.

$$\mathsf{Isom}(\mathfrak{g}) \cong \mathsf{O}(n).$$

Example: The n-th Heisenberg group

Consider the Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where

$$\mathfrak{g}_1 = \mathrm{span}\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}, \qquad \mathfrak{g}_2 = \mathrm{span}\{Z\}.$$

$$[X_i, X_j] = 0, \qquad [Y_i, Y_j] = 0, \qquad [X_i, Y_j] = \delta_{ij} Z.$$

$X_1, \ldots, X_n, Y_1, \ldots, Y_n$ form an orthonormal basis.

We need then a Carnot algebra isometry to be an isometry of $\mathfrak{g}_1$ and preserve the symplectic structure $\sum_{j=1}^{n} X_j^* \wedge Y_j^*$:

$$\mathrm{Isom}(\mathfrak{g}) \cong U(n).$$

Example: Engel algebra

Consider the Lie algebra

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 = \mathrm{span}\{X, Y\} \oplus \mathrm{span}\{Z\} \oplus \mathrm{span}\{W\},$$

with orthonormal basis X, Y and with only non-zero brackets

$$[X, Y] = Z, \qquad [X, Z] = W.$$

Then

$$\mathrm{Isom}(\mathfrak{g}) \cong \mathbb{Z}/2\mathbb{Z}$$

Summary

- The size of the symmetry group can vary much.
- These are the first order approximation of a sub-Riemannian space. The size of the isometry group of a Carnot group will affect the geometry of all spaces who has this as an infinitesimal approximation.

Thank you very much!