Lectures on gerbes, stacks, and strings

42th Winter School on Geometry and Physics, Srni, 2022

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Part II: Stacks

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We start with the modern definition of a presheaf.

A presheaf of sets on a category $\mathcal{C}$ is a functor

$$\mathcal{F} : \mathcal{C}^{op} \longrightarrow \mathcal{S}\mathrm{ets},$$

where $\mathcal{S}\mathrm{ets}$ is the category of sets.

If X is a topological space, let $\mathcal{C} := \mathcal{O}\mathrm{pen}_X$ be the category whose objects are the open sets of X, and whose morphisms are all the inclusions $U \hookrightarrow V$ of open sets. A presheaf on $\mathcal{O}\mathrm{pen}_X$ is what one usually finds in most textbooks.

A Grothendieck topology on a category $\mathcal{C}$ is a subclass $T \subseteq \mathrm{Mor}(\mathcal{C})$ of morphisms that

- contains all isomorphisms
- is closed under composition
- is closed under pullbacks along arbitrary morphisms, i.e., if $\pi : Y \to M$ is in T, and $\phi : N \to M$ is a morphism, then the pullback

$$\begin{array}{ccc} \phi^*Y & \xrightarrow{\Phi} & Y \\ {\scriptstyle \phi^*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow[\phi]{} & M \end{array}$$

  exists and $\phi^*\pi$ is in T.

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A category together with a Grothendieck topology T is called a site. The elements of T are called coverings. Coverings are those morphisms in $\mathcal{C}$ for which we want to discuss *descent*, or *gluing*.

For us, only a single example will be relevant, namely, where $\mathcal{C} = \mathcal{M}\mathrm{an}$ is the category of smooth manifolds, and T consists of all surjective submersions.

Let $\mathcal{C}$ be a site, $\mathcal{F}$ be a presheaf on $\mathcal{C}$, and $\pi : Y \to M$ be a covering. We define the set of gluing data:

$$\mathcal{G}\mathrm{lue}_\pi(\mathcal{F}) := \{f \in \mathcal{F}(Y) \mid \mathrm{pr}_2^* f = \mathrm{pr}_1^* f \text{ in } \mathcal{F}(Y^{[2]})\}$$

Note that the map $\pi^* : \mathcal{F}(M) \to \mathcal{F}(Y)$ lands in the gluing data, since

$$\mathrm{pr}_2^* \pi^* f = (\pi \circ \mathrm{pr}_2)^* f = (\pi \circ \mathrm{pr}_1)^* f = \mathrm{pr}_1^* \pi^* f.$$

Thus, we obtain a map

$$\mathrm{tear}_\pi : \mathcal{F}(M) \to \mathcal{G}\mathrm{lue}_\pi(\mathcal{F})$$

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A presheaf $\mathcal{F}$ on a site $\mathcal{C}$ is called separated presheaf, if for all coverings the map

$$\mathrm{tear}_\pi : \mathcal{F}(X) \to \mathcal{G}\mathrm{lue}_\pi(\mathcal{F})$$

is injective, and it is called sheaf, if it is a bijection.

The presheaves Ω^k of *k-forms* and Ω^k_{cl} of *closed k-forms* are sheaves.

Here is an application of this fact:

- Let $\mathcal{G} = (Y, \pi, B, P, \omega, \mu)$ be a bundle gerbe with connection over M. Recall the identity
$$\mathrm{curv}(\omega) = \mathrm{pr}_2^* B - \mathrm{pr}_1^* B$$
over $Y^{[2]}$. We obtain an equality
$$\mathrm{pr}_2^* \mathrm{d}B = \mathrm{pr}_1^* \mathrm{d}B$$
in $\Omega^3_{cl}(Y^{[2]})$; thus, $\mathrm{d}B \in \mathfrak{Glue}_\pi(\Omega^3_{cl})$.
- Hence, there exists a unique closed 3-form $H \in \Omega^3_{cl}(X)$ such that
$$\mathrm{tear}_\pi(H) = \pi^* H = \mathrm{d}B.$$
The 3-form H is called the curvature of $\mathcal{G}$.

If $\mathcal{F}$ is a presheaf, then we define another presheaf $\mathcal{F}^+$ by

$$\mathcal{F}^+(M) := \{(\pi, f) \mid \pi : Y \to M \text{ is a covering, } f \in \mathcal{G}\mathrm{lue}_\pi(\mathcal{F})\}/\sim ,$$

where

$$(\pi, f) \sim (\pi', f')$$

whenever f and f' coincide in $\mathcal{F}(Y \times_M Y')$.

For a map $\phi : N \to M$, we set

$$\phi^* : \mathcal{F}^+(M) \to \mathcal{F}^+(N) : [\pi, f] \mapsto [\phi^*\pi, \Phi^* f],$$

where Φ is the covering map in the pullback diagram

$$\begin{array}{ccc} \phi^* Y & \xrightarrow{\Phi} & Y \\ {\scriptstyle \phi^*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow[\phi]{} & M \end{array}$$

The passage $\mathcal{F} \mapsto \mathcal{F}^+$ is called Grothendieck's plus construction.

Theorem

If $\mathcal{F}$ is separated, then $\mathcal{F}^+$ is a sheaf.

Since $\mathcal{F}$ is separated, it remains to prove that the map

$$\mathrm{tear}_\xi : \mathcal{F}^+(M) \to \mathcal{G}\mathrm{lue}_\xi(\mathcal{F}^+)$$

is surjective for all coverings $\xi : Z \to M$.

We have, by definition,

$$\mathcal{G}\mathrm{lue}_\xi(\mathcal{F}^+) = \{[\pi, f] \in \mathcal{F}^+(Z) \mid \mathrm{pr}_2^*[\pi, f] = \mathrm{pr}_1^*[\pi, f] \text{ in } \mathcal{F}^+(Z^{[2]})\}.$$

Here, $\pi : Y \to Z$ and $f \in \mathcal{F}(Y)$. An exercise in computing fibre products reveals that the condition on $[\pi, f]$ is equivalent to the condition $\mathrm{pr}_2^* f = \mathrm{pr}_1^* f$ in $\mathcal{F}(Y \times_M Y)$. In other words,

$$[\xi \circ \pi, f] \in \mathcal{F}^+(M).$$

One can then check that $\xi^*[\xi \circ \pi, f] = [\pi, f]$; hence tear_ξ is surjective, QED.

A presheaf of categories on a category $\mathcal{C}$ is a 2-functor

$$\mathcal{F} : \mathcal{C}^{op} \to \mathcal{C}\mathrm{at},$$

where $\mathcal{C}\mathrm{at}$ is the 2-category of categories, functors, and natural transformations.

Thus, a presheaf of categories $\mathcal{F}$ assigns:

- to each smooth manifold M a category $\mathcal{F}(M)$,
- to each smooth map $\phi : N \to M$ a functor $\phi^* : \mathcal{F}(M) \to \mathcal{F}(N)$,
- and to each pair (ϕ, ψ) of composable smooth maps $\phi : N \to M$ and $\psi : M \to L$, a natural equivalence

  $$c_{\phi,\psi} : (\psi \circ \phi)^* \Rightarrow \phi^* \circ \psi^*,$$

  and these are required to satisfy a coherence axiom w.r.t. triples of composable smooth maps.

The category $\mathcal{G}\mathrm{lue}_\pi(\mathcal{F})$ of gluing data for $\mathcal{F}$ is defined as follows:

- The objects are pairs (P, ϕ) consisting of an object $P \in \mathcal{F}(Y)$ and of an isomorphism
$$\phi : \mathrm{pr}_2^* P \to \mathrm{pr}_1^* P$$
in $\mathcal{F}(Y^{[2]})$, which satisfying the cocycle condition
$$\mathrm{pr}_3^* P \xrightarrow{\mathrm{pr}_{23}^*\phi} \mathrm{pr}_2^* P \xrightarrow{\mathrm{pr}_{12}^*\phi} \mathrm{pr}_3^* P, \qquad \mathrm{pr}_3^* P \xrightarrow{\mathrm{pr}_{13}^*\phi} \mathrm{pr}_3^* P$$

- The morphisms $(P, \phi) \to (P', \phi')$ are morphisms $\psi : P \to P'$ in $\mathcal{F}(Y)$ which are compatible with ϕ and ϕ':
$$\begin{array}{ccc} \mathrm{pr}_2^* P & \xrightarrow{\phi} & \mathrm{pr}_1^* P \\ {\scriptstyle \mathrm{pr}_2^*\psi}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1^*\psi} \\ \mathrm{pr}_2^* P' & \xrightarrow[\phi']{} & \mathrm{pr}_1^* P \end{array}$$

Again, we find for each covering $\pi : Y \to M$ a functor

$$\mathrm{tear}_\pi : \mathcal{F}(M) \to \mathcal{G}\mathrm{lue}_\pi(\mathcal{F}) : P \mapsto (\pi^* P, \phi_{P,\pi})$$

where $\phi_{P,\pi}$ is the canonical morphism

$$\mathrm{pr}_2^* \pi^* P \xrightarrow{(c^{-1}_{\mathrm{pr}_2,\pi})_P} (\pi \circ \mathrm{pr}_2)^* P = (\pi \circ \mathrm{pr}_1)^* P \xrightarrow{(c_{\mathrm{pr}_1,\pi})_P} \mathrm{pr}_1^* \pi^* P,$$

A presheaf of categories is called prestack, if for all coverings $\pi : Y \to M$ the functor

$$\mathrm{tear}_\pi : \mathcal{F}(M) \to \mathcal{G}\mathrm{lue}_\pi(\mathcal{F})$$

is full and faithful, and it is called stack or sheaf of categories, if this functor is an equivalence of categories.

The most familiar stack probably is the stack $\mathcal{B}\mathrm{un}_G$ of principal G-bundles.

It is easy to see that $\mathcal{B}\mathrm{un}_G$ is a presheaf of categories. In a minute, we give a proof that it is a stack.

In order to get another example, we fix the following notation. If H is any group, we denote by $\mathcal{B}H$ the groupoid with a single object that has automorphism group H. Composition is multiplication.

The notation expresses the fact that the geometric realization of the groupoid $\mathcal{B}H$ is the classifying space BH.

Let G be a Lie group. Let $\underline{\mathcal{B}G}$ be the presheaf on $\mathcal{M}\mathrm{an}$ with

$$\underline{\mathcal{B}G}(M) := \mathcal{B}C^{\infty}(M, G).$$

We are going to prove that this is a prestack, but not a stack.

To see this, we analyze $\mathcal{G}\mathrm{lue}_\pi(\underline{\mathcal{B}G})$ for a covering $\pi : Y \to M$:

- objects are smooth maps $f : Y^{[2]} \to G$ satisfying the cocycle condition
$$f(y_2, y_3)f(y_1, y_2) = f(y_1, y_2).$$
- morphisms $f \to f'$ are smooth maps $h : Y \to G$ such that
$$h(y_2)f(y_1, y_2) = f'(y_1, y_2)h(y_1).$$

This category is equivalent to the category of principal G-bundles that trivialize when pulled back along $\pi : Y \to M$.

The functor
$$\mathrm{tear}_\pi : \underline{\mathcal{B}G}(M) \to \mathcal{G}\mathrm{lue}_\pi(\underline{\mathcal{B}G})$$
is the inclusion of the trivial bundle in this category. We see that it is full and faithful, but not essentially surjective.

This proves that $\underline{\mathcal{B}G}$ is a prestack, but not a stack.

The plus construction for presheaves of categories works as before: the objects of $\mathcal{F}^+(M)$ are pairs of

- a covering $\pi : Y \to M$
- an object (P, ϕ) in $\mathcal{G}\mathrm{lue}_\pi(\mathcal{F})$.

Theorem

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Theorem

If $\mathcal{F}$ is a prestack, then $\mathcal{F}^+$ is a stack.

Applying the plus construction to the prestack $\mathcal{B}\underline{G}$, it collects principal G-bundles trivializing over a given covering π, and then collects all possible coverings. This gives us *all* principal G-bundles, and we obtain

$$\mathcal{B}\underline{G}^+ \cong \mathcal{B}\mathrm{un}_G .$$

In particular, we have just proved that the presheaf $\mathcal{B}\mathrm{un}_G$ of principal G-bundles is a stack on the site of smooth manifolds.

Here is an example how to use this.

Suppose M is a smooth manifold on which a Lie group H acts smoothly, freely, and properly. This ensures that the projection $M \to M/H$ is a surjective submersion, and that

$$M \times_{M/H} M \cong M \times H.$$

This leads to the result that

$$\mathcal{G}\mathrm{lue}_{\pi}(\mathcal{B}\mathrm{un}_G) \cong \mathcal{B}\mathrm{un}_G(M)^H,$$

the category of H-equivariant principal G-bundles.

Thus, we not only automatically get the correct definition of H-equivariant G-bundles, we also get for free the theorem that

$$\mathcal{B}\mathrm{un}_G(M/H) \cong \mathcal{B}\mathrm{un}_G(M)^H.$$

Next we look at presheaves of bicategories and 2-stacks.

Bundle gerbes with connection over a smooth manifold M form a bicategory $\mathcal{G}\mathrm{rb}^{\nabla}(M)$.

Its objects are bundle gerbes with connections, its morphisms are the isomorphisms (Q, α) discussed in the first lecture, and its 2-morphisms are isomorphisms between those: connection-preserving bundle isomorphisms

$$Q \to Q'$$

over $Y \times_M Y'$ that are compatible with the isomorphisms α and α'.

Further, one can easily define the pullback of a bundle gerbe along a smooth map $\phi : N \to M$, and then complete this to a presheaf $\mathcal{G}\mathrm{rb}^{\nabla}$ of bicategories.

Similarly, bundle gerbes without connections form a presheaf $\mathcal{G}\mathrm{rb}$ of bicategories.

If $\mathcal{C}$ is a monoidal category, we denote by $\mathcal{BC}$ the bicategory with a single object and with category of endomorphisms $\mathcal{C}$. Composition is the monoidal structure.

Consider then the presheaf of bicategories $\mathcal{B}\mathcal{B}\mathrm{un}_{\mathrm{U}(1)}$ defined by

$$(\mathcal{B}\mathcal{B}\mathrm{un}_{\mathrm{U}(1)})(M) := \mathcal{B}(\mathcal{B}\mathrm{un}_{\mathrm{U}(1)}(M)).$$

The fact that $\mathcal{B}\mathrm{un}_{\mathrm{U}(1)}$ is a stack is equivalent to the statement that $\mathcal{B}\mathcal{B}\mathrm{un}_{\mathrm{U}(1)}$ is a pre-2-stack. We claim that

$$\mathcal{G}\mathrm{rb} = (\mathcal{B}\mathcal{B}\mathrm{un}_{\mathrm{U}(1)})^{+}.$$

Theorem

The presheaf $\mathcal{G}\mathrm{rb}$ of bundle gerbes is a 2-stack on the site of smooth manifolds.

Let us try to understand the claim

$$\mathcal{G}\mathrm{rb} = (\mathcal{B}\mathcal{B}\mathrm{un}_{\mathrm{U}(1)})^{+}.$$

The plus construction tells us to consider pairs of a covering $\pi : Y \to M$ and of an object in the bicategory $\mathcal{G}\mathrm{lue}_{\pi}(\mathcal{B}\mathcal{B}\mathrm{un}_{\mathrm{U}(1)})$.

This is:

- an object of $\mathcal{B}\mathcal{B}\mathrm{un}_{\mathrm{U}(1)}(Y)$: no information
- a 1-morphism in $\mathcal{B}\mathcal{B}\mathrm{un}_{\mathrm{U}(1)}(Y^{[2]})$: a principal $\mathrm{U}(1)$-bundle P over $Y^{[2]}$
- a 2-morphism in $\mathcal{B}\mathcal{B}\mathrm{un}_{\mathrm{U}(1)}(Y^{[3]})$: a bundle isomorphism μ over $Y^{[3]}$
- a condition in $\mathcal{B}\mathcal{B}\mathrm{un}_{\mathrm{U}(1)}(Y^{[4]})$: a commutative diagram of isomorphisms over $Y^{[4]}$.

For bundle gerbes with connection, one needs a slightly more elaborate pre-2-stack.

Consider the presheaf $\mathcal{T}$ for which $\mathcal{T}(M)$ is the bicategory with:

- Objects: 2-forms $B \in \Omega^2(M)$
- 1-morphisms $B_1 \to B_2$: principal $\mathrm{U}(1)$-bundles P over M with connection ω such that
$$\mathrm{curv}(\omega) = B_1 - B_2.$$
- 2-morphisms: connection-preserving bundle isomorphisms.

The new claim is now
$$\mathcal{G}\mathrm{rb}^\nabla = \mathcal{T}^+,$$
and it can be understood and proved as before.

This proves that the presheaf $\mathcal{G}\mathrm{rb}^\nabla$ is a 2-stack, too.

Every compact simple Lie group G has a simply-connected universal covering group, $G = \tilde{G}/Z$, where $Z \subseteq Z(\tilde{G})$ is a (finite) subgroup of the center. The quotient map

$$\pi : \tilde{G} \to G$$

is a surjective submersion.

Wess-Zumino-Witten models on the group G require a gauge field for strings, and the idea is to let the basic bundle gerbe $\mathcal{G}_{basic}$ descend to G.

In order to do so, one needs to promote $\mathcal{G}_{basic}$ to an object in $\mathcal{G}\mathrm{lue}_\pi(\mathcal{G}\mathrm{rb}^\nabla)$, and this turns out to be the same as equipping it with a Z-equivariant structure.

Such structures can be constructed explicitly and have been classified by *group cohomology* of the group Z.

For instance, when $G = \mathrm{PSO}(4n)$, then $\tilde{G} = \mathrm{Spin}(4n)$ and $Z = \mathbb{Z}_2 \times \mathbb{Z}_2$. In this case, there exist *two* different Z-equivariant structures on $\mathcal{G}_{basic}$, corresponding to two different bundle gerbes with connection on G. In turn, these correspond to *two different* Wess-Zumino-Witten models on $\mathrm{PSO}(4n)$ at each level.

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We are now in position to create other versions of bundle gerbes.

- Consider the stack $\mathcal{B}\mathrm{un}_A$ for any abelian Lie group A, and define

$$\mathcal{G}\mathrm{rb}_A := (\mathcal{B}\mathcal{B}\mathrm{un}_A)^+.$$

  For instance, when $A = \mathbb{Z}_2$, the spin lifting gerbe is a $\mathbb{Z}_2$-bundle gerbe.

- Let $\mathcal{L}\mathcal{B}\mathrm{un}_{\mathbb{C}}$ be the stack of complex line bundles. Then,

$$\mathcal{L}\mathcal{G}\mathrm{rb}_{\mathbb{C}} := (\mathcal{B}\mathcal{L}\mathcal{B}\mathrm{un}_{\mathbb{C}})^+$$

  gives the definition of a "line bundle gerbe".

  The associated bundle construction establishes a stack isomorphism

$$\mathcal{B}\mathrm{un}_{\mathbb{C}^\times} \cong \mathcal{L}\mathcal{B}\mathrm{un}_{\mathbb{C}}.$$

  Under the plus construction, it induces 2-stack isomorphism

$$\mathcal{G}\mathrm{rb}_{\mathbb{C}^\times} \cong \mathcal{L}\mathcal{G}\mathrm{rb}_{\mathbb{C}}.$$

Let $\mathcal{A}\mathrm{lg}_{\mathbb{C}}$ denote the bicategory whose objects are (unital, associative, complex) algebras, whose 1-morphisms $A \to B$ are B-A-bimodules, and whose 2-morphisms are bi-intertwiners.

Let $\mathcal{A}\mathrm{lg}\mathcal{B}\mathrm{un}_{\mathbb{C}}$ denote the corresponding bundle version of this bicategory. Then,

$$\text{2-}\mathcal{V}\mathrm{ect}\mathcal{B}\mathrm{un}_{\mathbb{C}} := (\mathcal{A}\mathrm{lg}\mathcal{B}\mathrm{un}_{\mathbb{C}})^{+}$$

yields the 2-stack of 2-vector bundles.

There is an inclusion

$$\mathcal{B}(\mathcal{L}\mathcal{B}\mathrm{un}_{\mathbb{C}}) \to \mathcal{A}\mathrm{lg}\mathcal{B}\mathrm{un}_{\mathbb{C}}$$

that sends (over a smooth manifold M) the single object to the trivial algebra bundle $M \times \mathbb{C}$.

This inclusion induces under the plus construction a morphism

$$\mathcal{L}\mathcal{G}\mathrm{rb}_{\mathbb{C}} \to \text{2-}\mathcal{V}\mathrm{ect}\mathcal{B}\mathrm{un}_{\mathbb{C}}$$

of 2-stacks. It thus embeds the theory of bundle gerbes into the much richer theory of 2-vector bundles.

In the context of higher gauge theory it turns out that the correct generalization of "abelian group" is not "group" but "2-group".

A (strict) Lie 2-group is a groupoid Γ in the category of Lie groups: it has a Lie group Γ_0 of objects and a Lie group Γ_1 of morphisms, and all structure maps

$$s, t : \Gamma_1 \to \Gamma_0 \quad , \quad \mathrm{id} : \Gamma_0 \to \Gamma_1 \quad , \quad \Gamma_1 \, {}_s\!\times_t \Gamma_1 \to \Gamma_1$$

are Lie group homomorphisms.

In a moment we explain that there is an isomorphism of categories

$$\left\{ \begin{array}{c} \text{Strict Lie} \\ \text{2-groups} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Crossed modules} \\ \text{of Lie groups} \end{array} \right\}$$

A crossed module of Lie groups consists of:

- two Lie groups G and H
- a Lie group homomorphism $t : H \to G$
- a smooth action $\alpha : G \times H \to H$ of G on H by Lie group homomorphisms,

such that

$$t(\alpha(g,h)) = gt(h)g^{-1} \quad \text{and} \quad \alpha(t(h),x) = hxh^{-1}$$

hold for all $g \in G$ and $h, x \in H$.

If Γ is a Lie 2-group, we obtain a crossed module by

$$G := \Gamma_0 \quad , \quad H := \ker(s) \subseteq \Gamma_1 \quad , \quad \alpha(g,\gamma) := \mathrm{id}_g \cdot \gamma \cdot \mathrm{id}_{g^{-1}}.$$

Conversely, if $t : H \to G$ is a crossed module, then we obtain a Lie 2-group by setting

$$\Gamma_0 := G \quad , \quad \Gamma_1 := H \ltimes_\alpha G.$$

- If A is an abelian Lie group, then $\mathcal{B}A$ is a Lie 2-group. The corresponding crossed module is
$$A \longrightarrow \{1\}$$

- If X is a set, we denote by X_{dis} the category whose set of objects is X, and which only has identity morphisms.

  If G is a Lie group, then G_{dis} is a Lie 2-group. The corresponding crossed module is
$$G \stackrel{\mathrm{id}}{\to} G$$
  with the conjugation action.

- If H is a Lie group, then there is a Lie 2-group $\mathcal{A}\mathrm{ut}(H)$ whose crossed module is
$$H \stackrel{t}{\to} \mathrm{Aut}(H),$$
  where $t(h)(x) := hxh^{-1}$, and $\alpha(\varphi, h) := \varphi(h)$.

Let $\Gamma = (H \xrightarrow{t} G)$ be a Lie 2-group.

A principal Γ-bundle over M is:

- a principal H-bundle P over M
- a smooth map $f : P \longrightarrow G$ that is G-anti-equivariant, i.e.:

$$f(ph) = t(h)^{-1} f(p)$$

The map f is also called the anchor of P.

Let (P, f) and (Q, g) be the principal Γ-bundles over M. The tensor product is:

$$P \otimes Q := \big(P \times_M Q\big) / \sim \qquad \text{where} \qquad (p \cdot h, q) \sim (p, q \cdot \alpha(f(p)^{-1}, h)).$$

Principal Γ-bundles form a stack $\mathcal{B}\mathrm{un}_\Gamma$ of monoidal categories.

The 2-stack of Γ-bundle gerbes is defined by

$$\mathcal{G}\mathrm{rb}_\Gamma := (\mathcal{B}\mathcal{B}\mathrm{un}_\Gamma)^+.$$

In our first two examples, we obtain the following:

- For $\Gamma = \mathcal{B}A$, we have $\mathcal{B}\mathrm{un}_{\mathcal{B}A} = \mathcal{B}\mathrm{un}_A$ and hence
$$\mathcal{G}\mathrm{rb}_{\mathcal{B}A} = \mathcal{G}\mathrm{rb}_A.$$
- For $\Gamma = G_{dis}$, we have an isomorphism of monoidal categories
$$\mathcal{B}\mathrm{un}_{G_{dis}}(X) \cong C^\infty(X, G)_{dis}$$
and thus obtain an equivalence
$$\mathcal{B}\mathcal{B}\mathrm{un}_{G_{dis}} \cong \underline{\mathcal{B}G}_{dis}.$$
From there we obtain
$$\mathcal{G}\mathrm{rb}_{G_{dis}} = (\mathcal{B}\mathcal{B}\mathrm{un}_{G_{dis}})^+ \cong (\underline{\mathcal{B}G}_{dis})^+ = (\underline{\mathcal{B}G}^+)_{dis} \cong (\mathcal{B}\mathrm{un}_G)_{dis}.$$

For $\Gamma = \mathcal{A}\mathrm{ut}(H)$, a principal Γ-bundle P is the same as a principal H-bibundle.

The additional left H-action on P is defined by

$$h \cdot p := p \cdot f(p)(h),$$

where $f : P \rightarrow (H)$ is the anchor of P.

The first non-abelian gerbes have been discussed for $\Gamma = \mathcal{A}\mathrm{ut}(H)$ in the setting of bibundles, in work of Breen-Messing and Aschieri-Cantini-Jurco.

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The 2-group $\mathcal{A}\mathrm{ut}(\mathrm{U}(1))$ has

$$\mathcal{A}\mathrm{ut}(\mathrm{U}(1))_0 = \mathrm{Aut}(\mathrm{U}(1)) = \mathbb{Z}_2 \quad \text{and} \quad \mathcal{A}\mathrm{ut}(\mathrm{U}(1))_1 = \mathrm{U}(1).$$

The corresponding bundle gerbes look like ordinary $\mathrm{U}(1)$-bundle gerbes with an additional $\mathbb{Z}_2$-anti-equivariance.

They have found an application in so-called *orientifold sigma models*, where a $\mathbb{Z}_2$-action flips the orientation of the worldsheet.

The *T-duality 2-group* $\mathcal{TD}_n$ is given by a crossed module

$$\mathrm{U}(1) \times \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{R}^n \times \mathbb{R}^n : (z, m, \hat{m}) \mapsto (m, \hat{m})$$

in which $\mathbb{R}^n \times \mathbb{R}^n$ acts on $\mathrm{U}(1) \times \mathbb{Z}^n \times \mathbb{Z}^n$ via

$$(a, \hat{a}) \cdot (z, m, \hat{m}) := (z \cdot e^{2\pi \mathrm{i} \hat{a} m}, m, \hat{m}).$$

One can show that there is an equivalence of bicategories

$$\mathcal{G}\mathrm{rb}_{\mathcal{TD}_n} \cong \left\{ \begin{array}{c} \text{Topological T-duality} \\ \text{correspondences for} \\ \mathbb{T}^n\text{-bundles} \end{array} \right\}$$

For a Lie 2-group $\Gamma = (H \xrightarrow{t} G)$, the groups

$$\pi_0\Gamma := G/t(H) \quad \text{and} \quad \pi_1\Gamma := \ker(t)$$

are called the homotopy groups of Γ.

- $\pi_1\Gamma$ is abelian.
- There is an action of $\pi_0\Gamma$ on $\pi_1\Gamma$.

We obtain a sequence of Lie 2-group homomorphisms

$$1 \longrightarrow \mathcal{B}\pi_1\Gamma \longrightarrow \Gamma \longrightarrow (\pi_0\Gamma)_{dis} \longrightarrow 1.$$

This is in fact an extension of Lie 2-groups in a certain homotopy-theoretical sense.

Such an extension is called central, if the action of $\pi_0\Gamma$ on $\pi_1\Gamma$ is trivial.

Any Lie 2-group homomorphism $\Gamma \to \Omega$ induces "extension" functors

$$\mathcal{B}un_\Gamma \to \mathcal{B}un_\Omega \quad \text{and} \quad \mathcal{G}rb_\Gamma \to \mathcal{G}rb_\Omega.$$

We shall describe these functors in case of the homomorphism $\Gamma \to \pi_0\Gamma_{dis}$.

The extension functor

$$\mathcal{B}un_\Gamma(M) \to \mathcal{B}un_{\pi_0\Gamma_{dis}}(M) = C^\infty(M, \pi_0\Gamma)_{dis}$$

is given as follows.

If P is a principal Γ-bundle over M with anchor map $f : P \to \Gamma_0$, then the map

$$M \to \pi_0\Gamma \quad \text{with} \quad \pi(p) \mapsto [f(p)]$$

is well-defined and smooth. We call it the base map of P and denote it by $\pi_0 P : M \to \pi_0\Gamma$.

Note that isomorphic principal Γ-bundles have the same base map, and that the base map of a tensor product gives the product of base maps.

If $\mathcal{P}$ is a Γ-bundle gerbe, the extension functor

$$\mathcal{G}\mathrm{rb}_{\Gamma}(M) \to \mathcal{G}\mathrm{rb}_{\pi_0\Gamma_{dis}}(M) \cong \mathcal{B}\mathrm{un}_{\pi_0\Gamma}(M)$$

is the following.

If $\pi : Y \to M$ is the surjective submersion of $\mathcal{P}$, then let $\pi_0 P : Y^{[2]} \to \pi_0\Gamma$ be the base map of its principal Γ-bundle P.

The bundle isomorphism

$$\mathrm{pr}_{23}^* P \otimes \mathrm{pr}_{12}^* P \cong \mathrm{pr}_{13}^* P$$

over $Y^{[3]}$ implies the cocycle condition for $\pi_0 P$, and hence, gluing produces a principal $\pi_0\Gamma$-bundle over M.

We denote it by $\pi_0\mathcal{P}$ and call it the base bundle of $\mathcal{P}$.

One can now pose the following lifting problem:

Given a principal $\pi_0\Gamma$-bundle P over M, does there exist a Γ-bundle gerbe $\mathcal{P}$ with $\pi_0\mathcal{P} \cong P$?

In other words, can the structure group of P be lifted along a central extension

$$1 \to \mathcal{B}\pi_1\Gamma \to \Gamma \to (\pi_0\Gamma)_{dis} \to 1.$$

Note that this is a generalization of the lifting problem considered in the first lecture, replacing a Lie group homomorphisms $\hat{G} \to \pi_0\Gamma$ by a 2-group homomorphisms $\Gamma \to \pi_0\Gamma$.

In the next lecture we will study a 2-group model $\Gamma = \mathrm{String}(d)$ for the string group, which is a central extension

$$1 \to \mathcal{B}\mathrm{U}(1) \to \mathrm{String}(d) \to \mathrm{Spin}(d)_{dis} \to 1.$$

Suppose M is a spin manifold, and let $P := \mathrm{Spin}(M)$ be its spin structure, which is a principal $\mathrm{Spin}(d)$-bundle.

A string structure on M is a lift of the structure group of $\mathrm{Spin}(M)$ to $\mathrm{String}(d)$.

Our last goal for today is to describe generalized lifting problems in terms of a *lifting gerbe*, analogous to the lifting gerbe from the first lecture.

Associated to any *central* Lie 2-group extension

$$1 \to \mathcal{B}\pi_1\Gamma \to \Gamma \to \pi_0\Gamma_{dis} \to 1$$

is a multiplicative $\pi_1\Gamma$-bundle gerbe over $\pi_0\Gamma$, which we denote by $\mathcal{G}_\Gamma$.

Its surjective submersion is the projection $\pi : G \to \pi_0\Gamma$. The double fibre product $G \times_{\pi_0\Gamma} G$ comes equipped with a central extension

$$1 \to \pi_1\Gamma \to H \ltimes_\alpha G \to G \times_{\pi_0\Gamma} G \to 1,$$

which is, in particular, a principal $\pi_1\Gamma$-bundle over $G \times_{\pi_0\Gamma} G$.

The bundle gerbe product μ can be provided in a straightforward way, and the multiplicative structure can be induced from the multiplicative structure we get from above sequence.

This construction defines a functor

$$\left\{ \begin{array}{c} \text{Central extensions} \\ \mathcal{B}\pi_1\Gamma \to \Gamma \to \pi_0\Gamma_{dis} \end{array} \right\} \to \left\{ \begin{array}{c} \text{Multiplicative } \pi_1\Gamma\text{-bundle} \\ \text{gerbes over } \pi_0\Gamma \end{array} \right\}$$

Theorem

Let $\mathcal{CS}(P,\mathcal{G}_\Gamma)$ be the Chern-Simons 2-gerbe associated to the bundle P and the multiplicative bundle gerbe $\mathcal{G}_\Gamma$. Then, there is an equivalence of bicategories

$$\left\{ \begin{matrix} \text{Lifts of } P \text{ to a} \\ \Gamma\text{-bundle gerbe } \mathcal{P} \end{matrix} \right\} \cong \left\{ \begin{matrix} \text{Trivializations of} \\ \mathcal{CS}(P,\mathcal{G}_\Gamma) \end{matrix} \right\}$$

http://www.konradwaldorf.de/docs/srni-2-slides.pdf
http://www.konradwaldorf.de/docs/srni-2-notes.pdf

Theorem

Let $\mathcal{CS}(P, \mathcal{G}_\Gamma)$ be the Chern-Simons 2-gerbe associated to the bundle P and the multiplicative bundle gerbe $\mathcal{G}_\Gamma$. Then, there is an equivalence of bicategories

$$\left\{ \begin{array}{c} \textit{Lifts of } P \textit{ to a} \\ \Gamma\textit{-bundle gerbe } \mathcal{P} \end{array} \right\} \cong \left\{ \begin{array}{c} \textit{Trivializations of} \\ \mathcal{CS}(P, \mathcal{G}_\Gamma) \end{array} \right\}$$

Sketch of a proof. Given a lift $\mathcal{P}$, consider its pullback $\pi^*\mathcal{P}$ along the projection $\pi : P \to M$. Then, the base bundle of $\pi^*\mathcal{P}$ is π^*P, and hence trivializable. A Γ-bundle gerbe with trivializable base bundle reduces to an abelian $\pi\Gamma_1$-bundle gerbe $\mathcal{Q}$ over P. This is the first ingredient of a trivialization of $\mathcal{CS}(P, \mathcal{G}_\Gamma)$.

Conversely, given a trivialization of $\mathcal{CS}(P, \mathcal{G}_\Gamma)$. Its first ingredient is an abelian $\pi_1\Gamma$-bundle gerbe $\mathcal{Q}$ over P. Extending along $i : B\pi_1\Gamma \to \Gamma$, we may regard it is a Γ-bundle gerbe. The remaining parts of the trivialization complete $i(\mathcal{Q})$ to an object in $\mathcal{G}\mathrm{lue}_\pi(\mathcal{G}\mathrm{rb}_\Gamma)$, it therefore descends and yields an appropriate Γ-bundle gerbe $\mathcal{P}$. □

The bundle gerbe $\mathcal{G}_{\mathrm{String}(d)}$ associated to the central 2-group extension

$$1 \to \mathcal{B}\mathrm{U}(1) \to \mathrm{String}(d) \to \mathrm{Spin}(d)_{dis} \to 1.$$

is the basic bundle gerbe over $\mathrm{Spin}(d)$,

$$\mathcal{G}_{\mathrm{String}(d)} \cong \mathcal{G}_{basic}.$$

This shows that the lifting Chern-Simons 2-gerbe is precisely the Chern-Simons 2-gerbe $\mathcal{CS}_{\mathrm{Spin}(M)}$ introduced at the end of Lecture I.

Corollary

A spin manifold M admits string structures if and only if

$$\frac{1}{2}p_1(M) = 0.$$

Moreover, there is an equivalence of bicategories

$$\left\{ \begin{array}{c} \textit{String structures} \\ \textit{on } M \end{array} \right\} \cong \left\{ \begin{array}{c} \textit{Trivializations of the} \\ \textit{Chern-Simons 2-gerbe } \mathcal{CS}_{\mathrm{Spin}(M)} \end{array} \right\}$$