On infinite number of diffeomorphic and Hamiltonian non-isotopic exact Lagrangian fillings for spherical spuns

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Motivation

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- First consider the case of Legendrian links in the standard contact 3-dimensional vector space. In these geometric settings, it has been positively answered first by Casals and Gao. Later the works of An-Bae-Lee, Casals-Zaslow, and Gao-Shen-Weng have continued to develop various cluster and sheaf-theoretic methods to detect infinitely many exact Lagrangian fillings for Legendrian links in the standard 3-dimensional contact vector space.

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- Today we would like to discuss Legendrian submanifolds in high dimensions that positively answer this question.

Main result

- In January 2021, Casals and Ng following the ideas of Kalman have found the first collection of Legendrian links in the standard contact vector space $(\mathbb{R}^3; \xi_{st})$ with the property that the Chekanov-Eliashberg algebra detects infinitely many diffeomorphic, but Hamiltonian non-isotopic exact Lagrangian fillings.

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Theorem (G)

For a given $m \geq 1$ and $k_i \geq 2$, where $i = 1, \ldots, m$, there is a Legendrian submanifold Λ in the standard contact vector space $\mathbb{R}^{2(k_1+\cdots+k_m+1)+1}_{st}$ diffeomorphic to the disjoint union of some number of $S^1 \times S^{k_1} \times \cdots \times S^{k_m}$ which admits an infinite number of exact Lagrangian fillings distinct up to Hamiltonian isotopy.

Contact preliminaries

- A contact manifold $(M;\xi)$ is a $(2n+1)$-dimensional manifold M equipped with a smooth maximally nonintegrable hyperplane field $\xi \subset TM$, i.e., $\xi = \ker \alpha$, where α is a 1-form which satisfies $\alpha \wedge (d\alpha)^n \neq 0$. ξ is a contact structure and α is a contact 1-form which defines ξ.

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Example

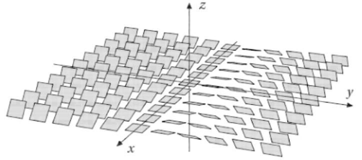

Figure: $(\mathbb{R}^{2n+1}, \xi_{st} = \ker(dz - \sum_i y_i dx_i))$, where $n = 1$.

Legendrian submanifolds and Reeb chords

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- A Reeb chord of Λ is a trajectory of ∂_z which starts and ends on Λ. The set of Reeb chords of Λ is denoted by $\mathcal{Q}(\Lambda)$.

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$$\partial(c^+) = \sum_{\dim \mathcal{M}^J(c^+;c_1^-,\dots,c_k^-)=1} \# \frac{M^J(c^+;c_1^-,\dots,c_k^-)}{\mathbb{R}} c_1^- \dots c_k^-,$$

where J is an almost complex structure on $(\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^s \alpha_{st}))$ such that $J : \xi \to \xi$, $d\alpha_{st}(v, Jv) > 0$ for $v \in \xi$, J is $\mathbb{R}$-invariant, $J(\partial_s) = \partial_z$ and $M^J(c^+; c_1^-, \dots, c_k^-)$ is a moduli space of punctured i-J holomorphic disks in $(\mathbb{R} \times \mathbb{R}^{2n+1}, J)$

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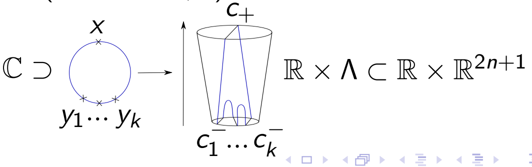


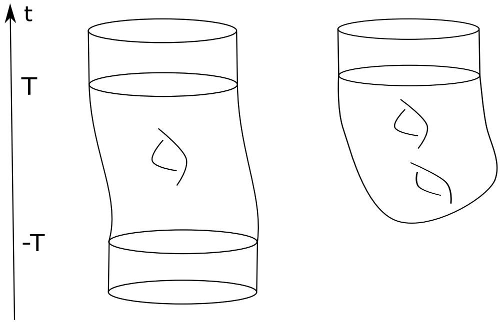
Exact Lagrangian cobordisms and fillings

Let Λ^- and Λ^+ be two Legendrian submanifolds of the standard contact vector space $\mathbb{R}^{2n+1}_{st} := (\mathbb{R}^{2n+1}, \alpha_{st} = dz - \sum y_i dx_i)$. We say that Λ^- is exact Lagrangian cobordant to Λ^+ if there is a smooth cobordism $(L; \Lambda^-, \Lambda^+)$ and an exact Lagrangian embedding $L \hookrightarrow S(\mathbb{R}^{2n+1}_{st})$, where $S(\mathbb{R}^{2n+1}_{st}) := (\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^t \alpha_{st}))$ and t is the coordinate on the first $\mathbb{R}$-factor, satisfying the following conditions:

- $L|_{(-\infty,-T)\times\mathbb{R}^{2n+1}_{st}} = (-\infty, -T) \times \Lambda^-$ and $L|_{(T,\infty)\times\mathbb{R}^{2n+1}_{st}} = (T, \infty) \times \Lambda^+$ for some $T \gg 0$,
- $L^c := L|_{[-T,T]\times\mathbb{R}^{2n+1}_{st}}$ is compact.
- There exists $f : L \to \mathbb{R}$ such that $e^t \alpha_{st}|_L = df$ and $f|_{(-\infty,-T)\times\Lambda^-}$, $f|_{(T,\infty)\times\Lambda^+}$ are constant functions.

If L is an exact Lagrangian cobordism with empty negative end and whose positive end is equal to Λ, then we say that L is an exact Lagrangian filling of Λ.

Exact Lagrangian fillings and augmentations

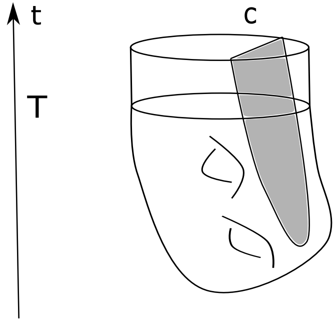


In general (by Ekholm-Honda-Kalman), having an exact Lagrangian cobirdism L from Λ^- to Λ^+, there is a DGA homomorphism

$$\mathcal{A}(\Lambda^+) \to \mathcal{A}(\Lambda^-).$$

In particular, if $\Lambda^- = \emptyset$, then there is a DGA homomorphism that we call augmentation

$$\varepsilon_L : \mathcal{A}(\Lambda^+) \to (\mathbb{Z}_2, 0).$$



For $c \in \mathcal{Q}(\Lambda)$, it is defined by

$$\varepsilon_L(c) = \#_{\mathbb{Z}_2}\mathcal{M}^J_L(c).$$

Spherical spinning

Let Λ be a closed, orientable Legendrian submanifold of $\mathbb{R}^{2n+1}_{st}$ parameterized by $f_\Lambda : \Lambda \to \mathbb{R}^{2n+1}$ with

$$f_\Lambda(p) = (x_1(p), y_1(p), \dots, x_n(p), y_n(p), z(p))$$

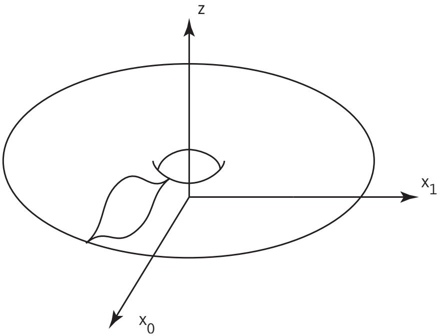
for $p \in \Lambda$. Without loss of generality assume that $x_1(p) > 0$ for all p. We define $\Sigma_{S^m}\Lambda$ to be the Legendrian submanifold of $\mathbb{R}^{2(m+n)+1}$ whose xz-projection is parametrized by $\Phi : \Lambda \times S^m \to \mathbb{R}^{n+m+1}$ with

$$\Phi(p, \theta, \overline{\phi}) = (\tilde{x}_{-m+1}(p, \theta, \overline{\phi}), \dots, \dots, \tilde{x}_1(p, \theta, \overline{\phi}), x_2(p), \dots, z(p)),$$

where $\theta \in [0, 2\pi)$, $\overline{\phi} = (\phi_1, \dots, \phi_{m-1}) \in [0, \pi]^{m-1}$ and

$$\begin{cases} \tilde{x}_{-m+1}(p, \theta, \overline{\phi}) = x_1(p) \sin\theta \sin\phi_1 \dots \sin\phi_{m-1}, \\ \tilde{x}_{-m+2}(p, \theta, \overline{\phi}) = x_1(p) \cos\theta \sin\phi_1 \dots \sin\phi_{m-1}, \\ \dots \\ \tilde{x}_1(p, \theta, \overline{\phi}) = x_1(p) \cos\phi_{m-1}. \end{cases}$$

Spherical spinning



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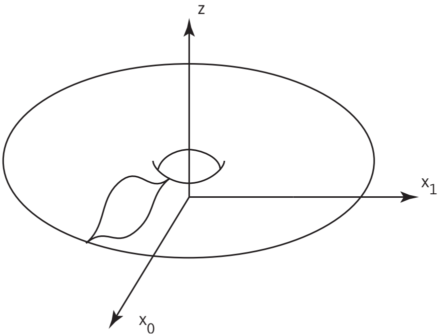

- From this definition it follows that $\Sigma_{S^m}\Lambda$ is diffeomorphic to $S^m \times \Lambda$.
- One can apply spherical spinning in a similar way to exact Lagrangian cobordisms and fillings. In particular, given an exact Lagrangian filling L of a Legendrian Λ, following the older result of [G], one gets an exact Lagrangian cobordism (filling) $\Sigma_{S^m}L$ of $\Sigma_{S^m}\Lambda$ such that $\Sigma_{S^m}L$ is diffeomorphic to $S^m \times L$.

An important statement that we need

We need the following statement for our consideration:

Theorem (Ekholm-Honda-Kalman, Karlsson)

Let L be an oriented, exact Lagrangian filling of closed Legendrian $\Lambda \subset \mathbb{R}^{2n+1}_{\xi_{st}}$ with Maslov number 0. Then L induces an augmentation $\varepsilon_L : (\mathcal{A}(\Lambda), \mathbb{Z}[H_1(L)]) \to \mathbb{Z}[H_1(L)]$ where $\mathbb{Z}[H_1(L)]$ lies entirely in grading 0. In addition, if L and L' are exact Lagrangian fillings of Λ which are isotopic through exact Lagrangian fillings of Λ, then there is a DGA homotopy between corresponding augmentations ε_L and $\varepsilon_{L'}$.

The class considered by Ng-Casals

- The class of Legendrian links considered by Casals and Ng is reasonably big, and in particular it contains the knot types 10_{139}, $m(10_{145})$, $m(10_{152})$, 10_{154}, and $m(10_{161})$.

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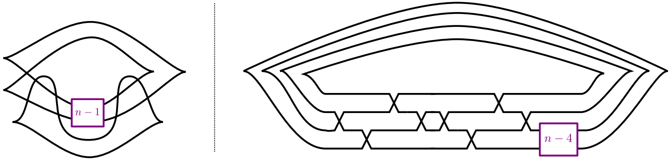

Figure: Two families of Legendrian links that are parts of the family considered by Casals-Ng. The boxes indicate a series of positive crossings.

Useful observation due to Ng-Casals

All Legendrians and exact Lagrangian fillings that appear in this class and that we consider have Maslov number 0. In addition, for all Legendrian submanifolds Λ all Reeb chords have non-negative degrees. In these settings, two DGA maps

$$(\mathcal{A}(\Lambda), \mathbb{Z}[H_1(L)]) \to \mathbb{Z}[H_1(L)]$$

are DGA homotopic iff they are equal. Hence if two fillings L, L' produce augmentations to $\mathbb{Z}[H_1(L)]$ that are distinct (under all isomorphisms identifying $H_1(L)$ and $H_1(L')$), then L, L' are not Hamiltonian isotopic.

Idea of proof

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- We assume that $\Sigma_{S^k}L$ and $\Sigma_{S^k}L'$ are Hamiltonian diffeomorphic, which implies that there is an isomorphism $\varphi : \mathbb{Z}[H_1(\Sigma_{S^k}L)] \to \mathbb{Z}[H_1(\Sigma_{S^k}L')]$ such that

$$\varphi \circ \varepsilon_{\Sigma_{S^k}L} = \varepsilon_{\Sigma_{S^k}L'}.$$

- In order to get contradiction we would like to reduce the previous equation to

$$\varphi \circ \varepsilon_L = \varepsilon_{L'}.$$

that will lead us to contradiction.

Idea of proof

We choose the perturbation of $\Sigma_{S^k}\Lambda$, we will still call it $\Sigma_{S^k}\Lambda$, for which there is a decomposition of Reeb chords $\mathcal{Q}(\Sigma_{S^k}\Lambda) = \mathcal{Q}_N \sqcup \mathcal{Q}_S$, where there is a canonical bijection between $\mathcal{Q}_S \simeq \mathcal{Q}(\Lambda)$ which preserves the index of the chords, and there is also a canonical bijection $\mathcal{Q}_N \simeq \mathcal{Q}(\Lambda)$, which increases the grading by k. Following the work of Dimitroglou Rizell-G we note that there is

$$i : \mathcal{A}(\Lambda) \hookrightarrow \mathcal{A}(\Sigma_{S^k}\Lambda),$$

which can be left inverted by the DGA map

$$\pi : \mathcal{A}(\Sigma_{S^k}\Lambda) \to \mathcal{A}(\Sigma_{S^k}\Lambda)/\langle \mathcal{Q}_N \rangle = \mathcal{A}(\Lambda)$$

which is given by taking the quotient of $\mathcal{A}(\Sigma_{S^k}\Lambda)$ with the two-sided ideal generated by $\mathcal{Q}_N$.

Idea of proof

- Let $Aug(\Lambda)$ and $Aug(\Sigma_{S^k}\Lambda)$ denote the sets of augmentations of $\mathcal{A}(\Lambda)$ and $\mathcal{A}(\Sigma_{S^k}\Lambda)$, respectively. Then it follows $i^* : Aug(\Sigma_{S^k}\Lambda) \to Aug(\Lambda)$ and $\pi^* : Aug(\Lambda) \to Aug(\Sigma_{S^k}\Lambda)$ are inverse maps and hence there is a one-to-one correspondence between augmentations of $\mathcal{A}(\Lambda)$ and $\mathcal{A}(\Sigma_{S^k}\Lambda)$, which to an augmentation ε on $\mathcal{A}(\Lambda)$ associates the augmentation $\tilde{\varepsilon}$ of $\mathcal{A}(\Sigma_{S^k}\Lambda)$ defined by $\tilde{\varepsilon}(c_S) = \varepsilon(c)$, $\tilde{\varepsilon}(c_N) = 0$ for $c \in \mathcal{Q}(\Lambda)$.

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- We prove the following Proposition $(*)$ saying that if ε is the augmentation induced by an exact Lagrangian filling L of Λ, then $\tilde{\varepsilon}$ is the augmentation induced by the exact Lagrangian filling $\Sigma_{S^k}L$ of $\Sigma_{S^k}\Lambda$.

Idea of proof

Then we apply i^* to the equation

$$\varphi \circ \varepsilon_L = \varepsilon_{L'}$$

and get

$$\varphi \circ i^*(\varepsilon_{\Sigma_{S^k} L}) = \varphi \circ \varepsilon_{\Sigma_{S^k} L} \circ i = i^*(\varphi \circ \varepsilon_{\Sigma_{S^k} L}) = i^*(\varepsilon_{\Sigma_{S^k} L'})$$

which using the Proposition $(*)$ transforms to

$$\varphi \circ \varepsilon_L = \varepsilon_{L'}.$$

Hence we get contradiction, and the main result follows.

Thank you for your attention!