UNIVERSITY OF BERGEN

Curvature in sub-Riemannian geometry

Lecture 2

42nd Winter School: Geometry and Physics

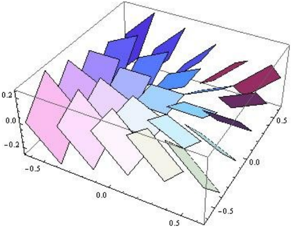
Erlend Grong, erlend.grong@uib.no

Srní, January 20, 2022

University of Bergen, Norway

Recall from last time, part I

- Sub-Riemannian manifolds considers manifolds with an inner product g is only defined on a subbundle E.
- They are very interesting.
- They do not have a compatible torsion-free connections.



A quick Riemannian introduction to Cartan connections

Why torsion-free, anyway?

Let (M, g) be a Riemannian manifold and let ∇ be a compatible connection. Important tensors

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \qquad R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z.$$

Recall that if $f(s, t)$ is a parametrized surface with a vector field $Z(s, t)$ along it then

$$\nabla_{\partial_s f}\partial_t f - \nabla_{\partial_t f}\partial_s f = T(\partial_s f, \partial_t f)$$
$$\nabla_{\partial_t f}\nabla_{\partial_t f} Z - \nabla_{\partial_t f}\nabla_{\partial_s f} Z = R(\partial_s f, \partial_t f)Z$$

Orthonormal frame bundle

- (M, g) a Riemannian manifold of dimension n.
- $\mathbb{R}^n$ with the euclidean metric has a trivial tangent bundle and a canonical basis $e_1, \ldots, e_n$. Let us steal these properties.

Orthonormal frame bundle

- (M, g) a Riemannian manifold of dimension n.
- $\mathbb{R}^n$ with the euclidean metric has a trivial tangent bundle and a canonical basis $e_1, \ldots, e_n$. Let us steal these properties.
- An orthonormal frame at $x \in M$ is a choice of orthonormal basis $u_1, \ldots u_n$ for T_xM. Equivalently, we can consider a frame as a linear isometry $u : \mathbb{R}^n \to T_xM$. The correspondence is given by

$$u_j = u(e_j).$$

We write the set of all such frames as $\mathrm{O}_x(M)$.

Orthonormal frame bundle

- For frame $u : \mathbb{R}^n \to T_xM$ and $a \in \mathrm{O}(n)$, we can define a new frame $u \cdot a = u \circ a : \mathbb{R}^n \to T_xM$ by precomposition. In other words, if $\tilde{u} = u \cdot a$, then

$$\tilde{u}_j = \sum_{i=1}^{n} a_{ij} u_i.$$

- We use the above action to construct a principal bundle called the frame bundle

$$\mathrm{O}(n) \to \mathrm{O}(M) \xrightarrow{\pi} M,$$

with fibers $\mathrm{O}_x(M)$.

Structures on the orthonormal frame bundle

(i) The vertical bundle: $\mathcal{V} = \ker \pi_*$ of rank n^2.

(ii) Canonical vertical vector field: If $A \in \mathfrak{so}(n)$, we define

$$\xi_A|_u = \frac{d}{dt} u \cdot e^{At}|_{t=0}.$$

We then have a globally defined vector field ξ_A.
Furthermore,

$$\mathcal{V}_u = \{\xi_A|_u \,:\, A \in \mathfrak{so}(n)\}, \qquad u \in \mathrm{O}(M).$$

Structures on the orthonormal frame bundle

(i) The vertical bundle: $\mathcal{V} = \ker \pi_*$ of rank n^2.

(ii) Canonical vertical vector field: If $A \in \mathfrak{so}(n)$, we define

$$\xi_A|_u = \frac{d}{dt} u \cdot e^{At}|_{t=0}.$$

We then have a globally defined vector field ξ_A. Furthermore,

$$\mathcal{V}_u = \{\xi_A|_u \,:\, A \in \mathfrak{so}(n)\}, \qquad u \in \mathrm{O}(M).$$

Exercise

Show that

$$\xi_A \cdot a = \xi_{\mathrm{Ad}(a^{-1})A}, \qquad A \in \mathfrak{so}(n), a \in \mathrm{O}(n).$$

Structures on the orthonormal frame bundle

(i) The vertical bundle: $\mathcal{V} = \ker \pi_*$ of rank n^2.

(ii) Canonical vertical vector field: If $A \in \mathfrak{so}(n)$, we define

$$\xi_A|_u = \frac{d}{dt} u \cdot e^{At}|_{t=0}.$$

We then have a globally defined vector field ξ_A. Furthermore,

$$\mathcal{V}_u = \{\xi_A|_u \,:\, A \in \mathfrak{so}(n)\}, \qquad u \in \mathrm{O}(M).$$

(iii) The tautological one-form: We define an $\mathbb{R}^n$-valued one-form $\theta = (\theta_1, \ldots, \theta_n)$. Then

$$\theta(w) = u^{-1}\pi_* w, \qquad w \in T_u\,\mathrm{O}(M).$$

Observe that $\ker \theta = \mathcal{V}$.

Connections on frame bundles

An Ehresmann connection $\mathcal{H}$ on $\pi : \mathrm{O}(M) \to M$ is a choice of complement to $\mathcal{V}$:

$$T\,\mathrm{O}(M) = \mathcal{H} \oplus \mathcal{V}.$$

Interpretation: $\pi_{*,u}|_{\mathcal{H}_u}$ is invertible and we can define an inverse h_u, so $h_u v$ is the unique element in $\mathcal{H}_u$ such that $\pi_* h_u v = v$.

An Ehresmann connection is called principal if it is invariant under the action $\mathrm{O}(n)$.

$$\mathcal{H}_u \cdot a = \mathcal{H}_{u \cdot a}, \qquad u \in \mathrm{O}(n), a \in \mathrm{O}(n)$$

Equivalently, $h_u v \cdot a = h_{u \cdot a} v$.

Connections on frame bundles

An Ehresmann connection is called principal if it is invariant under the action $\mathrm{O}(n)$.

$$\mathcal{H}_u \cdot a = \mathcal{H}_{u \cdot a}, \qquad u \in \mathrm{O}(n), a \in \mathrm{O}(n)$$

Equivalently, $h_u v \cdot a = h_{u \cdot a} v$.

Exercise

Define a one-form $\omega : T\mathrm{O}(n) \to \mathfrak{so}(n)$ such that

$$\omega(h_u v) = 0, \qquad \omega(\xi_A) = A.$$

Show that $\omega(w \cdot a) = \mathrm{Ad}(a^{-1})\omega(w)$, $w \in T\mathrm{O}(M)$, $a \in \mathrm{O}(n)$. ω is called the connection form of $\mathcal{H}$.

From affine to principal connections

Let ∇ be a compatible connection. Let $U(t)$ be a curve in $\mathrm{O}(M)$ such that $\pi(U(t)) = \gamma(t)$. Assume that $\dot{\gamma}(0) = v$, that $U(0) = u$ and that each $U_j(t)$ is parallel along $\gamma(t)$. We can then define

$$h_u v = \dot{U}(0),$$

so the derivative of a parallel frame moving the direction of v. We then define

$$\mathcal{H}_x = \{h_u v \,:\, u \in \mathrm{O}_x(M), v \in T_x M\}.$$

Exercise

Show that $\mathcal{H}$ is a principal Ehresmann connection.

A canonical basis

Now something interesting happens since for each point of $\mathrm{O}(M)$ is a frame. For any $p \in \mathbb{R}^n$, can define vector field H_p by

$$H_p|_u = \sum_{j=1}^{n} p^j h_u u_j.$$

Canonical basis H_p, ξ_A, $p \in \mathbb{R}^n$, $A \in \mathfrak{so}(n)$.

A canonical basis

Now something interesting happens since for each point of $O(M)$ is a frame. For any $p \in \mathbb{R}^n$, can define vector field H_p by

$$H_p|_u = \sum_{j=1}^{n} p^j h_u u_j.$$

Canonical basis H_p, ξ_A, $p \in \mathbb{R}^n$, $A \in \mathfrak{so}(n)$.

$$[H_p, H_q] = -\xi_{\bar{R}(u,v)} - H_{\bar{T}(p,q)}, \qquad [\xi_A, H_q] = H_{Aq}, \qquad [\xi_A, \xi_B] = \xi_{[A,B]}.$$

$$\bar{T}(p,q)|_u = u^{-1}T(u(p), u(q)), \qquad \bar{R}(p,q)|_u = u^{-1}R(u(p), u(q))u.$$

A canonical basis

$$[H_p, H_q] = -\xi_{\bar{R}(u,v)} - H_{\bar{T}(p,q)}, \qquad [\xi_A, H_q] = H_{Aq}, \qquad [\xi_A, \xi_B] = \xi_{[A,B]}.$$

$$\bar{T}(p,q)|_u = u^{-1}T(u(p), u(q)), \qquad \bar{R}(p,q)|_u = u^{-1}R(u(p), u(q))u.$$

Since

$$\omega(H_p) = 0, \qquad \omega(\xi_A) = A, \qquad \theta(H_p) = p, \qquad \theta(\xi_A) = 0.$$

A canonical basis

$$[H_p, H_q] = -\xi_{\bar{R}(u,v)} - H_{\bar{T}(p,q)}, \qquad [\xi_A, H_q] = H_{Aq}, \qquad [\xi_A, \xi_B] = \xi_{[A,B]}.$$

$$\bar{T}(p,q)|_u = u^{-1}T(u(p), u(q)), \qquad \bar{R}(p,q)|_u = u^{-1}R(u(p), u(q))u.$$

Since

$$\omega(H_p) = 0, \qquad \omega(\xi_A) = A, \qquad \theta(H_p) = p, \qquad \theta(\xi_A) = 0.$$

we can rewrite these equations as

$$d\theta + [\omega, \theta] = \Theta, \qquad d\omega + \frac{1}{2}[\omega, \omega] = \Omega.$$

$$\begin{aligned} \Theta(\xi_A, \cdot) &= 0 & \Omega(\xi_A, \cdot) &= 0, \\ \Theta(H_p, H_q) &= \bar{T}(p,q), & \Omega(H_p, H_q) &= \bar{R}(p,q). \end{aligned}$$

Values in $\mathfrak{se}(n)$

$$d\theta + [\omega, \theta] = \Theta, \qquad d\omega + \frac{1}{2}[\omega, \omega] = \Omega.$$

We can rewrite the following equation in $\mathfrak{se}(n)$. Can be considered as the space $\mathfrak{so}(n) \times \mathbb{R}^n$ with brackets,

$$[(A, p), (B, q)] = ([A, B], Aq - Bp), \qquad A, B \in \mathfrak{so}(n), p, q \in \mathbb{R}^n.$$

We can then consider $\psi = (\omega, \theta)$ with

$$d\psi + \frac{1}{2}[\psi, \psi] = (\Omega, \Theta).$$

A sidestep on integrating Lie-algebra valued forms

Let G be a Lie group with Lie algebra $\mathfrak{g}$. The left Maurer-Cartan form η is defined as

$$\eta(v) = a^{-1} \cdot v, \qquad v \in T_a G,$$

which is a $\mathfrak{g}$-valued one-form. We observe that this one-form satisfies

$$d\eta + \frac{1}{2}[\eta, \eta] = 0.$$

A sidestep on integrating Lie-algebra valued forms

We observe that this one-form satisfies

$$d\eta + \frac{1}{2}[\eta, \eta] = 0.$$

Let $f : M \to G$ be any map from a manifold M into G and define $\psi = f^*\omega$. Then by properties of the pull-back,

$$d\psi + \frac{1}{2}[\psi, \psi] = 0. \tag{1}$$

The converse is also true locally: Just like a real valued form α is locally integrable if and only if $d\alpha = 0$, a $\mathfrak{g}$-valued one-form can locally be written as $\psi = f^*\eta$ if and only if (1) holds.

Values in $\mathfrak{se}(n)$

$$d\theta + [\omega, \theta] = \Theta, \qquad d\omega + \frac{1}{2}[\omega, \omega] = \Omega.$$

We can rewrite the following equation in $\mathfrak{se}(n)$. Can be considered as the space $\mathfrak{so}(n) \times \mathbb{R}^n$ with brackets,

$$[(A, p), (B, q)] = ([A, B], Aq - Bp), \qquad A, B \in \mathfrak{so}(n), p, q \in \mathbb{R}^n.$$

We can then consider $\psi = (\omega, \theta)$ with

$$d\psi + \frac{1}{2}[\psi, \psi] = (\Omega, \Theta).$$

Torsion and curvature of the connection are the obstructions for integrating ψ to a map into the euclidean group $\mathrm{E}(n)$.

Values in $\mathfrak{se}(n)$

$$d\theta + [\omega, \theta] = \Theta, \qquad d\omega + \frac{1}{2}[\omega, \omega] = \Omega.$$

We can rewrite the following equation in $\mathfrak{se}(n)$. Can be considered as the space $\mathfrak{so}(n) \times \mathbb{R}^n$ with brackets,

$$[(A, p), (B, q)] = ([A, B], Aq - Bp), \qquad A, B \in \mathfrak{so}(n), p, q \in \mathbb{R}^n.$$

We can then consider $\psi = (\omega, \theta)$ with

$$d\psi + \frac{1}{2}[\psi, \psi] = (\Omega, \Theta).$$

Torsion and curvature of the connection are the obstructions for integrating ψ to a map into the euclidean group $\mathrm{E}(n)$.

Observe that since $\theta(v \cdot a) = a^{-1}\theta(w)$ and $\omega(w \cdot a) = \mathrm{Ad}(a^{-1})\omega(w)$ for $w \in T\mathrm{O}(M)$, $a \in \mathrm{O}(n)$,

$$\psi(w \cdot a) = \mathrm{Ad}(a^{-1})\psi(w).$$

Theorem (Flatness theorem)

A Riemannian manifold (M, g) is locally isometric to the Euclidean space if and only if the curvature of the Levi-Civita connection vanishes.

Theorem (Flatness theorem)

A Riemannian manifold (M, g) is locally isometric to the Euclidean space if and only if the curvature of the Levi-Civita connection vanishes.

$\Rightarrow$: Curvature is a local invariant. For $\Leftarrow$:

- If the curvature of the Levi-Civita connection is zero, then ψ is locally $\psi = f^*\eta$ for map into $\mathrm{E}(n)$,
- Since $\psi|_u : T_u\,\mathrm{O}(M) \to \mathfrak{se}(n)$ is bijective, f is a local diffeomorphism,
- Since $\psi(w \cdot a) = \mathrm{Ad}(a^{-1})\psi(w)$, $f(u \cdot a) = f(u) \cdot a$. This means that f decends to a local map $\check{f}$ from M to $\mathbb{R}^n$.
- Finally, $\check{f}$ is an isometry, since $v \mapsto h_u v \mapsto \theta(h_u v)$ is a linear isometry from $T_x M$ to $\mathbb{R}^n$.

Theorem (Flatness theorem)

A Riemannian manifold (M, g) *is locally isometric to the Euclidean space if and only if the curvature of the Levi-Civita connection vanishes.*

Observe that in this proof, we needed two steps

- Normalize the connection to remove of the torsion.
- Then we can set up the condition of the curvature vanishing afterwards.

Cartan connections in general

- Manifold M of dimension n.
- Let $\mathfrak{g}$ be a Lie algebra with subalgebra $\mathfrak{h}$ of codimension n.
- Let H be a Lie group with Lie algebra $\mathfrak{h}$.
- Ad a representation of H on $\mathfrak{g}$ extending usual adjoint action of H.
- Principal bundle $H \to P \xrightarrow{\pi} M$.

Cartan connections in general

A Cartan connection ψ on P modeled on $(\mathfrak{g}, \mathfrak{h})$ is a $\mathfrak{g}$-valued one form $\psi : TP \to \mathfrak{g}$, such that

(i) For each $p \in P$, $\psi|_p$ is a linear isomorphism from T_pP to $\mathfrak{g}$.

(ii) For each $a \in H$, $v \in TP$,

$$\psi(v \cdot a) = \mathrm{Ad}(a^{-1})\psi(v).$$

(iii) For every $D \in \mathfrak{h}$, $p \in P$, we have $\psi(\frac{d}{dt} p \cdot \exp_H(tD)|_{t=0}) = D$.

Cartan connections in general

A Cartan connection ψ on P modeled on $(\mathfrak{g}, \mathfrak{h})$ is a $\mathfrak{g}$-valued one form $\psi : TP \to \mathfrak{g}$, such that

(i) For each $p \in P$, $\psi|_p$ is a linear isomorphism from T_pP to $\mathfrak{g}$.

(ii) For each $a \in H$, $v \in TP$,

$$\psi(v \cdot a) = \mathrm{Ad}(a^{-1})\psi(v).$$

(iii) For every $D \in \mathfrak{h}$, $p \in P$, we have $\psi(\frac{d}{dt}p \cdot \exp_H(tD)|_{t=0}) = D$.

The curvature of a Cartan connection is represented by a smooth function $\kappa : P \to \wedge^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}$,

$$\kappa(\psi(\cdot), \psi(\cdot)) = d\psi + \frac{1}{2}[\psi, \psi].$$

Nipotentizations of sub-Riemannian manifolds

Recall from last time, part II

- A Cartan algebra is an algebra with a stratification $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ such that $[\mathfrak{g}_1, \mathfrak{g}_k] = \mathfrak{g}_{k+1}$ and with an inner product on $\mathfrak{g}_1$. Isometries are Lie algebra isomorphisms that preserve the brackets on the first layer. These can vary in dimensions.
- The corresponding simply connected Lie groups G with a sub-Riemannian structure given by left translation of $\mathfrak{g}_1$ with its inner product is called a Carnot group.

Nilpotentization of a sub-Riemannian manifold

Let (M, E, g) be a sub-Riemannian manifold. We will assume that E is equiregular. Have a flag of subbundles

$$E^0 = 0 = E^1 = E \subseteq E^2 \subseteq \cdots \subseteq E^s.$$

Define

$$\mathbf{symb}_x = E_x \oplus E_x^2/E_x \oplus \cdots \oplus E^s/E^{s-1}.$$

Then we can define a Lie algebra for $X_x \in E_x^i, Y_x \in E_x^j$

$$[\![X_x \mod E_x^{i-1}, Y_x \mod E_x^{j-1}]\!] = [X, Y]|_x \mod E_x^{i+j-1},$$

where X and Y are any vector field extending X_x and Y_x. This makes $(\mathbf{symb}_x, [\![\cdot, \cdot]\!])$ into a nilpotent Lie algebra with a stratification $\mathbf{symb}_{x,j} = E_x^j / E_x^{j-1}$ and with an inner product on $\mathbf{symb}_{x,1}$; in other words, a Carnot algebra.

Nilpotentization of a sub-Riemannian manifold

From the Carnot algebra symb_x, we get a corresponding Carnot groups $(\mathsf{Symb}_x, \tilde{E}, \tilde{g})$. This Carnot group is what (M, E, g) looks like when we “zoom in”. This can be made precise in terms of Gromov-Hausdorff convergence of metric spaces.

We say that (M, E, g) has constant symbol $\mathfrak{g}$ if symb_x is isometric to $\mathfrak{g}$ for any $x \in M$.

The Hopf fibration again

We consider again $\mathrm{SU}(2)$ with the sub-Riemannian structure $E = \mathrm{span}\{X, Y\}$,

$$[X, Y] = Z, \qquad [Y, Z] = X, \qquad [Z, X] = Y.$$

Then for any point $\mathbf{symb}_x = \mathrm{span}\{X, Y\} \oplus \mathrm{span}\{Z \mod E\}$, with

$$[\![X, Y]\!] = Z \mod E, \qquad [\![X, Z \mod E]\!] = [\![y, Z \mod E]\!] = 0.$$

We see that $\mathbf{symb}_x$ is the Heisenberg algebra.

Not constant symbol

Consider $\mathbb{R}^5$ with coordinates (x_1, x_2, y_1, y_2, z) with (E, g) given by an orthonormal basis

$$A_1 = \partial_{x_1}, \qquad B_1 = (1 + y_1^2)(\partial_{y_1} + x_1\partial_{x_1}.$$

$$A_2 = \partial_{x_2}, \qquad B_2 = \partial_{y_2} + x_1\partial_{x_1}.$$

Then $\mathbf{symb}_{x_1,x_2,y_1,y_2,z}$ is isometric to the 2nd Heisenberg algebra

$$[X_1, Y_1] = [X_2, Y_2] = Z,$$

but with an orthonormal basis given by $\sqrt{1+y_1^2}X_1, \sqrt{1+y_1^2}Y_2$, X_2, Y_2, where y_1 is now considered as a constant. These are not isometric for different values of y_1^2.

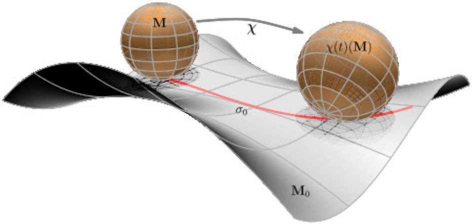
Example

If there is just one Carnot algebra in the class of growth vectors, then all sub-Riemannian manifolds with that growth vector will have constant symbol.

Consider two 2d Riemannian manifolds Σ and $\tilde{\Sigma}$, whose Gaussian curvature never coinsides. On $\hat{M} = \mathrm{O}(\Sigma) \times \mathrm{O}(\tilde{\Sigma})$, consider the sub-Riemannian structure $(\hat{E}, \hat{g})$ with orthogonal vector fields $H_{e_1} + \tilde{H}_{e_1}$ and $H_{e_2} + \tilde{H}_{e_2}$. Define $M = \hat{M}/\,\mathrm{O}(n)$ as the quotient the diagonal action and let (E, g) be the induced sub-Riemannian structure on (M, E, g). Then (M, E, g) has growth vector $(2, 3, 5)$. Since there is only one Cartan group with this growth vector, these have all constant symbol.

Example

Consider two 2d Riemannian manifolds Σ and $\tilde{\Sigma}$, whose Gaussian curvature never coinsides. On $\hat{M} = \mathrm{O}(\Sigma) \times \mathrm{O}(\tilde{\Sigma})$, consider the sub-Riemannian structure $(\hat{E}, \hat{g})$ with orthogonal vector fields $H_{e_1} + \tilde{H}_{e_1}$ and $H_{e_2} + \tilde{H}_{e_2}$. Define $M = \hat{M}/\,\mathrm{O}(n)$ as the quotient the diagonal action and let (E, g) be the induced sub-Riemannian structure on (M, E, g).



Nonholonomic frame bundle

- Assume (M, E, g) has constant symbol $\mathfrak{g}$. Define $G_0 = \mathsf{Isom}(\mathfrak{g})$ with Lie algebra $\mathfrak{g}_0 = \mathfrak{isom}(\mathfrak{g})$.
- $\mathfrak{g}_0$ consist of derivations of $\mathfrak{g}$ preserving the stratification and whose restriction to $\mathfrak{g}_1$ is skew symmetric.
- Define a new algebra $\hat{\mathfrak{g}} = \mathfrak{g}_0 \oplus \mathfrak{g}$ such that both $\mathfrak{g}_0$ and $\mathfrak{g}$ as subalgebras and with

$$[D, A] = DA.$$

Exercise

Show that $\hat{\mathfrak{g}}$ is the Lie algebra of isometries of the isometry algebra $\hat{G} = \mathsf{Isom}(G)$.

Nonholonomic frame bundle

- Define a vector bundle of symbols $\mathbf{symb} \to M$ over M. We will call this the non-holonomic tangent bundle.
- We now define a non-holonomic frame as a Carnot isomorphism $u : \mathfrak{g} \to \mathbf{symb}_x$. We will write a the set of all such frames as $\mathscr{F}_x$.
- We again have a right action of G_0 on $\mathscr{F}_x$ by precomposition. This gives us a principal bundle

$$G_0 \to \mathscr{F} \to M.$$

Cartan connections on the frame bundle

Let ψ be a $(\hat{\mathfrak{g}}, \mathfrak{g}_0)$ Cartan connection with values in $\hat{\mathfrak{g}}$. We can write it as $\psi = (\omega, \theta)$ where ω and θ has values in respectively $\mathfrak{g}_0$ and $\mathfrak{g}_-$. Observe.

- ω is a principal connection on the bundle $\mathscr{F}$. Corresponds to an affine connection ∇ on $\widetilde{\mathsf{symb}}$ such that parallel transport are Cartan isometries.

Cartan connections on the frame bundle

Let ψ be a $(\hat{\mathfrak{g}}, \mathfrak{g}_0)$ Cartan connection with values in $\hat{\mathfrak{g}}$. We can write it as $\psi = (\omega, \theta)$ where ω and θ has values in respectively $\mathfrak{g}_0$ and $\mathfrak{g}_-$. Observe.

- ω gives $\tilde{\nabla}$ on **symb**.
- θ correspond to a vector bundle isomorphism $I : TM \to \mathbf{symb}$ in the following way. For any $v \in T_xM$, and $u \in \mathscr{F}_x$,

$$I : v \mapsto h_u v \in \mapsto \theta(h_u v) \in \mathfrak{g} \mapsto u^{-1}\theta(h_u v) \in \mathbf{symb}_x . \qquad (2)$$

  We can see this map as a way choosing complements to $E^{k+1} = V^{k+1} \oplus E^k$ by $I^{-1}(E^{k+1}/E^k)$.
- From the previous structures, we can define a connection $\nabla = I^{-1}\tilde{\nabla} I$ on TM.

In summary, any choice of Cartan connection gives us an identification $I : TM \to \mathsf{symb}$ and connection ∇ on TM. We wil show next time that there is a preferred choice.

Thank you very much!