Tractors and mass of asymptotically hyperbolic manifolds

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The classical concept of mass

- Asymptotically Euclidean (AE) and hyperbolic (AH) Riemannian manifolds are defined via *charts at infinity*. These have values in in $E^n \setminus K$ respectively $\mathcal{H}^n \setminus K$, where K is compact, and the metric is assumed to be asymptotic to the standard metric in an appropriate sense.
- In the AE case, the ADM-mass is a number that is defined as the limit $r \to \infty$ of an integral of a coordinate expression over the sphere of radius r.
- In the AH-setting, one associates a mass integral to each solution of the KID (Killing initial data) equation. This leads to a "mass" which is a vector in $\mathbb{R}^{(n,1)}$.
- The geometric nature of mass, i.e. independence the of the choice of chart at infinity is very surprising and difficult to prove.

Our approach to mass-like invariants

- We work on a general manifold $\overline{M}$ with boundary ∂M and a class $\mathcal{G}$ of conformally compact metrics that are asymptotic to each other and associate a "relative mass" to two metrics in the class.
- While we impose rather strong restrictions on the order of asymptotics, we do not impose any restriction on the topology of $\overline{M}$ or ∂M and only weak assumptions on the metrics in $\mathcal{G}$.
- $\mathcal{G}$ induces a conformal structure on ∂M and our relative masses are tractor valued $(n-1)$-forms on ∂M. The construction does not use coordinates and satisfies strong compatibility conditions with diffeomorphisms.
- In case that $\mathcal{G}$ contains hyperbolic metrics, we can associate an invariant to a single metric in $\mathcal{G}$. If in addition $M = \mathcal{H}^n$ and $\mathcal{G}$ contains the Poincaré-metric, we recover the classical AH-mass by an integration procedure.

On a manifold $\overline{M} = M \cup \partial M$ with boundary, there is a coordinate-free concept of boundary asymptotics via local *defining functions*. For $U \subset \overline{M}$ open and a smooth defining function $\rho : U \to [0, \infty)$, any other defining function $\hat{\rho}$ is of the form $\hat{\rho} = e^h \rho$ for $h \in C^\infty(U, \mathbb{R})$.

For $f \in C^\infty(U, \mathbb{R})$, $f|_{\partial M} = 0$ iff $f = \rho f_1$ for $f_1 \in C^\infty(U, \mathbb{R})$ ("f is $\mathcal{O}(\rho)$"). For $k > 1$, f is $\mathcal{O}(\rho^k)$ if $f = \rho^k f_1$ for $f_1 \in C^\infty(U, \mathbb{R})$. We say that $f \in C^\infty(U \cap M, \mathbb{R})$ is $\mathcal{O}(\rho^{-k})$ iff $\rho^k f$ admits a smooth extension to all of U.

For $k \in \mathbb{Z}$, the fact that f is $\mathcal{O}(\rho^k)$ does not depend on the choice of ρ. But the functions f_1 respectively the boundary values of the extensions *do* depend on ρ.
This easily extends to tensor fields, e.g. via coordinate functions in charts that meet the boundary and compatibility of the concepts with tensorial operations is easy to analyze.

For tensor fields $t_1, t_2 : \overline{M} \to \mathbb{R}$ of the same type and $k > 0$, we write $t_1 \sim_k t_2$ iff $t_1 - t_2$ is $\mathcal{O}(\rho^k)$. This in particular implies that $t_1|_{\partial M} = t_2|_{\partial M}$.

Example: Let g_{ab} be a metric on $\overline{M}$ with Levi-Civita connection ∇_a and suppose that $\xi^b \in \mathfrak{X}(\overline{M})$ is $\mathcal{O}(\rho^k)$ for fixed $k > 0$. Then locally $\xi^b = \rho^k \tilde{\xi}^b$ for $\tilde{\xi}^b \in \mathfrak{X}(\overline{M})$. Putting $\rho_a := d\rho$, we get $\nabla_a \xi^b = k\rho^{k-1}\rho_a\tilde{\xi}^b + \rho^k \nabla_a \tilde{\xi}^b$ and the last term is $\mathcal{O}(\rho^k)$. Hence $\nabla_a \xi^b$ is $\mathcal{O}(\rho^{k-1})$ and $\nabla_a \xi^b \sim_k k\rho^{k-1}\rho_a\tilde{\xi}^b$.

Conformally compact metrics

For $\overline{M} = M \cup \partial M$, a Riemannian metric g on M is called *conformally compact* if g is $\mathcal{O}(\rho^{-2})$ and $\rho^2 g|_{\partial M}$ is non-degenerate. While $\rho^2 g|_{\partial M}$ depends on ρ, one obtains a well defined conformal class on ∂M ("*conformal infinity* of g").

If g is conformally compact, then $(\rho^2 g)^{-1}(d\rho, d\rho)$ is $\mathcal{O}(1)$ and its boundary value is independent of ρ. If this is $\equiv 1$, g is called *locally asymptotically hyperbolic (LAH)*. This is equivalent to all sectional curvatures of g being asymptotic to -1.

For a LAH-metric g, there is a class of distinguished defining functions, namely those for which $(\rho^2 g)^{-1}(d\rho, d\rho)$ is $\equiv 1$ on a neighborhood of the boundary.

Recall that conformal compactness of g is equivalent to the fact that $\sigma := \mathrm{vol}_g^{-1/n} \in \Gamma(\mathcal{E}[1])$ extends by 0 to a defining density for ∂M. Given the conformal class $[g]$ on $\overline{M}$, σ uniquely determines g.

Given $\overline{M}$ of dimension n, we can define an equivalence relation on conformally compact metrics on M by $g \sim h$ if $\rho^2 g \sim_n \rho^2 h$ and we let $\mathcal{G}$ be one equivalence class. Observe that

- All $g \in \mathcal{G}$ induce the same conformal infinity on ∂M ("$[\mathcal{G}]$").
- If some $g \in \mathcal{G}$ is LAH, then all metrics in $\mathcal{G}$ are LAH. We will assume this from now on.

Any $g \in \mathcal{G}$ determines a conformal $[g]$ on $\overline{M}$. Different metrics in $\mathcal{G}$ give rise to different conformal classes, but all these classes restrict to $[\mathcal{G}]$ on ∂M.

Recall that to $[g]$ one associates a *tractor bundle* $\mathcal{T}\overline{M}$ (rank $n+2$, endowed with a Lorentzian bundle metric and a canonical linear connection). Similarly, there is $\mathcal{T}\partial M$ associated to $[\mathcal{G}]$ and this can be naturally realized as a subbundle of the restriction $\mathcal{T}\overline{M}|_{\partial M}$. This also works for $n = 3$, which is most relevant for applications to GR.

Fix $\overline{M} = M \cup \partial M$ and a class $\mathcal{G}$ as above. For $g, h \in \mathcal{G}$ we can split $h_{ab} - g_{ab}$ into a trace-part fg_{ab} and a trace-free part $(h_{ab} - g_{ab})^0$. The trace-part is equivalently encoded by $\tau - \sigma \in \Gamma(\mathcal{E}[1])$ for the defining densities determined by h and g.

Let D be the tractor-D operator, ∇_a the tractor connection and $S : \Gamma(\mathcal{E}_{(ab)_0}[1]) \to \Gamma(\mathcal{E}_{aA})$ the BGG splitting operator for the conformal class $[g]$ and for $a_1, a_2 \in \mathbb{R}$ consider

$$\star_g \left(a_1 \nabla_a D_A(\tau - \sigma) + a_2 S(\sigma(h_{ab} - g_{ab})^0) \right) \in \Omega^{n-1}(M, \mathcal{T}M).$$

It is easy to see that this is smooth up to the boundary and its boundary value is a multiple of X^A, hence defining $c(g, h) \in \Omega^{n-1}(\partial M, \mathcal{T}\partial M)$. Indeed $c(g, h)$ can be computed explicitly, and using this, one shows that c is a cocycle in the sense that $c(h, g) = -c(g, h)$ and $c(g, k) = c(g, h) + c(h, k)$.

Consider a diffeomorphism $\Phi \in \mathrm{Diff}(\overline{M})$. If $\Phi^* g \in \mathcal{G}$ for one $g \in \mathcal{G}$, then $\Phi^*(\mathcal{G}) \subset \mathcal{G}$ and we denote by $\mathrm{Diff}_{\mathcal{G}}(\overline{M})$ the group of these diffeomorphisms.

For $\Phi \in \mathrm{Diff}_{\mathcal{G}}(\overline{M})$, $\Phi|_{\partial M}$ by construction is a conformal isometry of $(\partial M, [\mathcal{G}])$ and hence acts on $\Omega^{n-1}(\partial M, \mathcal{T}\partial M)$. Naturality of the construction readily implies $c(\Phi^* g, \Phi^* h) = (\Phi|_{\partial M})^*(c(g,h))$ for any $\Phi \in \mathrm{Diff}_{\mathcal{G}}(\overline{M})$.

There is a well defined concept of diffeomorphisms of $\overline{M}$ being asymptotic to each other to some order $k > 0$, and we denote this by $\Phi \sim_k \tilde{\Phi}$. Most easily, $\Phi \sim_k \tilde{\Phi} \iff f \circ \Phi \sim_k f \circ \tilde{\Phi}$ for any $f \in C^\infty(\overline{M}, \mathbb{R})$. Denote by $\mathrm{Diff}_0^k(\overline{M})$ the group of those Φ such that $\Phi \sim_k \mathrm{id}$. With some effort, one proves

Theorem

$\mathrm{Diff}_0^{n+1}(\overline{M}) \subset \mathrm{Diff}_{\mathcal{G}}(\overline{M})$ and it consists of those $\Phi \in \mathrm{Diff}_{\mathcal{G}}(\overline{M})$ such that $\Phi|_{\partial M} = \mathrm{id}$.

The key to this result is that elements of $\mathrm{Diff}_0^{n+1}(\overline{M})$ are closely related to distinguished defining functions. For an LAH metric g call a local defining function ρ *adapted to* g if $(\rho^2 g)^{-1}(d\rho, d\rho) \equiv 1$ on some neighborhood of the boundary. It is well known that one can locally find unique adapted defining functions ρ for any prescribed boundary value of $\rho^2 g$.

Given ρ adapted to g and ρ, we say that $h \in \mathcal{G}$ is *aligned with* (g, ρ) iff $g^{ac}\rho_c$ inserts trivially into $h_{ab} - g_{ab}$. This implies that ρ is adapted to h, too.

Key theorem

(1) Given $g, h \in \mathcal{G}$ and ρ adapted to g, there is $\Psi \in \mathrm{Diff}_0^{n+1}(\overline{M})$ such that $\Psi^* h$ is aligned with (g, ρ). This condition uniquely determines Ψ locally around the boundary.
(2) For the diffeomorphism Ψ from (1), the boundary quantities associated to $\Psi^* h - g$ can be explicitly computed from those associated to $h - g$.

Together with the explicit formula for c mentioned above this implies that putting $a_1 = \frac{a}{n}$ and $a_2 = \frac{a}{2}$ for some $a \in \mathbb{R}$, we obtain $c(g,h) = c(g, \Psi^* h)$ where Ψ is the diffeomorphism that aligns h to (g,ρ). Uniqueness of this diffeomorphism locally around the boundary easily implies an additional invariance property.

Corollary

For any $a \in \mathbb{R}$, $a_1 = \frac{a}{n}$ and $a_2 = \frac{a}{2}$ give rise to a cocycle c such that for any $g, h \in \mathcal{G}$ and $\Phi \in \mathrm{Diff}_0^{n+1}(\overline{M})$ we get $c(g,h) = c(g, \Phi^* h)$.

Assume that for each $x \in \partial M$, there is an open neighborhood U of x and a metric $g \in \mathcal{G}$ which is hyperbolic on U. By results of Fefferman-Graham, any two such metrics are locally diffeomorphic by a diffeomorphism fixing the boundary.
This implies that for $h \in \mathcal{G}$ and a_1, a_2 as above, $c(g,h)|_U$ is the same for all hyperbolic metrics g on U, so these glue together to an invariant $c(h) \in \Omega^{n-1}(\partial M, \mathcal{T}\partial M)$ such that $c(\Phi^* h) = (\Phi|_{\partial M})^* c(h)$ for any $\Phi \in \mathrm{Diff}_{\mathcal{G}}(M)$.

Integrating our invariants

Let us finally assume that M is a neighborhood of the boundary in hyperbolic space and $\mathcal{G}$ contains the Poincaré metric g. Then $[\mathcal{G}]$ is the round conformal structure on $\partial M = S^{n-1}$ and $\mathcal{T}\partial M$ is globally trivialized by parallel sections. For a parallel frame $\{s_i\}$ and $h \in \mathcal{G}$ we can expand $c(h) = \sum \varphi_i s_i$ for $\varphi_i \in \Omega^{n-1}(S^{n-1})$ and then consider $\sum(\int_{S^{n-1}} \varphi_i)s_i \in \Gamma(\mathcal{T}S^{n-1})$. This is independent of the choice of frame and hence defines a global invariant $C(h) := \int_{S^{n-1}} c(h)$, which is a parallel tractor on S^{n-1}.

Let $\langle\ ,\ \rangle$ be the tractor metric. Then for a global parallel frame $\{s_i\}$ as above, we can consider $\langle C(h), s_i\rangle = \int_{S^{n-1}} \langle c(h), s_i\rangle$. Now $c(h)$ is a boundary value anyway, so if we realize the s_i as boundary values of interior tractors $\tilde{s}_i$, we can write $\int_{S^{n-1}} \langle c(h), s_i\rangle$ as a limit of integrals over large spheres.

Recovering the AH mass

Let $\sigma \in \Gamma(\mathcal{E}[1])$ be the density determined by the Poincaré metric g. Then parallel sections of $\mathcal{T}S^{n-1}$ are exactly the boundary values of the parallel tractors $\tilde{s}_V := D(V\sigma)$, where V is a solution of the KID equation

$$\nabla_a \nabla_b V - g_{ab}\Delta V + (n-1)g_{ab}V = 0.$$

(Also $D(\sigma)$ is a parallel section of $\mathcal{T}\overline{M}$, but its boundary value is the normal tractor for $\partial M = S^{n-1}$.)

Writing $\tilde{c}(g,h)$ for the interior tractor valued form, we can explicitly compute $\langle \tilde{c}(g,h), \tilde{s}_V \rangle$ up to terms that vanish along the boundary. This then implies that, for the right choice of a, $\int_{\partial M} \langle c(h), s_V \rangle$ equals the limit of the mass integral of h associated to V by Chrusciel-Herzlich.

Thank you for your attention!