Lectures on gerbes, stacks, and strings

42th Winter School on Geometry and Physics, Srni, 2022

Konrad Waldorf

Part I: Gerbes

http://www.konradwaldorf.de/docs/srni-1-slides.pdf
http://www.konradwaldorf.de/docs/srni-1-notes.pdf

A gauge field consists of local gauge potentials $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$, gauge transformations on the overlaps,

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G,$$

and the passage between three open sets is consistent:

$$g_{\beta\gamma} g_{\alpha\beta} = g_{\alpha\gamma}.$$

Recall the condition for gauge transformations:

$$A_\beta = \mathrm{Ad}^{-1}_{g_{\alpha\beta}}(A_\alpha) + g^*_{\alpha\beta}\theta$$

When $G = \mathrm{U}(1)$, we identify $\mathfrak{g} \cong \mathbb{R}$; then, we have

$$g^*\theta = \mathrm{dlog}(g) := \frac{1}{2\pi \mathrm{i}} g^{-1}\mathrm{d}g.$$

The condition for gauge transformations simplifies to

$$A_\beta = A_\alpha + \mathrm{dlog}(g_{\alpha\beta}).$$

Taking exterior derivatives of the gauge potentials, we obtain

$$\mathrm{d}A_\alpha = \mathrm{d}A_\beta$$

and hence a globally defined "field strength"

$$F := \mathrm{d}A_\alpha \in \Omega^2(M).$$

This structure allows a consistent definition of the coupling between the gauge field and a charged particle moving on a trajectory

$$\gamma : [0,1] \to M.$$

Namely, the particle picks up a phase

$$\prod_{i=1}^{n} \exp\left(2\pi\mathrm{i}\int_{t_{i-1}}^{t_i} \gamma^* A_{\alpha_i}\right) g_{\alpha_i\alpha_{i+1}}(\gamma(t_i)) \in \mathrm{U}(1).$$

In differential geometry, we use for gauge fields a cover-independent but equivalent structure: a *principal G-bundle with connection*.

The field strength F is called the curvature.

The phase described above is a way to express the *parallel transport* along a path γ.

In order to replace *paths* by *strings*, we need to go one degree up:

- 2-forms $B_\alpha \in \Omega^2(U_\alpha)$
- 1-forms $A_{\alpha\beta} \in \Omega^1(U_\alpha \cap U_\beta)$ such that

$$B_\beta = B_\alpha + \mathrm{d}A_{\alpha\beta}$$

- gauge transformations

$$g_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to \mathrm{U}(1)$$

  such that

$$A_{\beta\gamma} = A_{\beta\gamma} + A_{\alpha\beta} + \mathrm{dlog}(g_{\alpha\beta\gamma}),$$

  and

$$g_{\beta\gamma\delta} \cdot g_{\alpha\gamma\delta} = g_{\alpha\gamma\delta} \cdot g_{\alpha\beta\gamma}$$

  holds on all 4-fold intersections.

This structure is called a *gauge field for strings*, or *B-field*.

The trajectory of a string be a smooth map

$$\phi : \Sigma \to M$$

where Σ is a closed oriented surface.

We need to triangulate the surface, with faces f, edges e, and vertices v. The phase becomes then:

$$\prod_f \exp\left(2\pi \mathrm{i} \int_f \phi^* B_{\alpha_f}\right) \prod_{e \in \partial f} \exp\left(2\pi \mathrm{i} \int_e \phi^* A_{\alpha_f \alpha_e}\right) \prod_{v \in \partial e} g_{\alpha_f \alpha_e \alpha_v}^{\epsilon(f,e,v)}(v),$$

where $\epsilon(f, e, v)$ is some sign.

A *bundle gerbe with connection* over M is the global differential-geometric object that corresponds to a B-field. Above phase is called the "surface holonomy" of the bundle gerbe.

http://www.konradwaldorf.de/docs/srni-1-slides.pdf
http://www.konradwaldorf.de/docs/srni-1-notes.pdf

We will allow two generalizations:

The first generalization replaces open covers $\{U_\alpha\}$ by surjective submersions.

Any open cover produces a surjective submersion

$$\coprod_{\alpha \in A} U_\alpha \to M.$$

We want to allow arbitrary surjective submersions $\pi : Y \to M$.

Intersections of open sets become replaced by the fibre products:

$$Y^{[k]} := \{(y_1, ..., y_k) \in Y^k \mid \pi(y_1) = ... = \pi(y_k)\}.$$

The second generalization replaces the gauge potentials $A_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ by a principal $\mathrm{U}(1)$-bundle $P_{\alpha\beta}$ with connection.

A *bundle gerbe* $\mathcal{G}$ *with connection* over M consists of:

- a surjective submersion $\pi : Y \to M$
- a 2-form $B \in \Omega^2(Y)$
- a principal $\mathrm{U}(1)$-bundle P over $Y^{[2]}$ with connection ω of curvature

$$\mathrm{curv}(\omega) = \mathrm{pr}_2^* B - \mathrm{pr}_1^* B$$

- a connection-preserving bundle isomorphism μ over $Y^{[3]}$; fibrewise

$$\mu_{y_1,y_2,y_3} : P_{y_2,y_3} \otimes P_{y_1,y_2} \to P_{y_1,y_3}$$

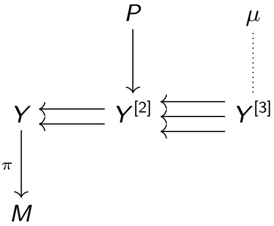
It is required that the diagram

$$\begin{array}{ccc} P_{y_3,y_4} \otimes P_{y_2,y_3} \otimes P_{y_1,y_2} & \xrightarrow{\mathrm{id}\otimes\mu_{y_1,y_2,y_3}} & P_{y_3,y_4} \otimes P_{y_1,y_3} \\ {\scriptstyle \mu_{y_2,y_3,y_4}\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mu_{y_1,y_3,y_4}} \\ P_{y_2,y_4} \otimes P_{y_1,y_2} & \xrightarrow[\mu_{y_1,y_2,y_4}]{} & P_{y_1,y_4} \end{array}$$

is commutative over $Y^{[4]}$.

http://www.konradwaldorf.de/docs/srni-1-slides.pdf
http://www.konradwaldorf.de/docs/srni-1-notes.pdf

Sketch:



Suppose we have open sets U_α allowing sections $s_\alpha : U_\alpha \to Y$. Then,

$$B_\alpha := s_\alpha^* B \in \Omega^2(U_\alpha).$$

On the overlaps, we have section $s_{\alpha\beta} := (s_\alpha, s_\beta) : U_\alpha \cap U_\beta \to Y^{[2]}$ and obtain a principal $\mathrm{U}(1)$-bundle

$$P_{\alpha\beta} := s_{\alpha\beta}^*$$

with a pullback connection $\omega_{\alpha\beta}$. If we choose the open sets well enough, $P_{\alpha\beta}$ will be trivializable, and admit sections $\sigma_{\alpha\beta} : U_\alpha \cap U_\beta \to P_{\alpha\beta}$. Then we get a 1-forms

$$A_{\alpha\beta} := \sigma_{\alpha\beta}^* \omega_{\alpha\beta} \in \Omega^1(U_\alpha \cap U_\beta).$$

On a 3-fold intersection, define a smooth map

$$g_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to \mathrm{U}(1)$$

by

$$\mu_{\alpha\beta\gamma}(\sigma_{\beta\gamma}(x) \otimes \sigma_{\alpha\beta}(x)) = \sigma_{\alpha\gamma}(x) \cdot g_{\alpha\beta\gamma}(x).$$

This yields precisely the data of a B-field; in particular, we can thereby define the "surface holonomy" of the bundle gerbe $\mathcal{G}$.

http://www.konradwaldorf.de/docs/srni-1-slides.pdf
http://www.konradwaldorf.de/docs/srni-1-notes.pdf

We may regard $g_{\alpha\beta\gamma}$ as a cocycle in the 2nd Čech cohomology of the cover $\{U_\alpha\}$ with values in the sheaf of smooth $\mathrm{U}(1)$-valued functions.

Thus, any bundle gerbe determines a class

$$[g] \in \check{\mathrm{H}}^2(M, \underline{\mathrm{U}(1)}) \cong \mathrm{H}^3(M, \mathbb{Z}).$$

This class is called the *Dixmier-Douady class of $\mathcal{G}$*.

We associate to each 2-form $B \in \Omega^2(M)$ a bundle gerbe with connection $\mathcal{I}_B$:

- Its surjective submersion is the identity, $\mathrm{id}_M : M \to M$.
- Its 2-form is B.
- Its principal $\mathrm{U}(1)$-bundle P is the trivial bundle, equipped with the trivial flat connection.
- Its isomorphism μ is the identity morphism between trivial bundles.

The surface holonomy of $\mathcal{I}_B$ around $\phi : \Sigma \to M$ is

$$\int_\Sigma \phi^* B.$$

The Dixmier-Douady class of $\mathcal{I}_B$ is $0 \in \mathrm{H}^3(M, \mathbb{Z})$.

One can define isomorphisms $\mathcal{G} \cong \mathcal{G}'$ between bundle gerbes with connection.

An *isomorphism* $\mathcal{G} \cong \mathcal{G}'$ is:

- a principal $\mathrm{U}(1)$-bundle Q over the common refinement $Z := Y \times_M Y'$ with connection χ of curvature

$$\mathrm{curv}(\chi) = \mathrm{pr}_{Y'}^* B' - \mathrm{pr}_Y^* B$$

- a connection-preserving bundle isomorphism α over $Z \times_M Z$; fibrewise

$$\alpha_{(y_1,y_1'),(y_2,y_2')} : Q_{y_2,y_2'} \otimes P_{y_1,y_2} \to P'_{y_1',y_2'} \otimes Q_{y_1,y_1'}$$

  such that the diagram

$$\begin{array}{ccc}
Q_{y_3,y_3'} \otimes P_{y_2,y_3} \otimes P_{y_1,y_2} & \xrightarrow{\mathrm{id} \otimes \mu_{y_1,y_2,y_3}} & Q_{y_3,y_3'} \otimes P_{y_1,y_3} \\
\Big\downarrow {\scriptstyle \alpha_{(y_2,y_2'),(y_3,y_3')} \otimes \mathrm{id}} & & \\
P'_{y_2',y_3'} \otimes Q_{y_2,y_2'} \otimes P_{y_1,y_2} & & \Big\downarrow {\scriptstyle \alpha_{(y_1,y_1'),(y_3,y_3')}} \\
\Big\downarrow {\scriptstyle \mathrm{id} \otimes \alpha_{(y_1,y_1'),(y_2,y_2')}} & & \\
P'_{y_2',y_3'} \otimes P'_{y_1',y_2'} \otimes Q_{y_1,y_1'} & \xrightarrow[\mu'_{y_1,y_2,y_3} \otimes \mathrm{id}]{} & P'_{y_1',y_3'} \otimes Q_{y_1,y_1'}
\end{array}$$

  is commutative.

If $B' \in \Omega^2(M)$, an isomorphism $\mathcal{G} \cong \mathcal{I}_{B'}$ is called a *trivialization with covariant derivative B'*.

Explicitly, this is the following:

- a principal $\mathrm{U}(1)$-bundle Q over Y with connection χ of curvature
$$\mathrm{curv}(\chi) = \pi^* B' - B$$
- a connection-preserving bundle isomorphism α over $Y^{[2]}$; fibrewise
$$\alpha_{y_1,y_2} : Q_{y_2} \otimes P_{y_1,y_2} \longrightarrow Q_{y_1}$$
such that the diagram
$$\begin{array}{ccc} Q_{y_3} \otimes P_{y_2,y_3} \otimes P_{y_1,y_2} & \xrightarrow{\mathrm{id} \otimes \mu_{y_1,y_2,y_3}} & Q_{y_3} \otimes P_{y_1,y_3} \\ {\scriptstyle \alpha_{y_2,y_3} \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \alpha_{y_1,y_3}} \\ Q_{y_2} \otimes P_{y_1,y_2} & \xrightarrow[\mathrm{id} \otimes \alpha_{y_1,y_2}]{} & Q_{y_1} \end{array}$$
is commutative.

There are bijections

$$\left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \text{bundle gerbes over } X \end{array} \right\} \cong \mathrm{H}^3(X,\mathbb{Z}) : [\mathcal{G}] \mapsto [g_{\alpha\beta\gamma}]$$

$$\left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \text{bundle gerbes with} \\ \text{connection over } X \end{array} \right\} \cong \hat{\mathrm{H}}^3(X) : [\mathcal{G}] \mapsto [B_\alpha, A_{\alpha\beta}, g_{\alpha\beta\gamma}].$$

Here, $\hat{\mathrm{H}}^k(X)$ denotes the differential (or Deligne) cohomology.

There is a tensor product between bundle gerbes, such that above bijections are group isomorphisms. The bundle gerbe $\mathcal{I}_0$ is the tensor unit.

Let G be a compact simple Lie group. A *Weyl alcove* is a simplex $\mathfrak{A} \subseteq \mathfrak{t}^*$ in the dual of the Lie algebra of a maximal torus $\mathfrak{t}$, whose elements correspond 1-1 to the conjugacy classes of G. Thus, there is a map

$$q : G \to \mathfrak{A}$$

sending $g \in G$ to the element that corresponds to the conjugacy class of g. This map is continuous.

Let ν_α be the vertices of $\mathfrak{A}$, and let f_α be the face opposite of ν_α. Define open sets

$$U_\alpha := q^{-1}(\mathfrak{A} \setminus f_\alpha) \subseteq G.$$

Each 2-fold intersection $U_\alpha \cap U_\beta$ can be identified with the coadjoint orbit in $\mathfrak{g}^*$ going through $\nu_{\alpha\beta} := \nu_\beta - \nu_\alpha \in \mathfrak{t}^*$. In good cases this coadjoint orbit is quantizable, for example if $G = \mathrm{SU}(n)$. In these cases we have a prequantum $\mathrm{U}(1)$-bundle $P_{\alpha\beta}$ over $U_\alpha \cap U_\beta$.

The equation

$$\nu_{\beta\gamma} + \nu_{\alpha\beta} = \nu_{\alpha\gamma}$$

induces the isomorphism $\mu_{\alpha\beta\gamma}$.

This defines a bundle gerbe $\mathcal{G}_{basic}$ over G called the *basic gerbe*.

Consider:

- a Lie group homomorphism $\phi : \hat{G} \to G$
- a principal G-bundle P over M.

A *lift of P along ϕ* is a principal $\hat{G}$-bundle $\hat{P}$ over M together with a smooth map $\Phi : \hat{P} \to P$ such that the diagram

$$\begin{array}{ccc} \hat{P} & \xrightarrow{\Phi} & P \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

is commutative, and the condition

$$\Phi(\hat{p}\hat{g}) = \Phi(\hat{p})\phi(\hat{g})$$

holds for all $\hat{p} \in \hat{P}$ and $\hat{g} \in \hat{G}$.

There is a sequence of homomorphisms of topological groups:

$$\mathrm{O}(d) \leftarrow \mathrm{SO}(d) \leftarrow \mathrm{Spin}(d) \leftarrow \mathrm{String}(d) \leftarrow \mathrm{Fivebrane}(d) \leftarrow \dots$$

For large d, these form a *Whitehead tower*:

	π_0	π_1	π_2	π_3	π_4	π_5	π_6	π_7
$\mathrm{O}(d)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\mathrm{SO}(d)$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\mathrm{Spin}(d)$	0	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\mathrm{String}(d)$	0	0	0	0	0	0	0	$\mathbb{Z}$
$\mathrm{Fivebrane}(d)$	0	0	0	0	0	0	0	0

Let $P = \mathrm{O}(M)$ be the orthogonal frame bundle of a Riemannian manifold M.

- A lift of P to $\mathrm{SO}(d)$ is called *orientation*.
- A further lift to $\mathrm{Spin}(d)$ is called *spin structure*.
- A further lift to $\mathrm{String}(d)$ is called *string structure*.

Let P be a principal G-bundle over X and let

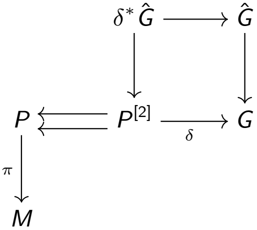
$$1 \to A \to \hat{G} \xrightarrow{\phi} G \to 1$$

be a central extension.

The *lifting gerbe* $\mathcal{L}_P$ is the following bundle gerbe:

- Its surjective submersion is the bundle projection $\pi : P \to X$.
- Its principal $\mathrm{U}(1)$-bundle is the pullback of the principal $\mathrm{U}(1)$-bundle $\hat{G}$ along the map $\delta : P^{[2]} \to G$ defined by $p_1 = p_2\delta(p_1, p_2)$.
- Its bundle gerbe product μ is just multiplication.

Sketch:



http://www.konradwaldorf.de/docs/srni-1-slides.pdf
http://www.konradwaldorf.de/docs/srni-1-notes.pdf

There is a bijection:

$$\left\{ \begin{array}{c} \text{Equivalence classes of} \\ \text{lifts } \Phi : \hat{P} \to P \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \text{trivializations of } \mathcal{L}_P \end{array} \right\}$$

Given $\Phi : \hat{P} \to P$, set $Q := \hat{P}$ and equip this with the A-action induced along $A \to \hat{G}$. Q becomes a principal A-bundle over P with projection map Φ. Define

$$\alpha_{p_1,p_2} : Q_{p_2} \otimes (\delta^* \hat{G})_{p_1,p_2} \longrightarrow Q_{p_1}$$

using the principal action of $\hat{G}$ on $\hat{P}$. This defines a trivialization (Q, α) of $\mathcal{L}_P$.

Conversely, if (Q, α) is a trivialization, set $\hat{P} := Q$ and let $\Phi : \hat{P} \to P$ be the bundle projection. Define a $\hat{G}$-action on $\hat{P}$ in the following way: for $q \in Q$ and $\hat{g} \in \hat{G}$ consider $p_2 := \Phi(q) \in P$ and $g := \phi(\hat{g}) \in G$; this makes $q \in Q_{p_2}$ and $\hat{g} \in \hat{G}_g$. Define $p_1 := p_2 g$. Then,

$$q\hat{g} := \alpha_{p_1,p_2}(q \otimes \hat{g}) \in P_{p_1}$$

defines the action, and it satisfies $\Phi(q\hat{g}) = p_1 = p_2 g = \Phi(q)\phi(\hat{g})$.

The spin group sits in a central extension

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(d) \to \mathrm{SO}(d) \to 1.$$

Let M be an oriented Riemannian manifold of dimension d, and $\mathrm{SO}(M)$ its frame bundle. The lifting bundle gerbe $\mathcal{L}_{\mathrm{Spin}(d)}$ is called the *spin lifting gerbe*.

It is trivializable if and only if M is a spin manifold, and its trivializations are precisely the spin structures on M.

Hence, the Dixmier-Douady class of $\mathcal{L}_{\mathrm{Spin}(d)}$ is the *second Stiefel-Whitney class* $w_2(M)$. In other words, $\mathcal{L}_{\mathrm{Spin}(d)}$ is a geometric representative of that class.

Side remark: it is a result of Grothendieck that a central extension

$$1 \to A \to \hat{G} \to G \to 1$$

is the same thing as a *multiplicative* principal A-bundle P over G.

This means that P is equipped with an associative bundle morphism

$$\phi : \mathrm{pr}_2^* P \otimes \mathrm{pr}_1^* P \to m^* P$$

over $G \times G$, where

$$\mathrm{pr}_1, \mathrm{pr}_2, m : G \times G \to G$$

are the projections and multiplication.

Fibrewise this means

$$\phi_{g_1,g_2} : P_{g_2} \otimes P_{g_1} \to P_{g_1 g_2},$$

and the associativity condition is that the diagram

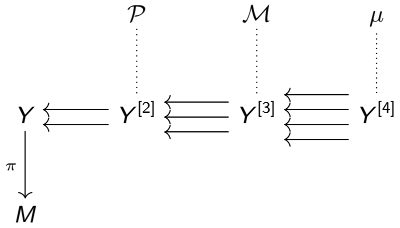
$$\begin{array}{ccc} P_{g_3} \otimes P_{g_2} \otimes P_{g_1} & \xrightarrow{\mathrm{id} \otimes \phi_{g_1,g_2}} & P_{g_3} \otimes P_{g_1 g_2} \\ {\scriptstyle \phi_{g_2,g_3} \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \phi_{g_1 g_2, g_3}} \\ P_{g_2 g_3} \otimes P_{g_1} & \xrightarrow[\phi_{g_1, g_2 g_3}]{} & P_{g_1 g_2 g_3} \end{array}$$

is commutative.

A *bundle 2-gerbe* over M consists of the following structure:

- a surjective submersion $\pi : Y \to M$
- a bundle gerbe $\mathcal{P}$ over $Y^{[2]}$
- an isomorphism
$$\mathcal{M} : \mathrm{pr}_{23}^*\mathcal{P} \otimes \mathrm{pr}_{12}^*\mathcal{P} \to \mathrm{pr}_{13}^*\mathcal{P}$$
of bundle gerbes over $Y^{[3]}$
- a "2-isomorphism" μ over $Y^{[4]}$, satisfying a pentagon condition over $Y^{[4]}$.

Sketch:



http://www.konradwaldorf.de/docs/srni-1-slides.pdf
http://www.konradwaldorf.de/docs/srni-1-notes.pdf

Let P be a principal G-bundle over M. We want to form a bundle 2-gerbe with surjective submersion $P \to M$, and consider the bundle gerbe

$$\mathcal{P} := \delta^* \mathcal{G}_{basic}.$$

The isomorphism $\mathcal{M}$ and the 2-isomorphism μ can be obtained from a *multiplicative structure* on the basic bundle gerbe $\mathcal{G}_{basic}$.

Such a structure consists of an isomorphism

$$\mathrm{pr}_1^* \mathcal{G} \otimes \mathrm{pr}_2^* \mathcal{G} \cong m^* \mathcal{G}$$

of bundle gerbes over $G \times G$, and an appropriate "higher" associativity law. In case of the basic bundle gerbe, it exists canonically.

The bundle 2-gerbe obtained like this is called the *Chern-Simons 2-gerbe* associated to P, and it is denoted by $\mathcal{CS}_P$.

Of special interest will be the Chern-Simons 2-gerbe $\mathcal{CS}_{\mathrm{Spin}(M)}$ associated to the spin structure $\mathrm{Spin}(M)$ of a spin manifold M.

Its Dixmier-Douady class is

$$\frac{1}{2}p_1(M) \in \mathrm{H}^4(M, \mathbb{Z}).$$

We will see in the next lecture that its trivializations can be seen as the *string structures* on M.