Weyl sturcutes for path geometries

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From 2nd order ODEs to path geometries

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- View F as an assignment

$$TN = T\mathbb{R} \times T\underline{N} \ni (1_t, X(x)) \mapsto (\dot{1}_t, F(t, x, X(x)) \in TTN$$

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- Arbitrary parametrization for $t \Rightarrow$ record the data as

$$PTN \ni \ell \mapsto \text{ a line } E_\ell \subseteq T_\ell PTN$$

From 2nd order ODEs to path geometries II

- The ODE-system yields a (possibly locally defined) line bundle

$$E \subseteq TPTN : T_\ell \pi(E_\ell) = \ell \text{ for all } \ell \in PTN$$

with $\pi : PTN \to N$, $V := \ker(T\pi)$.

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- Moreover, leaves of E in PTN descends to solution graphs in $\mathbb{R} \times \underline{N}$.

Generalized path geometries

Definition

A *generalized path geometry* $(M, E \oplus V)$ consists of a $(2n+1)$-dimensional manifold M and subbundles $E \oplus V \subseteq TM$ of rank 1 and n, respectively, such that
(i) $[\eta, \eta'] \in \Gamma(E \oplus V)$ for all $\eta, \eta' \in \Gamma(V)$;
(ii) The Levi bracket $\mathcal{L} : (E \oplus V) \times (E \oplus V) \to TM/(E \oplus V)$ is nondegenerate in each fiber.

Note: $\mathcal{L} :=$ (projection) $\circ$ (Lie bracket on vector fields) is an anti-symmetric tensorial map, thus
(i)&(ii)$\Leftrightarrow$ $\mathcal{L}(V, V) = 0$ and $\mathcal{L}|_{E \times V}$ in each fiber is the standard scalar multiplication $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$.

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- When $dim(M) \neq 5$, V is automatically involutive; when $dim(M) = 5$, assume additionally that V is involutive, then by general theory, M is locally isomorphic to a path geometry on PTN for some N.

The corresponding parabolic geometry

- Categorical equivalence

$$
\begin{aligned}
&\{\text{generalized path geometries of dimension } (2n+1)\} \\
\leftrightarrow\ &\{\text{normal regular parabolic geometries of type } (G,P)\}
\end{aligned}
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- We have some canonical information on $(M^{2n+1}, E \oplus V)$ encoded on a principal P-bundle $\mathcal{G} \to M$ via $\omega \in \Omega^1(\mathcal{G},\mathfrak{g})$. We want to interpret this information on M.

We indicate the grading of $\mathfrak{g} := \mathfrak{sl}(n+2, \mathbb{R})$, block size in $(1, 1, n)$

$$
\left(\begin{array}{c|c|c}
\mathfrak{g}_0 & \mathfrak{g}_1^E & \mathfrak{g}_2 \\
\hline
\mathfrak{g}_{-1}^E & \mathfrak{g}_0 & \mathfrak{g}_1^V \\
\hline
 & & \\
\mathfrak{g}_{-2} & \mathfrak{g}_{-1}^V & \mathfrak{g}_0 \\
 & &
\end{array}\right)
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- Frame bundle $\mathcal{G}_0 := \mathcal{G}/P_+$ modeling $(TM/(E \oplus V)) \oplus E \oplus V$ over $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}^E \oplus \mathfrak{g}_{-1}^V$, respecting components and the Levi bracket.
  P_+: the strictly block upper triangular matrices in G.
  Structure group of $\mathcal{G}_0$ is G_0, the block-diagonal matrices in G.
  $\Rightarrow E^*, V^*, (TM/(E \oplus V))^*$ model over $\mathfrak{g}_1^E, \mathfrak{g}_1^V, \mathfrak{g}_2$

Weyl structures

- G_0-equivariant sections $\sigma : \mathcal{G}_0 \to \mathcal{G}$ (*Weyl Structures*), this brings the information on ω down to M via $\sigma^*\omega$

$$\sigma^*\omega_{\mathfrak{g}_-} \Rightarrow TM \cong gr(TM) \text{ i.e. transversal bundle of } E \oplus V$$
$$\sigma^*\omega_{\mathfrak{g}_0} \Rightarrow \text{principal connection on } \mathcal{G}_0$$
$$\sigma^*\omega_{\mathfrak{g}_+} \Rightarrow P : TM \to gr(T^*M) \Rightarrow P : TM \times TM \to \mathbb{R}$$

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  $\sigma^*\omega_{\mathfrak{g}_0}$ induces a linear connection on any associated vector bundle (*Weyl connections*), it is equivalent to the Weyl connections on E and V.
- Can use E to parametrize Weyl structures $\{$nowhere vanishing sections of $E\} \leftrightarrow$ $\{$exact Weyl structures$\} \subseteq \{$Weyl structures$\}$

- Fix any Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$, any other one is written as

$$\hat{\sigma} : u \mapsto \sigma(u) exp(\Upsilon(u))$$

  for a unique equivariant maps $\Upsilon : \mathcal{G}_0 \to \mathfrak{g}_+$. Write $\Upsilon = \Upsilon_1^E + \Upsilon_1^V + \Upsilon_2$.
- Fix any nowhere vanishing $\xi_0 \in \Gamma(E)$. The relation of the two Weyl connections ($\sigma \Rightarrow \nabla, \hat{\sigma} \Rightarrow \hat{\nabla}$) on E:

$$\begin{aligned}
\hat{\nabla}\xi_0 =& \nabla\xi_0 + \Upsilon_1^E \xi_0 \text{ in direction } E \\
\hat{\nabla}\xi_0 =& \nabla\xi_0 - \Upsilon_1^V \xi_0 \text{ in direction } V \\
\hat{\nabla}\xi_0 =& \nabla\xi_0 + \Upsilon_2 \xi_0 + \frac{3}{2}\Upsilon_1^E \otimes \Upsilon_1^V \xi_0 \\
& \text{in the transversal } (\sigma) \text{ direction}
\end{aligned}$$

  In particular, we established an injective assignment $\xi_0 \mapsto \sigma : \nabla\xi_0 = 0$ from non-vanishing sections of E to Weyl structures.

Theory behind: E is a bundle of scales

- An element $(a, b, \frac{-a-b}{n}\mathbb{I}_n) \in \mathfrak{z}(\mathfrak{g}_0)$ is a scaling element $\Leftrightarrow$ $a, b, \frac{-a-b}{n} \in \mathbb{R}$ are mutually distinct.
- The corresponding element for the line bundle $E = \mathcal{G}_0 \times_{G_0} \mathfrak{g}^E_{-1}$ is a scaling element $A := \frac{1}{6}(-1, 1, 0) \in \mathfrak{z}(\mathfrak{g}_0)$ i.e. $tr_{\mathfrak{g}}(ad(A)ad(B)) = ad(B)|_{\mathfrak{g}^E_{-1}}$ for all $B \in \mathfrak{g}_0$. Hence E is a bundle of scales.
- General theory: nowhere-vanishing $\xi_0 \in \Gamma(E) \to$ unique Weyl structure with $\nabla \xi_0 = 0$.

Geometric information on a distinct Weyl structure

- The canonical Cartan connection is characterized by the fact that its curvature κ lies in $\ker(\partial^*) \subseteq \Omega^2(M, \mathcal{G} \times_P \mathfrak{g})$, where $\partial^* : \Omega^2(M, \mathcal{G} \times_P \mathfrak{g}) \to \Omega^1(M, \mathcal{G} \times_P \mathfrak{g})$ is tensorial.
- The Weyl structure σ pulls back the curvature to $\kappa_\sigma \in \ker(\partial^*) \subseteq \Omega^2(M, \mathcal{G}_0 \times_{G_0} \mathfrak{g})$, with identification $\mathcal{G}_0 \times_{G_0} \mathfrak{g} = Q \oplus E \oplus V \oplus End_0(gr(TM)) \oplus E^* \oplus V^* \oplus Q^*$, where $Q := TM/(E \oplus V)$
- General theory$\Rightarrow$ some components (those of homogeneity 1 and 2) of κ_σ have to be zero.
- Vanishing of these components and $\partial^* \kappa_\sigma = 0$ provides equations on the components of a Weyl structure.

The projection $\Pi : TM \to E \oplus V$

Let $q : TM \to TM/(E \oplus V) =: Q$ denote the natural projection.

- $\Pi : TM \to E \oplus V$ is the identity on $E \oplus V$, and
- For $\eta \in \Gamma(V)$, $[\xi_0, \eta]$ is a lift of $\mathcal{L}(\xi_0, \eta) \in \Gamma(Q)$. One computes that

$$\begin{aligned} \Pi([\xi_0, \eta]) &\in \Gamma(V) \\ \mathcal{L}(\xi_0, \Pi([\xi_0, \eta])) &= \frac{1}{2} q([\xi_0, [\xi_0, \eta]]) \end{aligned}$$

  for all $\eta \in \Gamma(V)$.

We decompose $\Pi = \Pi_E + \Pi_V$ for $\Pi_E : TM \to E$, $\Pi_V : TM \to V$

The Weyl connection ∇

- ∇ can be described by how it behaves on the bundles E and V.
- By assumption $\nabla\xi_0 = 0$. This determines ∇ on E.
- ∇ on V: let $\eta, \eta' \in \Gamma(V), \zeta \in \Gamma(\ker \Pi)$ one computes that

$$\mathcal{L}(\xi_0, \nabla_{\xi_0}\eta) = \frac{1}{2} q([\xi_0, [\xi_0, \eta]])$$
$$\mathcal{L}(\xi_0, \nabla_{\eta'}\eta) = q([\eta', [\xi_0, \eta]])$$
$$\mathcal{L}(\xi_0, \nabla_{\zeta}\eta) = \mathcal{L}(\xi_0, \Pi([\zeta, \eta])) + P(\eta, \xi_0) q(\zeta)$$

The Rho tensor P on $(E \oplus V) \times (E \oplus V)$

Let $\eta, \eta' \in \Gamma(V)$.

$$P(\eta, \eta')\xi_0 = P(\eta', \eta)\xi_0 = \Pi_E([\xi_0, \eta], \eta')$$

$$-P(\xi_0, \eta)\xi_0 = 2P(\eta, \xi_0)\xi_0 = \Pi_E([\xi_0, [\xi_0, \eta]])$$

$$P(\xi_0, \xi_0) = \frac{1}{n} tr(V \to V, \eta \mapsto \Pi_V(\left[\xi_0, [\xi_0, \eta] - \Pi([\xi_0, \eta])\right]))$$

The Rho tensor P on other components

Let R denote the curvature of ∇. Let $\eta \in \Gamma(V), \zeta \in \Gamma(\ker \Pi)$ such that $\mathcal{L}(\xi_0, \eta) = q(\zeta)$. P on some other components:

$$
\begin{aligned}
P(\xi_0, \zeta) =& \frac{1}{n+2}\Big(\xi_0.P(\eta, \xi_0) - \eta.P(\xi_0, \xi_0) + tr_{\ker \Pi}(R(\xi_0, \cdot)\zeta)\Big) \\
P(\zeta, \xi_0)\xi_0 =& \frac{1}{n+2} R(\zeta, \xi_0)\xi_0 \\
& + \frac{\xi_0}{n+2}(\xi_0.P(\eta, \xi_0) - \eta.P(\xi_0, \xi_0) \\
& + tr_V(\Pi_V([\zeta, (id - \Pi)([\xi_0, \cdot])])))
\end{aligned}
$$

and other components of P expressed in terms of R, Π and P