Lectures on gerbes, stacks, and strings

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Part III: Strings

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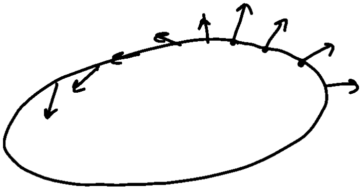
If M is a smooth manifold, then the loop space

$$LM := C^\infty(S^1, M)$$

is a Fréchet manifold.

A tangent vector at a loop $\tau : S^1 \to M$ is a vector field in M along τ:

$$T_\tau LM = \{X : S^1 \to TM \mid X(z) \in T_{\tau(z)}M\}.$$



Alternatively, the loop space can be treated in the "convenient setting", most prominently in the setting of diffeological spaces.

The loop space is the *configuration space* of closed strings in M:

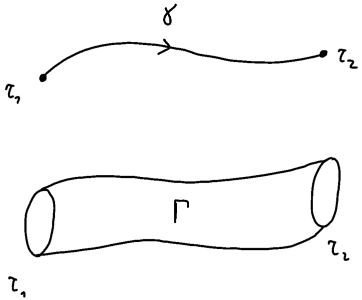


Figure: A path in the loop space.

Thus, a gauge field for strings should be a *principal bundle* over LM with connection.

On the other hand, as we explained in Lecture I, such a gauge field is a *bundle gerbe* with connection over M. The relation between these two structures is called transgression.

Transgression can easily be defined on the level of cohomology or differential forms.

Let

$$\mathrm{ev} : LM \times S^1 \to M : (\tau, z) \mapsto \tau(z)$$

be the evaluation map.

- Transgression of a *differential form* is

$$T : \Omega^k(M) \to \Omega^{k-1}(LM); \quad T(H) := \int_{S^1} \mathrm{ev}^* H.$$

- Transgression in *cohomology* is defined in a similar same way,

$$T : \mathrm{H}^k(M, \mathbb{Z}) \to \mathrm{H}^{k-1}(M, \mathbb{Z}); \quad T(\xi) := \int_{S^1} \mathrm{ev}^* \xi,$$

  where the fibre integral in cohomology is:

$$\mathrm{H}^k(X \times S^1, \mathbb{Z}) \twoheadrightarrow \mathrm{H}^{k-1}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \underbrace{\mathrm{H}^1(S^1, \mathbb{Z})}_{=\mathbb{Z}} = \mathrm{H}^{k-1}(X, \mathbb{Z}),$$

The transgression of a bundle gerbe $\mathcal{G}$ with connection over M is a principal $\mathrm{U}(1)$-bundle $\mathcal{T}(\mathcal{G})$ with connection over LM.

Its fibre over a loop $\tau : S^1 \to M$ is defined to be the set

$$\mathcal{T}(\mathcal{G})_\tau := \left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \text{trivializations of } \tau^*\mathcal{G} \end{array} \right\} = \mathrm{h}_0\mathcal{H}om(\tau^*\mathcal{G}, \mathcal{I}_0).$$

Note:

- $\tau^*\mathcal{G}$ is a bundle gerbe over S^1, and since $\mathrm{H}^3(S^1, \mathbb{Z}) = 0$ there exist trivializations.
- There are no 2-forms on S^1 and every trivialization necessarily has vanishing covariant derivative.

We need to establish a $\mathrm{U}(1)$-action on

$$\mathcal{T}(\mathcal{G})_\tau = \mathrm{h}_0\mathcal{H}om(\tau^*\mathcal{G}, \mathcal{I}_0).$$

The definition of bundle gerbe isomorphisms gives for trivial gerbes an equivalence of monoidal categories

$$\mathcal{H}om(\mathcal{I}_0, \mathcal{I}_0) = \mathcal{B}un^{\nabla}_{\mathrm{U}(1)}(X)^{flat}.$$

For $X = S^1$, every $\mathrm{U}(1)$-bundle is trivial and every connection is flat, and so the monodromy

$$\mathrm{h}_0\mathcal{B}un^{\nabla}_{\mathrm{U}(1)}(S^1)^{flat} \cong \mathrm{U}(1).$$

All together, we get an isomorphism of groups

$$\mathrm{h}_0\mathcal{H}om(\mathcal{I}_0, \mathcal{I}_0) \cong \mathrm{U}(1).$$

Now we define the $\mathrm{U}(1)$-action:

$$\mathcal{T}(\mathcal{G})_\tau \times \mathrm{U}(1) = \mathrm{h}_0\mathcal{H}om(\mathcal{G}, \mathcal{I}_0) \times \mathrm{h}_0\mathcal{H}om(\mathcal{I}_0, \mathcal{I}_0) \xrightarrow{\circ} \mathrm{h}_0\mathcal{H}om(\mathcal{G}, \mathcal{I}_0) = \mathcal{T}(\mathcal{G})_\tau$$

Theorem (Brylinski '93 – KW '07)

(a) *The fibres* $\mathcal{T}(\mathcal{G})_\tau$ *form a smooth principal* $\mathrm{U}(1)$*-bundle* $\mathcal{T}(\mathcal{G})$ *over* LM.

(b) *The bundle* $\mathcal{T}(\mathcal{G})$ *carries a unique connection such that*

$$\mathrm{Hol}_\gamma(\mathcal{T}(\mathcal{G})) = \mathrm{Hol}_\Gamma(\mathcal{G})$$

holds for all loops $\gamma : S^1 \to LM$ *and corresponding tori* $\Gamma : S^1 \times S^1 \to M$.

(c) *The assignment* $\mathcal{G} \mapsto \mathcal{T}(\mathcal{G})$ *extends to a monoidal functor*

$$\mathcal{T} : \mathrm{h}_1\mathcal{G}\mathrm{rb}^\nabla(M) \to \mathcal{B}\mathrm{un}^\nabla_{\mathrm{U}(1)}(LM).$$

(d) *Transgression is compatible with curvatures and characteristic classes:*

$$\begin{array}{ccccc}
\Omega^3(M) & \xleftarrow{\ \mathrm{curv}\ } & \mathrm{h}_0\mathcal{G}\mathrm{rb}^\nabla(M) & \xrightarrow{\ \mathrm{DD}\ } & \mathrm{H}^3(M,\mathbb{Z}) \\
\downarrow T & & \downarrow \mathcal{T} & & \downarrow T \\
\Omega^2(M) & \xleftarrow[\mathrm{curv}]{} & \mathrm{h}_0\mathcal{B}\mathrm{un}^\nabla(LM) & \xrightarrow[\mathrm{c}_1]{} & \mathrm{H}^2(LM,\mathbb{Z})
\end{array}$$

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The transgression bundle $\mathcal{T}(\mathcal{G})$ is frequently used in conformal field theory,

- to perform geometric quantization on LM
- to describe properly the parallel transport of a bundle gerbe
- to describe open strings coupled to D-branes

Let $\mathcal{G}_{basic}$ be the basic bundle gerbe over a compact simple Lie group G. It is multiplicative in the sense that it comes equipped with an isomorphism

$$\mathrm{pr}_1^*\mathcal{G}_{basic} \otimes \mathrm{pr}_2^*\mathcal{G}_{basic} \to m^*\mathcal{G}_{basic}$$

over $G \times G$, together with an "associator" 2-isomorphism over $G \times G \times G$.

Applying transgression yields a *multiplicative* principal $\mathrm{U}(1)$-bundle

$$L\widetilde{\mathrm{Spin}(d)} := \mathcal{T}(\mathcal{G}_{basic})$$

over the loop group LG. We discussed in Lecture I that this is the same as a central extension

$$1 \to \mathrm{U}(1) \to L\widetilde{\mathrm{Spin}(d)} \to LG \to 1.$$

This central extension is called the basic central extension. It is related to:

- representation theory (Pressley-Segal)
- Conformal field theory
- Twisted equivariant K-theory (Freed-Hopkins-Teleman)

The loop space perspective to strings motivates further constructions that are well-known for point-particles.

For instance, spin structures, i.e. lifts of the frame bundle of M along

$$\mathrm{Spin}(d) \to \mathrm{SO}(d).$$

Recall that $\mathrm{Spin}(d)$ has a representation Σ_d, the spinor representation, whose properties are suitable for modelling particles with spin.

The *spinor bundle* of M is the associated vector bundle

$$\mathbb{S}_d := \mathrm{Spin}(M) \times_{\mathrm{Spin}(d)} \Sigma_d .$$

Recall that the representation Σ_d is not a representation of $\mathrm{SO}(d)$; hence, the spin structure is necessary.

Suppose M is a spin manifold with a spin structure $\mathrm{Spin}(M)$.

Then, $L\mathrm{Spin}(M)$ is a principal $L\mathrm{Spin}(d)$-bundle over LM.

A spin structure on LM is a lift of the structure group of $L\mathrm{Spin}(M)$ along the basic central extension

$$1 \to \mathrm{U}(1) \to \widetilde{L\mathrm{Spin}(d)} \to L\mathrm{Spin}(d) \to 1.$$

Thus, a spin structure on LM is a principal $\widetilde{L\mathrm{Spin}(d)}$-bundle $\widetilde{L\mathrm{Spin}(M)}$ over LM together with a bundle morphism

$$\begin{array}{ccc} \widetilde{L\mathrm{Spin}(M)} & \longrightarrow & L\mathrm{Spin}(M) \\ & \searrow \quad \swarrow & \\ & LM & \end{array}$$

We have seen in Lecture I that every lifting problem has its *lifting gerbe*, a geometric representative of the obstruction against lifts.

We let $\mathcal{L}_{L\mathrm{Spin}(M)}$ be the lifting gerbe for spin structures on LM:

$$\mathcal{L}_{L\mathrm{Spin}(M)} = \left\{ \begin{array}{ccccc} & & P & \longrightarrow & \widetilde{L\mathrm{Spin}(d)} \\ & & \downarrow & & \downarrow \\ \mathrm{Spin}(M) & \leftleftarrows & \mathrm{Spin}(M)^{[2]} & \xrightarrow[\delta]{} & L\mathrm{Spin}(d) \\ {\scriptstyle\pi}\downarrow & & & & \\ LM & & & & \end{array} \right\}$$

Our Theorem about the lifting gerbe from Lecture I gives us a bijection:

$$\left\{ \begin{array}{c} \text{Equivalence classes of} \\ \text{spin structures on } LM \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \text{trivializations of } \mathcal{L}_{L\mathrm{Spin}(d)} \end{array} \right\}.$$

Spin structures on LM have been invented by Killingback in 1986, with the motivation to define anomaly-free supersymmetric string theories.

Killingback related the existence of spin structures on LM to the vanishing of the first Pontryagin class $\frac{1}{2}p_1(M)$, a class that was known to represented the so-called *fermionic anomaly*.

In 1992, McLaughlin proved indeed:

$$\frac{1}{2}p_1(M) = 0 \quad \Rightarrow \quad LM \text{ admits spin structures}$$

Over the mapping space $C^\infty(\Sigma, M)$ of a closed oriented surface Σ to a spin manifold M there is a Pfaffian line bundle L, associated to a family of Dirac operators twisted by maps $\phi : \Sigma \to M$.

It has a canonical section σ, whose value $\sigma(\phi)$ can be regarded as the fermionic action functional for a string $\phi : \Sigma \to M$. In order to use this as an integrand in the path integral, it is necessary to trivialize L, so that σ becomes a function.

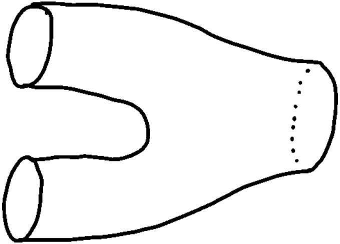
A result of Freed from 1986 shows that

$$\mathrm{c}_1(L) = \int_\Sigma \mathrm{ev}^* \left(\frac{1}{2} p_1(M)\right).$$

However, it is not enough to know that the Pfaffian bundle is trivial – we need to specify a trivialization. Killingback's hope was that spin structures on loop space would do this – but this has never been proved.

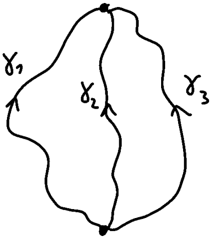
Principal $\mathrm{U}(1)$-bundles with connection over LM can to some extended be seen as an alternative to bundle gerbes with connection, but they do not carry the full information, e.g. needed in string theory.

The problem is the pair of pants:



This problem can be resolved by the observation that the principal bundles on LM that are transgressed bundle gerbes carry the an additional structure, called a fusion product.

We denote by $PM := C^\infty([0,1], M)$ the space of smooth paths in M. Consider three paths $\gamma_1, \gamma_2, \gamma_3 \in PM$ with a common initial point and a common end point:



The space of such triple of paths can be described as the 3-fold fibre product $PM^{[3]}$ of the surjective submersion $PM \to M \times M$ with itself.

Consider the loops

$$\gamma_i \cup \gamma_j := \overline{\gamma_j} \star \gamma_i \in LM.$$

A fusion product on a principal $\mathrm{U}(1)$-bundle P over LM is a bundle isomorphism

$$\lambda_{\gamma_1,\gamma_2,\gamma_3} : P_{\gamma_2 \cup \gamma_3} \otimes P_{\gamma_2 \cup \gamma_1} \longrightarrow P_{\gamma_1 \cup \gamma_3}.$$

over $PM^{[3]}$. Moreover, it has to be associative over $PM^{[4]}$.

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Given a principal $\mathrm{U}(1)$-bundle P over LM with fusion product λ, one can construct an (infinite-dimensional) bundle gerbe $\mathcal{R}_x(P,\lambda)$ over M, called the regression:

- Its surjective submersion is $P_xM \to M : \gamma \mapsto \gamma(1)$.
- Its principal $\mathrm{U}(1)$-bundle is the pullback of P along $\cup : P_xM^{[2]} \to LM$.
- Its bundle gerbe product is the fusion product λ.

The usual sketch of this bundle gerbe is:

$$\begin{array}{ccccc} & & \cup^* P & & \lambda \\ & & \downarrow & & \vdots \\ P_x M & \leftleftarrows & P_x M^{[2]} & \Lleftarrow & P_x M^{[3]} \\ \downarrow & & & & \\ M & & & & \end{array}$$

Theorem (KW '09)

(a) *If $\mathcal{G}$ is a bundle gerbe with connection over M, then its transgression $\mathcal{T}(\mathcal{G})$ comes equipped with a canonical fusion product λ. Moreover, that fusion product is connection-preserving.*

(b) *Regression is inverse to transgression: for any point $x \in M$, there is an isomorphism*

$$\mathcal{G} \cong \mathcal{R}_x(\mathcal{T}(\mathcal{G}), \lambda)$$

of bundle gerbes over M.

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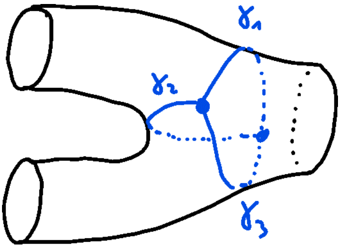
The fusion product resolves the problem with the pair of pants:



This is closely related to so-called *smooth functorial field theories*.

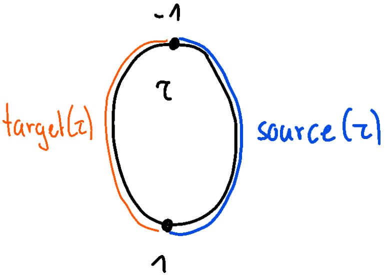
Adding fusion products is one step towards getting an equivalence between bundle gerbes with connection over M and a category of bundles over LM. Yet, one further structure is missing: a *thin homotopy equivariant structure*. Moreover, a couple of constraints have to be imposed on the parallel transport of the connections "superficial connections".

We are now in position to construct a certain (infinite-dimensional) Lie 2-group, the string 2-group. We use the fact that the basic central extension

$$1 \to \mathrm{U}(1) \to \widetilde{L\mathrm{Spin}(d)} \to LG \to 1.$$

can be obtained as the transgression of the basic gerbe $\mathcal{G}_{basis}$ over $\mathrm{Spin}(d)$, and hence comes equipped with a fusion product.

We want to see a loop as a morphism between paths:



From this perspective, fusion gives a notion of composition.

We recall from Lecture II the definition of a Lie 2-group:

A (strict) Lie 2-group is a groupoid Γ in the category of Lie groups: it has a Lie group Γ_0 of objects and a Lie group Γ_1 of morphisms, and all structure maps

$$s, t : \Gamma_1 \to \Gamma_0 \quad , \quad \mathrm{id} : \Gamma_0 \to \Gamma_1 \quad , \quad \Gamma_1 \, {}_s\!\times_t \Gamma_1 \to \Gamma_1$$

are Lie group homomorphisms.

The string 2-group $\mathrm{String}(d)$ is the following infinite-dimensional Lie 2-group:

- The Lie group of objects is $P_e\mathrm{Spin}(d)$.
- The Lie group of morphisms is the restriction of $\widetilde{L\mathrm{Spin}(d)}$ to $\Omega\mathrm{Spin}(d)$.
- Source and target maps are projection and splitting:

$$\widetilde{L\mathrm{Spin}(d)} \to \Omega\mathrm{Spin}(d) \rightrightarrows P_e\mathrm{Spin}(d).$$

- Composition is the fusion product λ.

We want to understand what kind of 2-group this is. Therefore, we discuss a bit the classification theory for Lie 2-groups. We look at:

- the group $\pi_0\Gamma$ of isomorphism classes of objects, and
- the group $\pi_1\Gamma := \mathrm{Aut}(1_{\Gamma_0})$, which is always abelian
- an actin of $\pi_0\Gamma$ on $\pi_1\Gamma$ induced by $\gamma \mapsto \mathrm{id}_g \cdot \gamma \cdot \mathrm{id}_{g^{-1}}$.

Inclusion and projection define Lie 2-group homomorphisms

$$1 \to \mathcal{B}\pi_1\Gamma \to \Gamma \to (\pi_0\Gamma)_{dis} \to 1.$$

This is in fact an extension of Lie 2-groups in the appropriate homotopy-theoretical sense.

The extension is called central, if the action of $\pi_0\Gamma$ on $\pi_1\Gamma$ is trivial.

In case of the *string 2-group* that we just defined, we have

$$\pi_0 \mathrm{String}(d) = P_e \mathrm{Spin}(d)/\sim \qquad \text{with} \qquad \beta_1 \sim \beta_2 \Leftrightarrow \beta_1(1) = \beta_2(1)$$

and so we can identify this with $\mathrm{Spin}(d)$.

Moreover, $\pi_1 \mathrm{String}(d) = \mathrm{U}(1)$.

The action of $\mathrm{Spin}(d)$ on $\mathrm{U}(1)$ is trivial since $\mathrm{U}(1) \hookrightarrow L\widetilde{\mathrm{Spin}(d)}$ is central.

Thus, the string 2-group is a central extension

$$1 \to \mathcal{B}\mathrm{U}(1) \to \mathrm{String}(d) \to \mathrm{Spin}(d)_{dis} \to 1.$$

We go a bit further in the classification.

Associated to any central Lie 2-group extension

$$1 \to \mathcal{B}\mathrm{U}(1) \to \Gamma \to G_{dis} \to 1$$

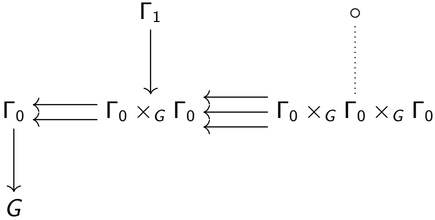
is a multiplicative $\mathrm{U}(1)$-bundle gerbe over G, which we denote by $\mathcal{G}_\Gamma$:

- Its surjective submersion is the projection $\pi : \Gamma_0 \to G$.
- The double fibre product $\Gamma_0 \times_G \Gamma_0$ comes equipped with a central extension

$$1 \to \mathrm{U}(1) \to \Gamma_1 \stackrel{s,t}{\to} \Gamma_0 \times_G \Gamma_0 \to 1.$$

- The bundle gerbe product μ is the composition in Γ.

The sketch is



The bundle gerbe $\mathcal{G}_\Gamma$ serves us a Dixmier-Douady class $[\mathcal{G}_\Gamma] \in \mathrm{H}^3(G, \mathbb{Z})$.

One can now do the following: geometric realization gives a fibre sequence of topological spaces,

$$1 \to \mathrm{BU}(1) \to |\Gamma| \to G \to 1,$$

which in turn induces a long exact sequence of homotopy groups.

Since $\mathrm{BU}(1)$ is a $K(\mathbb{Z}, 2)$, its only non-trivial homotopy group is $\pi_2(\mathrm{BU}(1)) = \mathbb{Z}$.

Hence, the long exact sequence gives isomorphisms $\pi_k|\Gamma| \cong \pi_k G$ in degrees 0, 1, and $k \geq 4$. The remaining sequence is

$$0 \to \pi_3|\Gamma| \to \pi_3 G \to \pi_2\mathrm{BU}(1) \to \pi_2|\Gamma| \to 0,$$

and it turns out that the arrow in the middle is the composition

$$\pi_3 G \to \mathrm{H}_3(G) \stackrel{[\mathcal{G}_\Gamma]}{\to} \mathbb{Z}.$$

$$\mathcal{G}_{\mathrm{String}(d)} = \left\{ \begin{array}{ccccc} & & \Gamma_1 & & \circ \\ & & \downarrow & & \vdots \\ \Gamma_0 & \leftleftarrows & \Gamma_0 \times_G \Gamma_0 & \Lleftarrow & \Gamma_0 \times_G \Gamma_0 \times_G \Gamma_0 \\ \downarrow & & & & \\ G & & & & \end{array} \right\}$$

$$= \left\{ \begin{array}{ccccc} & & L\widetilde{\mathrm{Spin}(d)} & & \lambda \\ & & \downarrow & & \vdots \\ P_e\mathrm{Spin}(d) & \leftleftarrows & P_e\mathrm{Spin}(d)^{[2]} & \Lleftarrow & P_e\mathrm{Spin}(d)^{[3]} \\ \downarrow & & & & \\ \mathrm{Spin}(d) & & & & \end{array} \right\}$$

$$= \mathcal{R}_e(\mathcal{T}(\mathcal{G}_{basic}), \lambda)$$

$$\cong \mathcal{G}_{basic}$$

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Since the class of the basic gerbe $[\mathcal{G}_{basic}] \in \mathrm{H}^3(\mathrm{Spin}(d), \mathbb{Z})$ is a generator, the corresponding map $[\mathcal{G}_\Gamma] : \mathrm{H}_3(G) \to \mathbb{Z}$ is the identity.

Since also $\pi_3 G \to \mathrm{H}_3(G)$ is the identity, the relevant sequence is

$$0 \to \pi_3|\Gamma| \to \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \to \pi_2|\Gamma| \to 0.$$

This shows that $\pi_3|\Gamma| = \pi_2|\Gamma| = 0$.

Comparing this with the Whitehead tower of the orthogonal group $\mathrm{O}(d)$, we see that $|\mathrm{String}(d)|$ has the correct homotopy type of the string group.

	π_0	π_1	π_2	π_3	π_4	π_5	π_6	π_7
$\mathrm{O}(d)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\mathrm{SO}(d)$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\mathrm{Spin}(d)$	0	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\mathrm{String}(d)$	0	0	0	0	0	0	0	$\mathbb{Z}$
$\mathrm{Fivebrane}(d)$	0	0	0	0	0	0	0	0

Any Lie 2-group homomorphism $\Gamma \to \Omega$ induces an "extension" functor

$$\mathcal{G}\mathrm{rb}_{\Gamma} \to \mathcal{G}\mathrm{rb}_{\Omega}.$$

In particular, if $G = \pi_0\Gamma$, there is a functor

$$\mathcal{G}\mathrm{rb}_{\Gamma}(M) \to \mathcal{G}\mathrm{rb}_{G}(M) \cong \mathcal{B}\mathrm{un}_{G}(M).$$

If $\mathcal{P}$ is a Γ-bundle gerbe, we denote the corresponding G-bundle by $\pi_0(\mathcal{P})$.

One can now pose the following *lifting problem*:

Given a principal G-bundle P over M, does there exist a Γ-bundle gerbe $\mathcal{P}$ with $\pi_0\mathcal{P} \cong P$?

Note that this is a generalization of the lifting problem considered in Lecture I: instead of Lie group homomorphisms $\hat{G} \to G$ (i.e., 2-group homomorphism $\hat{G}_{dis} \to G_{dis}$), we now allow general 2-group homomorphisms $\Gamma \to G_{dis}$.

Definition

A *string structure* on a spin manifold M is a lift of its spin structure along the central extension

$$1 \to \mathcal{B}\mathrm{U}(1) \to \mathrm{String}(d) \to \mathrm{Spin}(d)_{dis} \to 1.$$

Thus, a string structure on a spin manifold M is a $\mathrm{String}(d)$-bundle gerbe $\mathcal{S}$ together with an isomorphism $\pi_0\mathcal{S} \cong \mathrm{Spin}(M)$.

This is a clear, geometric definition of a string structure, and it puts string structures in one row with orientations and spin structures.

The problem of lifting a principal G-bundle along a central 2-group extension

$$1 \longrightarrow \mathcal{B}\mathrm{U}(1) \longrightarrow \Gamma \longrightarrow G_{dis} \longrightarrow 1$$

can be treated by a *lifting 2-gerbe*.

Theorem (Nikolaus-KW '13)

Let $\mathcal{CS}(P, \mathcal{G}_\Gamma)$ be the Chern-Simons 2-gerbe associated to the bundle P and the multiplicative bundle gerbe $\mathcal{G}_\Gamma$. Then, there is a bijection

$$\left\{ \begin{array}{c} \textit{Equivalence classes of lifts} \\ \textit{of } P \textit{ to a } \Gamma\textit{-bundle gerbe } \mathcal{P} \end{array} \right\} \cong \left\{ \begin{array}{c} \textit{Isomorphism classes of} \\ \textit{trivializations of } \mathcal{CS}(P, \mathcal{G}_\Gamma) \end{array} \right\}$$

Thus, Chern-Simons 2-gerbes are the lifting gerbes belonging to our generalized lifting problem.

In case of string structures, we get

$$\mathcal{CS}(\mathrm{Spin}(M), \mathcal{G}_{\mathrm{String}(d)}) \cong \mathcal{CS}(\mathrm{Spin}(M), \mathcal{G}_{basic}) = \mathcal{CS}_{\mathrm{Spin}(M)}.$$

Thus, the lifting 2-gerbe for string structures is precisely the Chern-Simons 2-gerbe $\mathcal{CS}_{\mathrm{Spin}(M)}$ considered in Lecture I, whose Dixmier-Douady class is:

$$\frac{1}{2}p_1(M) \in \mathrm{H}^4(M, \mathbb{Z}).$$

Corollary

A spin manifold admits string structures if and only if $\frac{1}{2}p_1(M) = 0$. *Moreover, there is a bijection*

$$\left\{ \begin{array}{c} \textit{String structures} \\ \textit{on } M \end{array} \right\} \cong \left\{ \begin{array}{c} \textit{Trivializations of the} \\ \textit{Chern-Simons 2-gerbe } \mathcal{CS}_{\mathrm{Spin}(M)} \end{array} \right\}$$

Transgression generalizes from bundle gerbes with connections to bundle 2-gerbes with connections. More precisely, it becomes a functor

$$\mathcal{T} : \left\{ \begin{array}{c} \text{Bundle 2-gerbes with} \\ \text{connection over } M \end{array} \right\} \to \left\{ \begin{array}{c} \text{Bundle gerbes with} \\ \text{connection over } LM \end{array} \right\}$$

We only have to identify the result in case of the Chern-Simons 2-gerbe:

Theorem (KW '14)

The transgression of the Chern-Simons 2-gerbe is the spin lifting gerbe on loop space,

$$\mathcal{T}(\mathcal{CS}_{\mathrm{Spin}(M)}) \cong \mathcal{L}_{L\mathrm{Spin}(M)}.$$

In particular, trivializations of the Chern-Simons 2-gerbe transgress to spin structures on LX.

Finally, we have achieved the following picture:

$$\left\{ \begin{array}{c} \text{String} \\ \text{structures} \\ \text{on } M \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{c} \text{Trivializations of the} \\ \text{Chern-Simons} \\ \text{2-gerbe } \mathcal{CS}_{\mathrm{Spin}(M)} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Spin structures} \\ \text{on } LM \end{array} \right\}$$

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This can be sharpened by including a *fusion product* in the spin structures on loop space:

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In 2010, Bunke proved that the transgression of the Chern-Simons 2-gerbe to the mapping space of any closed oriented surface yields the Pfaffian bundle L, and thereby proved ultimatively that (geometric) string structures trivialize the anomaly of the supersymmetric sigma model.

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In 2019, Peter Kristel and I constructed from a fusive spin structure on LM a 2-vector bundle on M, the so-called *stringor bundle*. Its existence was conjectured in 2005 by Stolz and Teichner.

Big open questions:

(a) Construct the Dirac operator on loop space.

(b) Compute its index, which is supposed to take values in *tmf*.

(c) Prove the Stolz conjecture.