UNIVERSITY OF BERGEN

Curvature in sub-Riemannian geometry

Lecture 3

42nd Winter School: Geometry and Physics

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Recall from last time

Let (M, E, g) be a sub-Riemannian manifold. We will assume that E is equiregular. Have a flag of subbundles

$$E^0 = 0 = E^1 = E \subseteq E^2 \subseteq \cdots \subseteq E^s.$$

Define

$$\mathbf{symb}_x = E_x \oplus E_x^2/E_x \oplus \cdots \oplus E^s/E^{s-1}.$$

Then we can define a Lie algebra for $X_x \in E_x^i, Y_x \in E_x^j$

$$[\![X_x \mod E_x^{i-1}, Y_x \mod E_x^{j-1}]\!] = [X, Y]|_x \mod E_x^{i+j-1},$$

where X and Y are any vector field extending X_x and Y_x. This makes $(\mathbf{symb}_x, [\![\cdot, \cdot]\!])$ into a nilpotent Lie algebra with a stratification $\mathbf{symb}_{x,j} = E_x^j/E_x^{j-1}$ and with an inner product on $\mathbf{symb}_{x,1}$; in other words, a Carnot algebra.

Recall from last time

From the Carnot algebra $\mathbf{symb}_x$, we get a corresponding Carnot groups $(\mathrm{Symb}_x, \tilde{E}, \tilde{g})$. This Carnot group is what (M, E, g) looks like when we "zoom in". This can be made precise in terms of Gromov-Hausdorff convergence of metric spaces.

We say that (M, E, g) has constant symbol $\mathfrak{g}$ if $\mathbf{symb}_x$ is isometric to $\mathfrak{g}$ for any $x \in M$.

Erlend Grong,
Canonical connections on sub-riemannian manifolds with constant symbol.
arXiv:2010.05366, 2020.

Sub-Riemannian frame bundles

Nonholonomic frame bundle

- Assume (M, E, g) has constant symbol $\mathfrak{g}$. Define $G_0 = \mathsf{Isom}(\mathfrak{g})$ with Lie algebra $\mathfrak{g}_0 = \mathfrak{isom}(\mathfrak{g})$.
- $\mathfrak{g}_0$ consist of derivations of $\mathfrak{g}$ preserving the stratification and whose restriction to $\mathfrak{g}_1$ is skew symmetric.
- Define a new algebra $\hat{\mathfrak{g}} = \mathfrak{g}_0 \oplus \mathfrak{g}$ such that both $\mathfrak{g}_0$ and $\mathfrak{g}$ as subalgebras and with

$$[D, A] = DA.$$

Exercise

Show that $\hat{\mathfrak{g}}$ is the Lie algebra of isometries of the isometry algebra $\hat{G} = \mathsf{Isom}(G)$.

Nonholonomic frame bundle

- Define a vector bundle of symbols **symb** $\to M$ over M. We will call this the nonholonomic tangent bundle.
- We now define a nonholonomic frame as a Carnot isomorphism $u : \mathfrak{g} \to \mathbf{symb}_x$. We will write a the set of all such frames as $\mathscr{F}_x$.
- We again have a right action of G_0 on $\mathscr{F}_x$ by precomposition. This gives us a principal bundle

$$G_0 \to \mathscr{F} \to M.$$

Cartan connections on the frame bundle

Let ψ be a $(\hat{\mathfrak{g}}, \mathfrak{g}_0)$ Cartan connection with values in $\hat{\mathfrak{g}}$. We can write it as $\psi = (\omega, \theta)$ where ω and θ has values in respectively $\mathfrak{g}_0$ and $\mathfrak{g}$. Observe.

- ω is a principal connection on the bundle $\mathscr{F}$. Corresponds to an affine connection $\tilde{\nabla}$ on **symb** such that parallel transport are Cartan isometries.

Cartan connections on the frame bundle

Let ψ be a $(\hat{\mathfrak{g}}, \mathfrak{g}_0)$ Cartan connection with values in $\hat{\mathfrak{g}}$. We can write it as $\psi = (\omega, \theta)$ where ω and θ has values in respectively $\mathfrak{g}_0$ and $\mathfrak{g}_-$. Observe.

- ω gives $\tilde{\nabla}$ on **symb**.
- θ correspond to a vector bundle isomorphism $I : TM \to \mathbf{symb}$. For any $v \in T_xM$, and $u \in \mathscr{F}_x$,

$$I : v \mapsto h_u v \in \mapsto \theta(h_u v) \in \mathfrak{g} \mapsto u^{-1}\theta(h_u v) \in \mathbf{symb}_x . \quad (1)$$

  We can see this map as a way choosing complements to $E^{k+1} = V^{k+1} \oplus E^k$ by $I^{-1}(E^{k+1}/E^k)$.
- We can define a connection $\nabla = I^{-1}\tilde{\nabla} I$ on TM.

Summary: any choice of Cartan connection gives us an identification $I : TM \to \mathsf{symb}$ and connection ∇ on TM. Properties

- $I(E^k) = \mathsf{symb}_1 \oplus \cdots \oplus \mathsf{symb}_k$. Recall that this gives a decomposition $TM = V_1 \oplus \cdots \oplus V_s$ such that $E^{k+1} = E^k \oplus V^{k+1}$.

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- The connection ∇ is compatible with sub-Riemannian metric (E, g),
- Each vector bundle V^k is parallel,
- Define a tensor $\mathbb{T}$ such that

$$\mathbb{T}(v, w) = -I^{-1}[\![Iv, Iw]\!].$$

Then $\nabla\mathbb{T} = 0$.

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We say that such an I is an E-grading. We will call (E, g, I) a graded sub-Riemannian and say that a connection satisfying the above is strongly compatible with (E, g, I).

Theorem (Partial flatness theorem)
Let (M, E, g, I) is a graded sub-Riemannian manifold. If ∇ is strongly compatible with (E, g, I), and if

$$R = 0, \qquad T = \mathbb{T},$$

then (M, E, g) is locally isometric to a Carnot group.

To get the converse, we need a canonical way to choose I and ∇.

Tohru Morimoto.
Geometric structures on filtered manifolds.
Hokkaido Math. J., 22(3):263–347, 1993.,

Tohru Morimoto.
Cartan connection associated with a subriemannian structure.
Differential Geom. Appl., 26(1):75–78, 2008.

Canonical structures

- We can introduce an inner product on $\hat{\mathfrak{g}}$: We can see $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ as a surjection from the truncated tensor algebra $T^s(\mathfrak{g}_1)$, and induce the inner product from there. We now have that $\mathfrak{g}_0$ consist of skew-symmetric mappings on an inner product space, which have their own induced inner product.

- On the set of <u>linear</u> k-forms on $\mathfrak{g}$ with values in $\hat{\mathfrak{g}}$, we define the Spencer differential or the Lie algebra cohomology differential by $\partial : \wedge^k \mathfrak{g}^* \otimes \hat{\mathfrak{g}} \to \wedge^{k+1} \mathfrak{g}^* \otimes \hat{\mathfrak{g}}$,

$$
\begin{aligned}
&(\partial\alpha)(A_0, \ldots, A_k) \\
&= \sum_{i=0}^{n} (-1)^i [A_i, \alpha(A_0, \ldots, \hat{A}_i, \ldots, A_k)] \\
&\qquad + \sum_{i<j} (-1)^{i+j} \alpha([A_i, A_j], A_0, \ldots, \hat{A}_i, \ldots, \hat{A}_j, \ldots, A_k),
\end{aligned}
$$

- We can write ∂^* for the dual of ∂.

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\end{aligned}$$

- We can write ∂^* for the dual of ∂.

Theorem

There is a unique $(\hat{\mathfrak{g}}, \mathfrak{g}_0)$-Cartan connection $\psi : T\mathscr{F} \to \mathfrak{g}$ with $\kappa : \mathscr{F} \to \wedge^2 \mathfrak{g}^* \otimes \hat{\mathfrak{g}}$ satisfying $\partial^* \kappa = 0$.

How to make the canonical choice of a grading I and connection ∇.

- By I we can get an Carnot algebra structure on each T_xM. From this structure, we can extend the metric g to a Riemannian metric g_I.
- Define a subbundle $\mathfrak{s}$ of $\operatorname{End} TM$ consisting of isometry algebras $\mathfrak{s}_x = \mathfrak{isom}(T_xM)$ on each fiber.

- $\mathfrak{s}_x = \mathfrak{isom}(T_x M)$.
- Finally define $\chi : TM \to \wedge^2 TM$ by
  $\langle \chi(v), w_1 \wedge w_2 \rangle_{g_l} = -\langle v, \mathbb{T}(w_1, w_2) \rangle_{g_l}$.

Theorem

There is a unique grading and strongly compatible connection such that the torsion T and curvature R satisfies for any $D \in \mathfrak{s}$ and any $v \in E^i$, $w \in E^j$ with $0 \leq j < i \leq s$,

$$\langle R(\chi(v)), D \rangle_{g_l} = \langle T(v, \cdot), D \rangle_{g_l} \tag{2}$$

$$\langle T(\chi(v)), w \rangle_{g_i} = -\langle T(v, \cdot), \mathbb{T}(w, \cdot) \rangle_{g_l}. \tag{3}$$

We can ue this to see the difference between sub-Riemannian manifolds.

Contact manifolds

The n-th Heisenberg algebra $\mathfrak{h}_n = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where

$$\mathfrak{g}_2 = \mathrm{span}\{Z\}, \qquad \mathfrak{g}_1 = \mathrm{span}\{X_1, \dots, X_n, Y_1, \dots, Y_n\},$$

Only non-zero brackets

$$[X_j, Y_j] = Z, \qquad j = 1, 2 \dots, n.$$

For any vector $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ such that $1 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, we define

$$\langle X_j, X_j \rangle_{\mathfrak{g}_1} = \langle Y_j, Y_j \rangle_{\mathfrak{g}_1} = \lambda_j^2.$$

We write this Carnot algebra $\mathfrak{h}_n(\lambda)$. All Carnot algebra structures on the Heisenberg groups are of this form.

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$$\langle X_j, X_j \rangle_{\mathfrak{g}_1} = \langle Y_j, Y_j \rangle_{\mathfrak{g}_1} = \lambda_j^2.$$

The isometry algebra $\mathfrak{g}_0 = \mathfrak{isom}(\mathfrak{h}_n(\lambda))$ is given by

$$\mathfrak{g}_0 = \mathrm{span}\{D_{ij} \,:\, i < j, \lambda_i = \lambda_j\} \cup \{Q_{ij} \,:\, i \leq j, \lambda_i = \lambda_j\}.$$

$$\begin{aligned} D_{ij}(X_k) &= \delta_{ki} X_j - \delta_{kj} X_i, & Q_{ij}(X_k) &= \frac{1}{2}\delta_{ki} Y_j + \frac{1}{2}\delta_{kj} Y_i, \\ D_{ij}(Y_k) &= \delta_{ki} Y_j - \delta_{kj} Y_i, & Q_{ij}(Y_k) &= -\frac{1}{2}\delta_{ki} X_j - \frac{1}{2}\delta_{kj} X_i, \\ D_{ij} Z &= 0, & Q_{ij} Z &= 0. \end{aligned}$$

Contact manifolds

Let (M, E, g) be a sub-Riemannian manifold of dimension $2n+1$ assume that E has rank $2n$. We assume that E is a contact distribution, that is $X \wedge Y \mapsto [\![X, Y]\!] = [X, Y] \mod E$ is non-degenerate.

Working locally, we can assume that $E = \ker \theta$ for a one-form θ. We normalize θ by requiring that the maximal imaginary part of the eigenvalues of $d\theta|_E$ is one.

Contact manifolds

We can then write

$$d\theta(v,w) = \langle v, \Lambda^{-1} J w\rangle_g, \qquad v,w \in E, J^2 = -\operatorname{id}_E .$$

where $\Lambda|_x$ is symmetric on E and has eigenvalues $1 = \lambda_{1,x} \leq \cdots \leq \lambda_n$, each appearing twice. The symbol at each point $\mathfrak{h}_n(\lambda_x)$. Hence (M, E, g) only has constant symbol if $\lambda_x = \lambda$ is constant.

Contact manifolds

$$d\theta(v,w) = \langle v, \Lambda^{-1} J w\rangle_g, \qquad v,w \in E, J^2 = -\operatorname{id}_E .$$

$1 = \lambda_1 \leq \cdots \leq \lambda_n$, $\mathsf{symb}_x \cong \mathfrak{h}_n(\lambda)$.

Contact manifolds

$$d\theta(v,w) = \langle v, \Lambda^{-1} J w\rangle_g, \qquad v,w \in E, J^2 = -\operatorname{id}_E.$$

$1 = \lambda_1 \le \cdots \le \lambda_n$, $\operatorname{symb}_x \cong \mathfrak{h}_n(\lambda)$.

- Let $1 = \lambda[1] < \lambda[2] < \cdots < \lambda[k]$ be the eigenvalues without repetition, with corresponding decomposition $E = E[1] \oplus \cdots \oplus E[k]$. Let $\operatorname{pr}[1], \ldots, \operatorname{pr}[k]$ be the corresponding projections.
- Reeb vector field Z:

$$\theta(Z) = 1, \qquad d\theta(Z, \cdot) = 0.$$

Define I such that $V^1 \oplus V^2 = E \oplus \operatorname{span}\{Z\}$ with g_I defined such that Z is a unit vector field.

$$\langle \tau_X Y_1, Y_2 \rangle = \frac{1}{2} \sum_{j=1}^{k} (\mathcal{L}_{X - \mathrm{pr}[j]X} g_I)(\mathrm{pr}[j]Y_1, \mathrm{pr}[j]Y_2),$$

$$\nabla Z = \nabla' Z = 0,$$

$$\nabla_X Y = \sum_{j=1}^{k} \mathrm{pr}[j] \nabla^{g_I}_{\mathrm{pr}[j]X} \mathrm{pr}[j]Y + \sum_{j=1}^{k} \mathrm{pr}[j][X - \mathrm{pr}[j]X, \mathrm{pr}[j]Y] + \tau_X Y,$$

$$\nabla'_X Y = \nabla_X Y + \frac{1}{2}(\nabla_X J)JY,$$

Theorem

A contact manifold with constant symbol is a Carnot group if and only if ∇' has curvature $R' = 0$ and torsion $T' = \mathbb{T}$.

Note that ∇' is not the connection that can be deduced from the previous theorem, but a simplification of that one.

For (2,3,5) manifolds

Flatness theorem for $(2,3,5)$-manifolds

Assume (M, E, g) that E has growth vector $(2,3,5)$. Let X_1, X_2 be any local orthonormal basis of E and define $X_3 = [X_1, X_2]$, $X_4 = [X_1, X_3]$ and $X_5 = [X_2, X_3]$ with $[X_i, X_j] = \sum_{k=1}^{5} c_{ij}^k X_k$. Define vector fields Z, Y_1 and Y_2 by

$$
\begin{aligned}
Z &= X_3 + (c_{23}^3 + c_{24}^4 + c_{25}^5)X_1 - (c_{13}^3 + c_{14}^4 + c_{15}^5)X_2, \\
Y_1 &= X_4 - (c_{14}^4 + c_{15}^5)X_3 \\
&\qquad + (c_{24}^3 - X_2(c_{14}^4 + c_{15}^5) + c_{24}^4(c_{14}^4 + c_{15}^5) + c_{24}^5(c_{24}^4 + c_{25}^5))X_1 \\
&\qquad - (c_{14}^3 - X_1(c_{14}^4 + c_{15}^5) + c_{14}^4(c_{14}^4 + c_{15}^5) + c_{14}^5(c_{24}^4 + c_{25}^5))X_2, \\
Y_2 &= X_5 - (c_{24}^4 + c_{25}^5)X_3 \\
&\qquad + (c_{25}^3 - X_2(c_{25}^4 + c_{25}^5) + c_{25}^4(c_{14}^4 + c_{15}^5) + c_{25}^5(c_{24}^4 + c_{25}^5))X_1 \\
&\qquad - (c_{15}^3 - X_1(c_{25}^4 + c_{25}^5) + c_{15}^4(c_{14}^4 + c_{15}^5) + c_{15}^5(c_{24}^4 + c_{25}^5))X_2,
\end{aligned}
$$

Then $V_1 \oplus V_2 \oplus V_3 = \mathrm{span}\{X_1, X_2\} \oplus \mathrm{span}\{Z\} \oplus \mathrm{span}\{Y_1, Y_2\}$ is independent of basis chosen.

Flatness theorem for $(2, 3, 5)$-manifolds

Let $\bar{g}$ be the Riemannnian metric making X_1, X_2, Z, Y_1, Y_2 into an orthonormal basis with Levi-Civita connection $\nabla^{\bar{g}}$.

Define a connection ∇ making $V_1 \oplus V_2 \oplus V_3$ parallel and further determined by the rules $\nabla Z = 0$ and

$$\langle \nabla_{X_i} X_j, X_k \rangle_{\bar{g}} = \langle \nabla_{X_i} Y_j, Y_k \rangle_{\bar{g}} = \langle \nabla^{\bar{g}}_{X_i} X_j, X_k \rangle_{\bar{g}},$$

$$\langle \nabla_Z X_j, X_k \rangle_{\bar{g}} = \langle \nabla_Z Y_j, Y_k \rangle_{\bar{g}} = \langle [Z, X_j], X_k \rangle_{\bar{g}} + \frac{1}{2} (\mathcal{L}_Z \bar{g})(X_j, X_k),$$

$$\langle \nabla_{Y_i} X_j, X_k \rangle_{\bar{g}} = \langle \nabla_{Y_i} Y_j, Y_k \rangle_{\bar{g}} = \langle [Y_i, X_j], X_k \rangle_{\bar{g}} + \frac{1}{2} (\mathcal{L}_{Y_i} \bar{g})(X_j, X_k).$$

Then (M, E, g) is locally isometric to the Carnot group with growth vector $(2, 3, 5)$ if and only if the curvature R vanishes and the only non-zero parts of the torsion T are given by

$$T(X_2, X_1) = Z, \qquad T(Z, X_j) = Y_j.$$

Concluding remarks

- For most cases, an explicit formula for the canonical connection and grading is still unknown.
- There has been very little consideration of constant curvature models.

Erlend Grong.
Model spaces in sub-Riemannian geometry.
Comm. Anal. Geom. 29 (2021), no. 1.

Eirik Berge and Erlend Grong.
On G2 and Sub-Riemannian Model Spaces of Step and Rank Three.
Math. Z. 298 (2021), no. 3-4, 1853–1885.

Concluding remarks

- For most cases, an explicit formula for the canonical connection and grading is still unknown.
- There has been very little consideration of constant curvature models.
- Are the normalization conditions

$$\langle R(\chi(v)), D\rangle_{g_l} = \langle T(v,\cdot), D\rangle_{g_l}$$
$$\langle T(\chi(v)), w\rangle_{g_i} = -\langle T(v,\cdot), \mathbb{T}(w,\cdot)\rangle_{g_l}.$$

really the best one? (Chitour, G., Jean, Kokkonen, Ann. Inst. Fourier, 2019)

Děkuji mnohokrát
Uvidíme se příště v Srní