Lie Superalgebra Cohomology: new insights from pseudoforms

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Outline

- Lie Algebra Cohomology (Chevalley-Eilenberg, Koszul, Hochschild-Serre, etc.)
- Forms on Supermanifolds (Berezin, Bernstein, Leites, Manin, Penkov, Witten, etc.)
- Lie Superalgebra Cohomology: superforms (Kac, Fuks, etc.)
- Lie Superalgebra Cohomology: integral and pseudoforms

Introducing Lie Algebra Cohomology

Let $\mathfrak{g}$ be an ordinary finite dimensional Lie algebra defined over the field $\mathbb{K} = \mathbb{C}, \mathbb{R}$, and let V be a $\mathfrak{g}$-module; a *p-chain* of $\mathfrak{g}$ valued in V is an alternating $\mathbb{K}$-linear map

$$C_p(\mathfrak{g}, V) := \wedge^p \mathfrak{g} \otimes V \ ,$$

where $\wedge^p \mathfrak{g}$ is for $\mathfrak{g}$ considered a vector space. This can be lifted to a complex by introducing the differential

$$\begin{aligned} \partial : C_p(\mathfrak{g}, V) &\rightarrow C_{p-1}(\mathfrak{g}, V) \\ fX^1 \wedge \ldots \wedge X^p &\mapsto \sum_{i=1} (-1)^{i+1} (X_i f) X_1 \ldots \wedge \hat{X}_i \wedge \ldots \wedge X_p + \\ &+ \sum_{i<j}^{p} (-1)^{i+j+1} f \, [X_i, X_j] \wedge X_1 \ldots \wedge \hat{X}_i \wedge \ldots \wedge \hat{X}_j \wedge \ldots \wedge X_p \ . \end{aligned}$$

On the dual side, we can define *p-cochains* as

$$\Omega^p(\mathfrak{g}, V) := Hom_k\left(\wedge^p \mathfrak{g}, V\right) = \wedge^p \mathfrak{g}^* \otimes V \ .$$

Again, we can introduce a differential

$$\begin{aligned} d : \Omega^p(\mathfrak{g}, V) &\rightarrow \Omega^{p+1}(\mathfrak{g}, V) \\ fX^{*1} \wedge \ldots \wedge X^{*p} &\mapsto \sum_{i=1}^{n} (X_i f)\, X^{*i} \wedge X^{*1} \wedge \ldots \wedge X^{*p} + \end{aligned}$$

$$+ \sum_{i=1}^{p} \sum_{j<k} (-1)^{i+1} f \cdot \left(X^{*i}\left[X_j, X_k\right]\right) X^{*j} \wedge X^{*k} \wedge X^{*1} \wedge \ldots \wedge \hat{X}^{*i} \wedge \ldots \wedge X^{*p} \ .$$

The operator d is nilpotent as a consequence of Jacobi identities.

One can define the *Chevalley-Eilenberg complexes of* $\mathfrak{g}$ as the pairs $(C_\bullet(\mathfrak{g},\mathbb{K}),\partial)$ on the homological side and $(\Omega^\bullet(\mathfrak{g},\mathbb{K}),d)$ on the cohomological side. The homology/cohomology is defined in the usual way: we call *Chevalley-Eilenberg cycles/cocylces* the elements of the vector space

$$Z_p(\mathfrak{g},\mathbb{K}) := \{f \in C_p(\mathfrak{g},\mathbb{K}) : \partial f = 0\} \ , \ Z^p(\mathfrak{g},\mathbb{K}) := \{f \in \Omega^p(\mathfrak{g},\mathbb{K}) : df = 0\} \ ,$$

and *Chevalley-Eilenberg boundaries/coboundaries* the elements in the vector space

$$\begin{aligned} B_p(\mathfrak{g},\mathbb{K}) &:= \{f \in C_p(\mathfrak{g},\mathbb{K}) : \exists g \in C_{p+1}(\mathfrak{g},\mathbb{K}) : f = \partial g\} \ , \\ B^p(\mathfrak{g},\mathbb{K}) &:= \{f \in \Omega^p(\mathfrak{g},\mathbb{K}) : \exists g \in \Omega^{p-1}(\mathfrak{g},\mathbb{K}) : f = dg\} \ . \end{aligned}$$

The *Chevalley-Eilenberg p-homology/p-cohomology group* of $\mathfrak{g}$ is defined as the quotient vector space

$$H_p(\mathfrak{g},\mathbb{K}) := Z_p(\mathfrak{g},\mathbb{K}) \big/ B_p(\mathfrak{g},\mathbb{K}) \ , \ H^p(\mathfrak{g},\mathbb{K}) := Z^p(\mathfrak{g},\mathbb{K}) \big/ B^p(\mathfrak{g},\mathbb{K}) \ .$$

The previous definitions hold in the case of $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ a *superalgebra* (with obvious corrections to signs due to parity): let us denote with $\mathcal{Y}_A := \{\mathcal{P}_i | \mathcal{Q}_\alpha\}$ a basis of $\mathfrak{g}$, and with $\mathcal{Y}^{*A} := \{\mathcal{V}^i | \psi^\alpha\}$ the dual basis, i.e., $\mathcal{Y}_A \left(\mathcal{Y}^{*B}\right) = \delta_A^B$. We have

$$[\mathcal{Y}_B, \mathcal{Y}_C] = f_{BC}^A \mathcal{Y}_A \implies d = f_{BC}^A \mathcal{Y}^{*B} \wedge \mathcal{Y}^{*C} \iota_{\mathcal{Y}_A} \ ,$$

$$d \left(\omega \wedge \eta\right) = \left(d\omega\right) \wedge \eta + (-1)^{|\omega|} \omega \left(d\eta\right) \ , \ \forall \omega, \eta \in \Omega^{(\bullet|0)} \left(\mathfrak{g}, \mathbb{K}\right) \ .$$

where now $[\cdot, \cdot]$ are *supercommutators*. We are then denoting

$$C^\bullet(\mathfrak{g}) := S^\bullet \mathfrak{g}^* \cong \Omega^\bullet(\mathfrak{g}, \mathbb{K}) \ .$$

Forms on Supermanifolds

On a supermanifold $\mathcal{SM}$ of dimension $dim\mathcal{SM} = (m|n)$, the complex of *superforms*, i.e., $\left(\Omega^{(\bullet|0)}\left(\mathcal{SM}, C^{\infty}_{\mathbb{R}^m}\left[\theta^1, \ldots, \theta^n\right]\right), d_{dR}\right)$ is *unbounded from above*:

$$0 \xrightarrow{\mathrm{d}} \Omega^{(0|0)}_{\mathcal{SM}} \xrightarrow{\mathrm{d}_{dR}} \Omega^{(1|0)}_{\mathcal{SM}} \xrightarrow{\mathrm{d}_{dR}} \ldots \xrightarrow{\mathrm{d}_{dR}} \Omega^{(m|0)}_{\mathcal{SM}} \xrightarrow{\mathrm{d}_{dR}} \ldots$$

as a consequence of the commutation relations

$$dx^i \wedge dx^j = -dx^j \wedge dx^i \ , \ dx^i \wedge d\theta^\alpha = d\theta^\alpha \wedge dx^i \ , \ d\theta^\alpha \wedge d\theta^\beta = d\theta^\beta \wedge d\theta^\alpha \ .$$

The notion of *top form* has to be found in the *Berezinian line bundle*, the super-analogous of the *Determinant line bundle*. With the Berezinian bundle one defines the complex of *integral forms*, which is unbounded from below:

$$\ldots \xrightarrow{\delta} \Omega^{(0|n)}_{\mathcal{SM}} \xrightarrow{\delta} \ldots \xrightarrow{\delta} \Omega^{(m-1|n)}_{\mathcal{SM}} \xrightarrow{\delta} \Omega^{(m|n)}_{\mathcal{SM}} \xrightarrow{\delta} 0 \ ,$$

where we denoted $\Omega^{(p|n)}_{\mathcal{SM}} := \mathcal{B}er(\mathcal{SM}) \otimes S^{n-p}(\Pi\mathcal{T}(\mathcal{SM}))$.

Berezinian: Polyvector Fields Realisation

The Berezinian of a vector superspace can be constructed via the cohomology of a suitable generalization of the Koszul complex. Given a supermanifold $\mathcal{SM}$ and a set $x^i|\theta^\alpha$, $i = 1, \dots, n$ and $\alpha = 1, \dots, m$ of local coordinates, one finds that the Berezinian line bundle is (locally) generated by

$$\mathcal{B}er(\mathcal{SM}) \cong \mathcal{O}_{\mathcal{SM}} \cdot [dx^1 \wedge \dots \wedge dx^n \otimes \partial_{\theta^1} \wedge \dots \wedge \partial_{\theta^m}] \ .$$

The differential δ is defined via Lie derivative.
In our setting, by denoting with $\mathcal{Y}_A = \{X_i, \chi_\alpha\}$ and $\mathcal{Y}^A = \left\{\mathcal{V}^i, \psi^\alpha\right\}$ the bases of vectors and MC forms, respectively, the *Haar Berezinian* (valued in $\mathbb{K}$) in this realisation reads

$$\mathcal{B}er_{inv} := \mathbb{K} \cdot \left[\mathcal{V}^1 \wedge \dots \wedge \mathcal{V}^m \otimes \chi_1 \wedge \dots \wedge \chi_n\right] \ .$$

Integral forms are then defined as

$$C^p_{int}(\mathfrak{g}) := \mathcal{B}er_{inv}(\mathfrak{g}) \otimes S^{n-p}(\Pi\mathcal{T}(\mathfrak{g})) \ .$$

Berezinian: Distributional Realisation

One way to realise integral forms is as (compactly supported) *generalised functions* on $\mathrm{Tot}\,\Pi\mathcal{T}^*(\mathcal{SM})$ [1], that is elements

$$\omega(x^1, \dots, x^n, d\theta^1, \dots, d\theta^m | \theta^1, \dots, \theta^m, dx^1, \dots dx^n) \in \Pi\mathcal{T}(\mathscr{M}) \ ,$$

where $x^i|\theta^\alpha$ are local coordinates for $\mathcal{SM}$, which only allow a *distributional dependence* supported in $d\theta^1 = \dots = d\theta^m = 0$. A generic $(p|n)$-integral form locally reads

$$\omega^{(p|n)} = \omega_{[i_1 \dots i_q j_1 \dots j_n]}(x, \theta)\, dx^{i_1} \wedge \dots dx^{i_q} \wedge \delta^{(j_1)}\left(d\theta^1\right) \wedge \dots \wedge \delta^{(j_n)}\left(d\theta^n\right) \ .$$

We are denoting $\delta^{(j)}\left(d\theta^\alpha\right) := \left(\iota_\alpha\right)^j \delta\left(d\theta^\alpha\right)$. These distribution satisfy the relations

$$\left|\delta^{(j)}\left(d\theta^\alpha\right)\right| = 1 \ , \ \forall j \in \mathbb{N} \cup \{0\} \ , \ \delta\left(d\theta^\alpha\right) \wedge \delta\left(d\theta^\beta\right) = -\delta\left(d\theta^\beta\right) \wedge \delta\left(d\theta^\alpha\right) \ ,$$

$$d\theta^\alpha \delta^{(j)}\left(d\theta^\alpha\right) = -j\delta^{(j-1)}\left(d\theta^\alpha\right) \ , \ \delta\left(\lambda d\theta^\alpha\right) = \frac{1}{\lambda}\delta\left(d\theta^\alpha\right) \ .$$

[1] See e.g. Witten.

The differential δ is defined as $\delta = d_{dR}$, acting on integral forms via the previous formal relations.
By restricting to trivial modules, the *Haar Berezinian* is now defined as

$$\omega^{(m|n)} \equiv \omega_{\mathfrak{g}}^{top} := \mathcal{V}^1 \dots \mathcal{V}^n \delta(\psi^1) \dots \delta(\psi^m) \ .$$

Any integral form can be obtained by acting on $\omega_{\mathfrak{g}}^{top}$ with contractions:

$$\omega^{(m-p|n)} = \iota_{\mathcal{Y}_{A_1}} \dots \iota_{\mathcal{Y}_{A_p}} \omega_{\mathfrak{g}}^{top} \ ,$$

thus reproducing the structure $\mathcal{B}er(\mathcal{SM}) \otimes S^{n-p}(\Pi\mathcal{T}(\mathcal{SM}))$.

Dictionary: Dirac $\leftrightsquigarrow$ Koszul

In both realisations we have that integral forms are defined as

$$C^p_{int}(\mathfrak{g}) := \mathcal{B}er(\mathfrak{g}) \otimes S^p \Pi \mathfrak{g} \ .$$

We have

$$\mathcal{B}er : \left(\prod_{i=1}^{m} \mathcal{V}^i \right) \left(\prod_{\alpha=1}^{m} \delta(\psi^\alpha) \right) \quad \leftrightsquigarrow \quad \left[\mathcal{V}^1 \wedge \ldots \wedge \mathcal{V}^m \otimes \chi_1 \wedge \ldots \wedge \chi_n \right]$$

$$\omega^{m-p}_{int} \left(\equiv \omega^{(m-p|n)} \right) : \left(\prod_{i=1}^{p} \iota_{\mathcal{Y}_i} \right) \mathcal{B}er \quad \leftrightsquigarrow \quad \mathcal{B}er \otimes \left(\bigwedge_{i=1}^{p} \pi \mathcal{Y}^i \right)$$

$$d_{CE} : d = f^A_{BC} \mathcal{Y}^{*B} \wedge \mathcal{Y}^{*C} \iota_{\mathcal{Y}_A} \quad \leftrightsquigarrow \quad \delta = 1 \otimes \sum_A \iota_{\mathcal{Y}^{*A}} \mathcal{L}_{\mathcal{Y}_A}$$

Distributional Realisation: Pseudoforms

The distributional realisation suggests the construction of a different type of forms, with *non-maximal* and *non-zero* number of delta's:

$$\omega^{(p|s)} = \omega_{[i_1 \ldots i_q j_1 \ldots j_n]}(x,\theta)\, dx^{i_1} \wedge \ldots dx^{i_q} \wedge \delta^{(j_1)}\left(d\theta^1\right) \wedge \ldots \wedge \delta^{(j_s)}\left(d\theta^s\right) \ , \ 0 < s < n \, .$$

These objects are not well defined; for example, they do not behave tensorially

$$d\theta^\alpha \mapsto \Lambda^\alpha_\mu dx^\mu + \Lambda^\alpha_\beta d\theta^\beta \ , \ \delta\left(d\theta^\alpha\right) \mapsto \delta\left(\Lambda^\alpha_\mu dx^\mu + \Lambda^\alpha_\beta d\theta^\beta\right) = \ldots ?$$

Example

$$d\theta^1 \mapsto d\theta^1 + d\theta^2 \implies \delta\left(d\theta^1\right) \mapsto \delta\left(d\theta^1 + d\theta^2\right) = \sum_{i=0}^{\infty} \frac{\left(d\theta^2\right)^i}{i!} \delta^{(i)}\left(d\theta^1\right) \ .$$

Some hints to define pseudoforms: Manin, Witten.

Fuks' Theorems

Theorem

If $m \geq n$, the natural inclusions

$$\begin{aligned} \mathfrak{gl}(m) &\to \mathfrak{gl}(m|n)_0 &\subset \mathfrak{gl}(m|n) \ , \\ \mathfrak{sl}(m) &\to \mathfrak{sl}(m|n)_0 &\subset \mathfrak{sl}(m|n) \ , \end{aligned}$$

induce an isomorphism in cohomology with trivial coefficients.

Theorem

$$H^\bullet(\mathfrak{osp}(m|n)) = \begin{cases} H^\bullet(\mathfrak{so}(m)) \ , \ \textit{if } m \geq 2n \ , \\ H^\bullet(\mathfrak{sp}(n)) \ , \ \textit{if } m < 2n \ . \end{cases}$$

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Only a part of the bosonic subalgebra contributes to the CE cohomology. The CE cohomology is related to the superalgebra invariants (or analogously to its rank): it looks like "some invariants get lost".

The Berezinian Complement Isomorphism

We define the "Berezinian complement" map $\star$ as

$$\begin{aligned} \star : \Omega^p_{diff}(\mathfrak{g}) &\rightarrow \Omega^{m-p}_{int}(\mathfrak{g}) \\ \omega &\mapsto \star\omega^{(p|0)} = (\star\omega)^{(m-p|n)} := \left(\prod_{i=1}^{p} \iota_{\mathcal{Y}_{A_i}} \right) \omega^{top}_{\mathfrak{g}} \ , \end{aligned}$$

where $\left(\prod_{i=1}^{p} \iota_{\mathcal{Y}_{A_i}} \right) \omega = 1$. This map induces a cohomology isomorphism:

$$\star : H^{\bullet}_{diff}(\mathfrak{g}) \xrightarrow{\cong} H^{m-\bullet}_{int}(\mathfrak{g}) \ .$$

The isomorphism is verified when $\mathfrak{g}$ admits non-degenerate invariant bilinear form, hence it holds e.g. for "basic Lie superalgebras".

Spectral Sequences

The idea is to reconstruct the cohomology of a Lie algebra starting from the (eventually known) cohomology of substructure.

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k} \ , \ \mathfrak{k} = \mathfrak{g}/\mathfrak{h} \ , \ \mathfrak{h} \text{ sub-algebra.}$$

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h} \ , \ [\mathfrak{h}, \mathfrak{k}] \subseteq \mathfrak{h} + \mathfrak{k} \ , \ [\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{h} + \mathfrak{k} \ .$$

The cohomology is calculated via *approximations*: we can split the differential d as

$$d = d_0 + d_1 + d_2 + \dots$$

then, calculate the cohomology w.r.t. d_0, then d_1, then d_2 etc., up to *convergence*.

- Trivial modules: Koszul
- General modules: Hochschild-Serre

Given a Lie algebra $\mathfrak{g}$ and a Lie sub-algebra $\mathfrak{h}$ (we denote $\mathfrak{g}/\mathfrak{h} = \mathfrak{k}$), we define the filtration

$$F^p\Omega^q(\mathfrak{g}) = \left\{ \omega \in \Omega^q(\mathfrak{g}) : \forall \xi_i \in \mathfrak{h}, \iota_{\xi_{i_\mathbf{1}}} \dots \iota_{\xi_{i_{q+\mathbf{1}-p}}} \omega = 0 \right\} \ ,$$

which is a filtration in the sense that

$$dF^p\Omega^q(\mathfrak{g}) \subseteq F^p\Omega^{q+1|2}(\mathfrak{g}) \ , \ \forall p, q \in \mathbb{Z} \ .$$

There exists a spectral sequence $(E_s^{\bullet,\bullet}, d_s), d_s : E_s^{p,q} \to E_s^{p+s,q+1-s}$ that converges to $H(\mathfrak{g})$. The first space (page zero) reads

$$E_0^{m,n} := F^m\Omega^{(m+n)}(\mathfrak{g}) / F^{m+1}\Omega^{(m+n)}(\mathfrak{g}) \ .$$

The differentials d_s are induced by the Koszul differential:

$$d = V_\mathfrak{h} V_\mathfrak{h} \iota_\mathfrak{h} + V_\mathfrak{h} V_\mathfrak{k} \iota_\mathfrak{h} + V_\mathfrak{k} V_\mathfrak{k} \iota_\mathfrak{h} + V_\mathfrak{h} V_\mathfrak{k} \iota_\mathfrak{k} + V_\mathfrak{k} V_\mathfrak{k} \iota_\mathfrak{k} \ ,$$

reflecting the structure

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \ , \ [\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{h} + \mathfrak{k} \ , \ [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{h} + \mathfrak{k} \ .$$

The following pages of the spectral sequence are defined as

$$E_s^{\bullet,\bullet} := H\left(E_{s-1}^{\bullet,\bullet}, d_{s-1}\right) \ .$$

In particular, the first differential formally reads

$$d_0 = V_{\mathfrak{h}} V_{\mathfrak{h}} \iota_{\mathfrak{h}} + V_{\mathfrak{h}} V_{\mathfrak{k}} \iota_{\mathfrak{k}} \ .$$

Theorem (Koszul, Hochschild-Serre)

If $\mathfrak{h}$ is reductive in $\mathfrak{g}$ (i.e., $[\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}$), hence

$$E_1^{m,n} = H^n(\mathfrak{h}) \otimes \left(\Omega^m(\mathfrak{k})\right)^{\mathfrak{h}} \ .$$

$$E_2^{m,n} = H^n(\mathfrak{h}) \otimes H^m(\mathfrak{g}, \mathfrak{h}) \ .$$

In the super-setting, Fuks classified the superform cohomology of classical Lie superalgebras with this technique, choosing $\mathfrak{h} = \mathfrak{g}_0$. New objects can be found by repeating the procedure for *sub-superalgebras*. Fixed a certain picture number, we can introduce two *inequivalent* filtrations

$$\begin{aligned} F^p \Omega^{(q|l)}(\mathfrak{g}) &:= \left\{ \omega \in \Omega^{(q|l)}(\mathfrak{g}) : \forall \xi_i \in \mathfrak{h}, \iota_{\xi_{i_1}} \dots \iota_{\xi_{i_{q+1-p}}} \omega = 0 \right\} , \\ \tilde{F}^p \Omega^{(q|l)}(\mathfrak{g}) &:= \left\{ \omega \in \Omega^{(q|l)}(\mathfrak{g}) : \forall \xi_i^* \in \mathfrak{h}^*, \xi^*_{i_1} \wedge \dots \wedge \xi^*_{i_{q+1-p}} \wedge \omega = 0 \right\} . \end{aligned}$$

- The two filtrations coincide if $\mathfrak{g}$ is a bosonic Lie algebra;
- if $\mathfrak{h}$ has non-trivial odd part, the second filtration is empty on superforms;
- If $\mathfrak{h}$ has non-trivial odd part, the first filtration is empty on integral forms, which are then kept into account by the second filtration only.

Page zero of the spectral sequence is defined, for any l, as

$$\mathcal{E}_0^{m,n} := E_0^{m,n} \oplus \tilde{E}_0^{m,n} := \frac{F^m \Omega^{(m+n|l)}(\mathfrak{g})}{F^{m+1}\Omega^{(m+n|l)}(\mathfrak{g})} \oplus \frac{\tilde{F}^{m+2n-r}\Omega^{(m+n|l)}(\mathfrak{g})}{\tilde{F}^{m+2n-r+1}\Omega^{(m+n|l)}(\mathfrak{g})} \ .$$

Suppose $l = \dim \mathfrak{h}_1 = \dim \mathfrak{g}_1/2$, then we find

$$\mathcal{E}_0^{m,n} = \left[\Omega^{(m|\dim \mathfrak{h}_1)}(\mathfrak{k}) \otimes \Omega^{(n|0)}(\mathfrak{h})\right] \oplus \left[\Omega^{(m|0)}(\mathfrak{k}) \otimes \Omega^{(n|\dim \mathfrak{h}_1)}(\mathfrak{h})\right] \ .$$

This means that we can introduce new objects, namely *pseudoforms* since they have non-maximal and non-zero picture number, as integral forms (i.e., well defined!) of the sub-superstructures $\mathfrak{h}$ and $\mathfrak{k}$.

Theorem

If $\mathfrak{h}$ is reductive in $\mathfrak{g}$, hence

$$\mathcal{E}_1^{m,n} = \left[\left(\Omega^{(m|l)}(\mathfrak{k})\right)^{\mathfrak{h}} \otimes H^{(n|0)}(\mathfrak{h})\right] \oplus \left[\left(\Omega^{(m|0)}(\mathfrak{k})\right)^{\mathfrak{h}} \otimes H^{(n|l)}(\mathfrak{h})\right] \ .$$

$$\mathcal{E}_2^{m,n} = \left[H^{(m|l)}(\mathfrak{g},\mathfrak{h}) \otimes H^{(n|0)}(\mathfrak{h})\right] \oplus \left[H^{(m|0)}(\mathfrak{g},\mathfrak{h}) \otimes H^{(n|l)}(\mathfrak{h})\right] \ .$$

Example ($\mathfrak{g} = \mathfrak{osp}(2|2)$)

4 even generators, 4 odd generators
From Fuks', we have

$$H^{(\bullet|0)}\left(\mathfrak{osp}(2|2)\right) = H^{\bullet}\left(\mathfrak{sp}(2)\right) = \left\{1, \omega^{(3)}\right\} \ .$$

The abelian factor $\mathfrak{so}(2)$ is the "lost part". We can choose

$$\mathfrak{h} = \mathfrak{osp}(1|2) \ , \ \text{3 bosons, 2 fermions} \implies \text{picture 2 integral forms;}$$
$$\mathfrak{k} = \mathfrak{g}/\mathfrak{h} \ , \ \text{1 boson, 2 fermions} \implies \text{picture 2 integral forms.}$$

One finds

$$H^{(\bullet|2)}\left(\mathfrak{osp}(2|2)\right) = \left\{\omega^{(0|2)}, \omega^{(1|2)}, \omega^{(3|2)}, \omega^{(4|2)}\right\} \ ,$$

$\omega^{(1|2)}$ encodes the abelian factor. If we use the distributional realisation of pseudo/integral forms, we can find the same results.

There are other ways to approach the problem:

- brute force; but pseudoforms live in infinite dimensional spaces, computations may be arbitrarily difficult
- Molien-Weyl integrals: it is possible to extend the bosonic formula to the super setting. Pseudoforms are not standard representations $\rightarrow$ infinite-dimensional representations $\rightarrow$ extremely rich, almost unexplored land

The best way is to complement every method with the other, to have a complete understanding of the problems and the results.

General Philosophy

- New point of view to approach old problems, e.g., self-duality in Physics $\implies$ possibility of introducing new objects, fields, pairings etc. (see FDA)
- Maybe a suggestion to define pseudoforms in the geometric context (maybe supergroups as starting point...?)

Thank you for your attention!