$N = 2$ spinning particle BRST and R-R fields

E Boffo
joint work with I Sachs and D Bockisch

42th Winter School Geometry and Physics, Srní

January 16, 2022

MATEMATICKO-FYZIKÁLNÍ FAKULTA
Univerzita Karlova

Table of Contents

Particle in worldline approach

Classical action:

$x^\mu(\tau), p_\mu(\tau) : I \subset \mathbb{R} \mapsto M$ (4-dim pseudo-Riemannian manifold), $e(\tau)\mathrm{d}\tau \in \Omega^1(I)$

$$\underbrace{S = \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \, \left(p_\mu \dot{x}^\mu + \frac{e}{2} p^2\right)}_{1st\ order}, \qquad \underbrace{S = \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \, \frac{1}{2e} \dot{x}^\mu \dot{x}_\mu}_{2nd\ order} . \tag{1}$$

Invariance under reparametrization $\tau \mapsto \tau'(\tau)$.

Pure spinor: $p_{\alpha\dot{\alpha}} = p_\mu \sigma^\mu_{\alpha\dot{\alpha}}$, $\sigma^\mu = (\mathrm{id}, \sigma^i)$ 2×2 Pauli matrices,

$$S = \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \, p_{\alpha\dot{\alpha}} \left(\dot{x}^{\alpha\dot{\alpha}} - 2\lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}\right), \tag{2}$$

for λ^α spinor.

Spinning particle in worldline approach

Supersymmetrizing along the worldline with Grassmann odd θ (easier in 2nd order action) can be regarded as endowing the particle with a half-integer spin at the classical level:
$x^\mu(\tau) \mapsto (x^\mu(\tau) + \theta\psi^\mu(\tau)) : \mathbb{R}^{(1|1)} \mapsto M$,
$e(\tau)\mathrm{d}\tau \mapsto (e(\tau) + \theta\chi(\tau))\,\mathrm{d}\tau$,
$\frac{\mathrm{d}}{\mathrm{d}\tau} \mapsto \frac{\mathrm{d}}{\mathrm{d}\theta} + \theta\frac{\mathrm{d}}{\mathrm{d}\tau}$ and integrate in $\mathrm{d}\theta$ using standard integration rules for θ:

$$S = \int_{\tau_i}^{\tau_f} \mathrm{d}\tau\, \left(p_\mu \dot{x}^\mu + \mathrm{i}\psi^\mu \dot{\psi}_\mu + \frac{e}{2}p^2 + \chi\psi^\mu p_\mu \right). \tag{3}$$

Now invariance under supersymmetry transformations too.
ψ^μ and χ have odd parity.
Equivalently:

$$S = \int_{\tau_i}^{\tau_f} \mathrm{d}\tau\, p_{\alpha\dot\alpha} \left(\dot{x}^{\alpha\dot\alpha} + 2\mathrm{i}(\theta^\alpha \dot{\tilde{\theta}}^{\dot\alpha} - \tilde{\theta}^{\dot\alpha}\dot{\theta}^\alpha) - 2\lambda^\alpha \tilde{\lambda}^{\dot\alpha} \right) - \mathrm{i}\rho(\lambda\tilde{\theta} + \tilde{\lambda}\theta - \zeta). \tag{4}$$

Dirac covariant quantization with constraints:

- represent the symplectic Lie superalgebra $[x^\mu, p_\nu] = \delta^\mu_\nu, \{\psi^\mu, \psi^\nu\} = g^{\mu\nu}$
- conservation of charges $p^2 =: H, \psi^\mu p_\mu =: q$, respecting superalgebra $\{q, q\} = H, [q, H] = 0$, gives dynamical equations for physical states.

Two sectors (representations):

- $|phys\rangle = u(p)e^{-\mathrm{i}p\cdot x}|0\rangle$ for $u(p)$ Dirac spinor, with $p^2|phys\rangle = 0$ and $\not{p}u(p) = 0$;

- focus on $SU(2) \subset Cliff(4)$ by introducing $\psi^i_\pm := \frac{1}{\sqrt{2}}\left(\psi^{2i} \pm \mathrm{i}\psi^{2i+1}\right), \quad i = 0, 1, \quad \psi^i_+|0, p\rangle = 0,$
  so the physical states are the massless and tranverse $|0, p\rangle, \psi^i_-|0, p\rangle$ and $(\psi^i_-\psi^j_- - \psi^j_-\psi^i_-)|0, p\rangle$.

Yang-Mills (N=2 spinning particle) *Dai-Huang-Siegel 2008*

Now 2 Grassmann wordline coordinates θ^i, supersymmetric partners of τ
Convenient to arrange Clifford Gamma matrices in linear complex combinations (*annihilation-creation pair* $(\psi, \bar{\psi})$) as before, $\{\psi^\mu, \bar{\psi}^\nu\} = 2g^{\mu\nu}$
For BRST: consider algebra of $SO(D-1,1) \times Osp(1,1|2)$ with generators:

$$\Sigma^{\mu\nu} := \psi^\mu \bar{\psi}^\nu - \psi^\nu \bar{\psi}^\mu, \tag{5}$$

$$S^\mu := \mathrm{i}\gamma\bar{\psi}^\mu + \mathrm{i}\bar{\gamma}\psi^\mu, \tag{6}$$

$$S := 2\gamma\bar{\gamma}. \tag{7}$$

Brackets of the ghosts (*Weyl algebra*):

$$[\gamma, \bar{\beta}] = [\bar{\gamma}, \beta] = \{b, c\} = 1. \tag{8}$$

Choose polarization that annihilates highest weight vector of the module
$(\bar{\psi}, \bar{\gamma}, \bar{\beta}, b)\,|0\rangle = 0$

$U(1)$ current J

$$J = \psi \cdot \bar{\psi} - \gamma \bar{\beta} + \bar{\gamma} \beta - 1,$$

can further constrain the polynomials in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^4) \otimes \mathit{Weyl} \otimes \mathit{Cliff}$ constructed by acting with arbitrary numbers of ψ, γ, β and at most one c on the vacuum.

A spin 1 gauge field respecting Yang-Mills equations is found in the cohomology of

$$Q = cH + \gamma \bar{q} + \bar{\gamma} q - \gamma \bar{\gamma} b. \tag{9}$$

BRST differential with a curved target space background $\nabla = p + A$, $[\nabla_\mu, \nabla_\nu] = \mathrm{i} F_{\mu\nu} : \Rightarrow Q_I = \frac{1}{2} c \left(\nabla^2 - \mathrm{i} F_{\mu\nu} \Sigma^{\mu\nu} \right) + S^\mu \nabla_\mu + Sb$
$H^\bullet(Q_I, \ker(J))$, $\ker(J) \subset \mathcal{H}$.
Then $Q_I^2 = 0$ if

$$\nabla \star F = 0. \tag{10}$$

Proceed with vertices $Q_I - Q \equiv V \Rightarrow Q_I^2 = 0$ yields $\{Q, V\} + V^2 = 0$. Scattering amplitudes with loops are then easily computed knowing the vertices.

Einstein gravity and SUGRA (N=4 spinning particle)

Bonezzi-Meyer-Sachs 2018,2020

Repeat the construction with 4 Grassmann odd θ^i now $\rightsquigarrow \psi_i^\mu, \bar{\psi}_i^\mu, \ i = 1, 2$

As ψ_1^μ *and* ψ_2^ν generate states from a (Fock) vacuum, a spin-2 field is there

Consistency of BRST differential Q (together with $\mathfrak{so}(4)$ current J now) implies Einstein gravity or (g, H, ϕ) Supergravity eom's

Which of the two occurs depends on which subalgebra of $\mathfrak{so}(4)$ is gauged.

The worldline approach once again lets us calculate scattering amplitude (just tree level so far, but loops are possible)

However just the "bosonic" NS-NS sector could be seen in the cohomology and subsequently used to deform Q. What about "fermionic" R-R?

Table of Contents

"Spin field" for the particle: an attempt

Consider the set of graded elements $\{x^\mu, -\frac{\mathrm{d}}{\mathrm{d}x^\mu} \equiv p_\mu, \theta^\alpha, \uparrow, \lambda_\alpha\}$ with brackets:

$$[x^\mu, p_\nu] = \delta^\mu_\nu,\ [\lambda_\alpha, \theta^\beta] = \delta^\beta_\alpha \uparrow,$$
$$[\lambda_\alpha, \uparrow] = 0 = [\lambda_\alpha, \lambda_\beta] = \{\theta^\alpha, \theta^\beta\},\ \ \{\theta^\alpha, \uparrow\} \neq 0, \tag{11}$$

θ^α and λ_α are 2 dimensional Weyl spinors. Add to this set the chiral pair $\tilde{\lambda}^{\dot{\alpha}}$ and $\tilde{\theta}_{\dot{\beta}}$, and with $\tilde{\sigma}^\mu \equiv (\mathrm{id}, -\sigma^j)$ define

$$\psi^\mu := \theta^\alpha \sigma^\mu_{\alpha\dot{\beta}} \tilde{\lambda}^{\dot{\beta}} + \tilde{\theta}_{\dot{\gamma}} \tilde{\sigma}^{\mu\, \dot{\gamma}\delta} \lambda_\delta, \tag{12}$$

$$\Downarrow$$

then one could treat this as $N = 1$ *spinning particle in 4d, but with* ψ^μ *as in* (12).

Introduce ghosts γ, β, c and b as usual and create states from bosonic vacuum $|0\rangle$ with $(\theta, \tilde{\theta}, \uparrow, c, \gamma)$.

BRST cohomology of N=1

Study cohomology of $Q = cH + \gamma\psi \cdot p + \gamma^2 b$ with ψ as in (12):

Nilpotency: $Q^2 = \gamma^2 \left(-2\mathcal{I}p^2 + H\right) + \gamma c[\psi \cdot p, H], \quad \mathcal{I} := \left(\theta^\alpha \lambda_\alpha \uparrow + \tilde{\theta}_{\dot{\alpha}} \tilde{\lambda}^{\dot{\alpha}} \uparrow\right)$

Always nilpotent if we restrict to $\ker(\mathcal{I} - 1) \subset \mathcal{H}$

Generic element $|\Psi\rangle \in \ker(\mathcal{I} - 1)$ has arbitrary powers of γ:

$$
\begin{aligned}
|\Psi\rangle = &\varphi_\alpha(x)\theta^\alpha |0\rangle + \varphi^\uparrow_\alpha(x)\theta^\alpha \uparrow |0\rangle + \underbrace{B_{\alpha\beta}(\theta^\alpha\theta^\beta - \theta^\alpha \uparrow \theta^\beta \uparrow) |0\rangle}_{=:B^{\alpha\beta}|\alpha\beta\rangle} + \dots \\
&+ \left(\gamma^n \phi^{(n)}_\alpha \theta^\alpha + c\gamma^n \phi^{(n)c}_\alpha \theta^\alpha\right) |0\rangle \, . \qquad (13)
\end{aligned}
$$

Infinite dimensional field space! $\Longrightarrow$ *Siegel gauge:* $b\,|\Psi\rangle = 0 = \beta\,|\Psi\rangle$ kills last row in (13).

The **physical states** in $\ker Q \cap \ker(\mathcal{I} - 1)$ are:

- $n_\theta = 1$: spinors fulfilling Weyl equation $(\bar{\partial}\varphi)^{\dot{\beta}} = 0$.
- $n_\theta = 2$: $B_{\alpha\beta} = B_{\mu\nu}(\sigma^{\mu\nu})_{\alpha\beta} + \chi\epsilon_{\alpha\beta}$ and $A_\mu\sigma^\mu(\theta \uparrow \tilde{\theta} - \theta\tilde{\theta} \uparrow)$, are exact $0, 1, 2$-forms $\Leftrightarrow$ R-R field strengths.

Non-flat target space

Which kind of backgrounds can be coupled to the worldline?

An allowed deformation of Q is

$$\gamma\delta q = \gamma\phi\left(\theta^\alpha\lambda_\alpha + \tilde{\theta}_{\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}}\right). \tag{14}$$

It corresponds to a mass term for the massless spinors or form fields.

N=2 as N=1 × N=1

- As usual, consider $SU(4) \subset Cliff(4)$ gamma matrices:
$$\psi^\mu = \tilde{\theta}_{\dot\alpha}\tilde{\sigma}^{\mu\;\dot\alpha\alpha}\lambda_\alpha, \quad \bar{\psi}^\mu = \theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\tilde{\lambda}_{\dot\alpha}.$$
Notice that $\{\psi^\mu, \bar{\psi}^\nu\} \neq 0$ but it is not just $\sim g^{\mu\nu}$, could retain $\sigma^{\mu\nu} := \mathrm{i}(\tilde{\sigma}^{[\mu}\sigma^{\nu]})/2$ as well.
- Define a number operator
$$\mathcal{I}' := \tilde{\theta}_{\dot\alpha}\tilde{\lambda}^{\dot\alpha} \uparrow - \gamma\bar{\beta} + \beta\bar{\gamma} - 1, \tag{15}$$
then the space of fields ker$\mathcal{I}'$ is finite. Elements $\in$ ker$\mathcal{I}'$ are arbitrary in θ, with different total n_θ number,
  - $n_\theta = 1$: $|\tilde{\chi}\rangle = \left(\tilde{\chi}^{\dot\beta}\tilde{\theta}_{\dot\beta} + \beta\varphi_\alpha\theta^\alpha \uparrow + c\beta F_\alpha\theta^\alpha \uparrow\right)|0\rangle$ + antifields;
  - $n_\theta = 2$: $B^{\dot\alpha\dot\beta}\,|\dot\alpha\dot\beta\rangle + \beta g_\beta{}^{\dot\gamma}(x)\,|e^\beta{}_{\dot\gamma}\rangle + c\beta F_\beta{}^{\dot\gamma}(x)\,|e^\beta{}_{\dot\gamma}\rangle$ + antifields,

  where $|e^\beta{}_{\dot\gamma}\rangle = \theta^\beta(\uparrow\tilde{\theta}_{\dot\gamma} - \tilde{\theta}_{\dot\gamma}\uparrow)\,|0\rangle$.
- Cohomology of
$$Q = cp^2 + \gamma\bar{\psi}^\nu p_\nu + \bar{\gamma}\psi^\mu p_\mu + \gamma\bar{\gamma}b, \tag{16}$$
nilpotent on the states above: the physical states have to satisfy, respectively,
•$n_\theta = 1$: $F_\alpha = \mathrm{i}(\partial\!\!\!/\tilde{\chi})_\alpha$, with gauge trafo $\delta\tilde{\chi}^{\dot\beta} = -\mathrm{i}(\partial\!\!\!/\varphi)^{\dot\beta}$
•$n_\theta = 2$: in gauge $b\,|B\rangle = 0 = \bar{\beta}\,|B\rangle$, linearized gauge theory for $B^{\dot\alpha\dot\beta}$, being a scalar potential or a self-dual 2-form potential. If not in gauge slice, it is off-shell theory.

Next step: deform the BRST differential with R-R background coupling (scalar + 2-form)

$p_\mu \to \Pi_\mu = p_\mu + \frac{\mathrm{i}}{2}\tilde{\partial}^\gamma_{\dot\beta} B^{\dot\alpha\dot\beta}\sigma_{\mu\,\gamma\dot\alpha}$

$$\Downarrow$$

$$\gamma\bar\gamma\left(\{\psi^{(\mu},\bar\psi^{\nu)}\}\Pi_\mu\Pi_\nu + \frac{1}{2}[\psi^\mu,\bar\psi^\nu][\Pi_\mu,\Pi_\nu]\right) = -\gamma\bar\gamma H \tag{17}$$

$$\gamma c[\bar\psi\cdot\Pi, H] = 0 = \bar\gamma c[\psi\cdot\Pi, H] \tag{18}$$

fulfilled with no conditions on the background field $B^{\dot\alpha\dot\beta}$.

- *R-R vector is also in the cohomology for $n_\theta = 2$: take*

$$|A\rangle = A^{\dot\alpha}{}_\beta\,|e_{\dot\alpha}{}^\beta\rangle + \left(c\beta f_{\alpha\beta} + \beta G_{\alpha\beta}\right)|\alpha\beta\rangle\,. \tag{19}$$

  However the theory is again off-shell (no dynamical equations) when we are not in Siegel gauge.
- A R-R vector background is also supported.
- *$N = 2$ as $N = 1 \times N = 1$: by changing polarization, i.e. trading β for $\bar\gamma \Longrightarrow$ now the interpretation of $|A\rangle$ and $|B\rangle$ as R-R field strengths is consistent.*

Conclusions:

- the graded set $\theta, \lambda, \uparrow$ gives us a "spin field";
- the BRST-BV cohomology is that of a spinor, or can give closure (Bianchi) of 0,1,2-form field strengths (from bispinor decomposition);
- non-flat backgrounds (with these form fields) are also supported, but no dynamics yet.