On boundary values of Poisson transforms for differential forms

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Overview

Motivation

Let $G = \mathrm{SO}_0(n+1,1)$ and $K \cong \mathrm{SO}(n+1)$ its maximal compact subgroup. Then G/K is the real hyperbolic space of dimension $n+1$, which has a natural boundary sphere S^n at infinity.

The Poisson transform Φ_w assigns to each complex parameter $w \in \mathbb{C}$ and smooth function $f \in C^\infty(S^n)$ a smooth map $\Phi_w(f) \in C^\infty(G/K)$ which is an eigenfunction for the Laplace. Moreover, taking the limit of $\Phi_w(f)$ to a boundary point $u \in S^n$ we recover the initial data $f(u)$ up to a nontrivial factor.

Let $\mathcal{E}[w] \to G/P$ denote the bundle of w-densities. Then we can view the Poisson transform as a map $\Gamma(\mathcal{E}[w]) \to C^\infty(G/K)$, which is equivariant with respect to the natural G-actions.

In [4], Gaillard generalized this construction to intertwining operators

$$\Phi_k \colon \Omega^k(G/P, \mathcal{E}[w]) \to \Omega^k(G/K)$$

whose image consist of coclosed differential forms which are eigenforms of the Laplace operator. Moreover, it was shown that these transforms recover the initial boundary value (up to a factor) if and only if $k < \frac{n}{2}$.

More generally: G connected semisimple Lie group, $K \subset G$ maximal compact subgroup, $P \subset G$ parabolic subgroup, G/K of non-compact type. Then we can view G/P as (part of) the boundary of the symmetric space G/K at infinity.

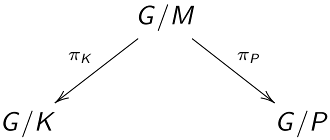
Question: Given a generalized Poisson transform

$$\Phi \colon \Omega^k(G/P) \to \Omega^\ell(G/K),$$

does Φ have boundary values?

Construction of Poisson transforms I

Via the Iwasawa decomposition we obtain $G/K \times G/P \cong G/M$ with $M := K \cap P$ and thus a double fibration



with G-equivariant projections.

The product structure induces a pointwise decomposition

$$\Lambda^k T^*(G/M) \cong \bigoplus_{p+q=k} (\Lambda^p T^*(G/K)) \otimes (\Lambda^q T^*(G/P)) .$$

Thus, we have a natural notion of a (p,q)-form on G/M. In particular, for $n := \dim(G/P)$ we can integrate (ℓ, n)-forms over the compact fibre of π_K, which is isomorphic to G/P.

Construction of Poisson transforms II

Fix a differential form $\varphi \in \Omega^{\ell,n-k}(G/M)$. For $\alpha \in \Omega^k(G/P)$ we construct $\Phi(\alpha) \in \Omega^\ell(G/K)$ as follows:

1. Consider the pullback $\pi_P^*\alpha \in \Omega^{0,k}(G/M)$.
2. Form the wedge product $\varphi \wedge \pi_P^*\alpha \in \Omega^{\ell,n}(G/M)$,
3. Integrate over G/P.

Definition (Poisson transform)

For all $\varphi \in \Omega^{\ell,n-k}(G/M)^G$ we call the intertwining operator

$$\Phi \colon \Omega^k(G/P) \to \Omega^\ell(G/K), \qquad \alpha \mapsto \int_{G/P} \varphi \wedge \pi_P^*\alpha.$$

the *Poisson transform* associated to the *Poisson kernel* φ.

Reduction to representation theory

Theorem

Let $\mathfrak{g} = Lie(G)$ and $\mathfrak{m} = Lie(M)$. There is a bijective correspondence between:

1. *Poisson transforms $\Phi\colon \Omega^k(G/P) \to \Omega^\ell(G/K)$,*
2. *M-invariant elements in $\Lambda^{\ell,n-k}(\mathfrak{g}/\mathfrak{m})^*$,*
3. *G-equivariant maps $\Lambda^{0,k}T^*(G/M) \to \Lambda^{\ell,0}T^*(G/M)$,*
4. *M-equivariant maps $\Lambda^{0,k}(\mathfrak{g}/\mathfrak{m})^* \to \Lambda^{\ell,0}(\mathfrak{g}/\mathfrak{m})^*$.*

For boundary values: focus on PT which descend to the BGG-complex.

BGG-complex I

The cotangent bundle $T^*(G/P)$ is naturally a bundle of Lie algebras. Thus, differentials in Lie algebra homology induce G-equivariant bundle maps

$$\partial^* = \partial_k^* \colon \Lambda^k T^*(G/P) \to \Lambda^{k-1} T^*(G/P),$$

called the *Kostant codifferential*.

The subbundles $\ker(\partial_k^*) \supset \operatorname{im}(\partial_{k+1}^*)$ are G-invariant, so their quotient bundle

$$\mathcal{H}_k(G/P) := \ker(\partial_k^*)/\operatorname{im}(\partial_{k+1}^*)$$

is also a homogeneous bundle which is completely reducible.

BGG-complex II

For all k there is a unique natural differential operator

$$L = L_k \colon \Gamma(\mathcal{H}_k(G/P)) \to \Gamma(\ker(\partial^*))$$

with $\pi \circ L = \mathrm{id}_{\mathcal{H}_k(G/P)}$ and $\partial^* \circ d \circ L = 0$. ("splitting operator").

$$\begin{array}{ccccccc}
\cdots & \underset{\partial^*}{\overset{d}{\rightleftarrows}} & \Omega^k(G/P) & \underset{\partial^*}{\overset{d}{\rightleftarrows}} & \Omega^{k+1}(G/P) & \underset{\partial^*}{\overset{d}{\rightleftarrows}} & \cdots \\
 & & \cup\!\mid & & \cup\!\mid & & \\
\cdots & & \Gamma(\ker(\partial^*)) & & \Gamma(\ker(\partial^*)) & & \cdots \\
 & \nearrow^{d\circ L} & \pi \downarrow \; \uparrow L & \nearrow^{d\circ L} & \pi \downarrow \; \uparrow L & \nearrow^{d\circ L} & \\
\cdots & \xrightarrow[D_{k-1}]{} & \Gamma(\mathcal{H}_k(G/P)) & \xrightarrow[D_k]{} & \Gamma(\mathcal{H}_{k+1}(G/P)) & \xrightarrow[D_{k+1}]{} & \cdots
\end{array}$$

The operator $D_k := \pi \circ d \circ L_k$ is called the *k-th BGG-operator*, and the lower line is called the *BGG-complex*.

Intertwining operators adapted to BGG-complexes

Assume there exists a smooth intertwining operator

$$\Phi\colon \Omega^k(G/P) \to \Omega^\ell(G/K)$$

which vanishes on $\mathrm{im}(\partial^*)$ and $\mathrm{im}(d\partial^*)$. Then Φ factors to

$$\underline{\Phi}\colon \Gamma(\mathcal{H}_k(G/P)) \to \Omega^\ell(G/K)$$

which is G-equivariant and satisfies:

1. For $\sigma \in \Gamma(\mathcal{H}_k(G/P))$ we have $\underline{\Phi}(\sigma) := \Phi(\alpha)$ for any $\alpha \in \pi^{-1}(\sigma)$,
2. For $\tau \in \Gamma(\mathcal{H}_{k-1}(G/P))$ we have $\underline{\Phi}(D_{k-1}\tau) = \Phi(d\beta)$ for any $\beta \in \pi^{-1}(\tau)$.

Theorem (H., 2018)

T.f.a.e:

1. $\Delta \circ \Phi = 0$
2. $\Phi \circ \partial^* = 0$ *and* $\Phi \circ d \circ \partial^* = 0$.

In this case we say that Φ *is BGG-compatible.*

PT for complex hyperbolic space.

Let $G = \mathrm{SU}(n+1,1)$ with maximal compact subgroup K and parabolic $P \subset G$. Then G/K is the complex hyperbolic space in (complex) dimension $n+1$ and $G/P \cong S^{2n+1}$ is the CR-sphere.

Let $H \subset T(G/P)$ be the contact subbundle and put $Q := T(G/P)/H$. Then for all $1 \leq k \leq 2n$ we have the short exact sequence

$$0 \longrightarrow \Lambda^{k-1}H^* \otimes Q^* \longrightarrow \Lambda^k T^*(G/P) \longrightarrow \Lambda^k H^* \longrightarrow 0.$$

The Kostant codifferential induces a bundle map $\Lambda^k H^* \to \Lambda^{k-2}H^* \otimes Q^*$ which is surjective for $k \leq n$ and injective for $k \geq n+1$. Thus, $\mathcal{H}_k$ are subbundles of $\Lambda^k H^*$ for $k \leq n$ and quotients of $\Lambda^{k-2}H^* \otimes Q$ for $k \geq n+1$.

After complexfication the irreducible components of $\mathcal{H}_{p+q} \otimes \mathbb{C}$ for $p+q \leq n$ are the bundles $\Lambda_0^{p,q}H^*$ of tracefree elements with respect to the Levi bracket.

Theorem (Čap, H., Julg, 2020)

Let $G = \mathrm{SU}(n+1,1)$, $K \subset G$ *maximal compact and* $P \subset G$ *parabolic.*

1. *If* $p+q \leq n$, *there is a unique BGG-compatible Poisson transform* $\Omega^{p+q}(G/P, \mathbb{C}) \to \Omega^{p,q}(G/K)$. *These induce intertwining operators*

$$\underline{\Phi}_{p,q} \colon \Gamma(\Lambda_0^{p,q} H^*) \to \Omega^{p,q}(G/K)$$

   whose image consist of harmonic, coclosed and primitive forms.

2. *If* $p+q \geq n+1$, *there is a 2-parameter family of Poisson transforms* $\Omega^{p+q}(G/P, \mathbb{C}) \to \Omega^{p,q}(G/K)$. *The image of the induced maps*

$$\underline{\Phi}^{\alpha,\beta}_{p,q} \colon \Gamma(\mathcal{H}_{p+q} \otimes \mathbb{C}) \to \Omega^{p,q}(G/K)$$

   consist of harmonic and coprimitive forms.

Boundary values of $\underline{\Phi}_{p,q}$

Consider the Poincaré ball model of G/K, i.e. open unit ball $B \subset \mathbb{C}^{n+1}$ endowed with the Bergman metric

$$g(z)(\xi, \eta) = \frac{\langle \xi, \eta \rangle}{1 - |z|^2} + \frac{\langle \xi, z \rangle \langle z, \eta \rangle}{(1 - |z|^2)^2}.$$

The topological boundary S^{2n+1} of B naturally identifies with G/P. We identify G/M with $B \times S^{2n+1}$.

Theorem (H., 2021)

For $0 \leq p, q \leq n$ with $k := p + q \leq n$ let $\alpha \in \Gamma(\Lambda_0^{p,q} H^*)$. For all $u \in S^{2n+1}$ let $\gamma \colon [0, T) \to \overline{B}$ be a continous curve with $\gamma(t) \in B$ for $t > 0$ and $\gamma(0) = u$. Then

$$\lim_{t \to 0} \underline{\Phi}_{p,q}(\alpha)(\gamma(t)) = C \cdot \alpha(u)$$

with a constant $C \neq 0$ depending on p, q and n.

Sketch of BV-proof I, $\alpha(u) = 0$

By K-equivariance of $\underline{\Phi}$: w.l.o.g. $u = eP$.

Case 1: $\alpha(u) = 0$.

1. Let $\phi_{p,q} \colon \Lambda^{0,k} T^*(G/M) \to \Lambda^{k,0} T^*(G/M)$ the G-equivariant map so that

$$\underline{\Phi}_{p,q}(\alpha) = \fint_{G/P} \phi_{p,q}(\pi_P^* \alpha)\,\mathrm{vol}_M$$

   with G-invariant form $\mathrm{vol}_M \in \Omega^{0,n}(G/M)^G$.

2. Let $\pi_K \colon G/M \to G/K$ projection. Choose an invariant metric g_M on G/M so that $g_M = \pi_K^* g$. Then $\phi_{p,q}$ is conformal with respect to g_M.

Sketch of BV-proof II, $\alpha(u) = 0$

3. For $u \in S^{2n+1}$ compute the change of $\|\alpha(u)\| \, \mathrm{vol}_M$ along the fibre $\pi_P^{-1}(u) = B \times \{u\}$. This is governed by $P(z,u)^{n+1-\frac{k}{2}}$, where

$$P(z,u)^{n+1} = \left(\frac{1 - |z|^2}{|1 - \langle z, u \rangle|^2} \right)^{n+1}$$

is the *Poisson-Szegö kernel*. The function $P(z,u)^{n+1-\frac{k}{2}}$ is bounded on B if and only if $0 \leq k \leq n$.

4. Estimate

$$\|\Phi_{p,q}(\alpha)(z)\| \leq \int_{S^{2n+1}} P(z,v)^{n+1-\frac{k}{2}} \|\alpha(v)\| dv$$

and apply a simple ϵ, δ argument.

Sketch of BV-proof III; $\alpha(u) \neq 0$

Case 2: $\alpha(u) \neq 0$.

By Case 1 $\Rightarrow$ boundary value of α only depends on its value in $u \in S^{2n+1}$. Therefore: replace α with a "constant" differential form.

1. Let $\mathfrak{k} := \mathrm{Lie}(K)$ and put $\mathbb{V}_{p,q} := \Lambda_0^{p,q}(\mathfrak{g}/\mathfrak{k})^*$. Then the vector space $\mathrm{Hom}_M(\mathbb{V}_{p,q}, \Lambda_0^{p,q} H^*)$ is 1-dimensional. By Frobenius reciprocity this is isomorphic to

   $$\mathrm{Hom}_K(\mathbb{V}_{p,q}, \Gamma(\Lambda_0^{p,q} H^*))$$

   and for a fixed generator σ we can find $v \in \mathbb{V}_{p,q}$ so that $\sigma(v)$ coincides with α at $u \in S^{2n+1}$.

   Put w.l.o.g.: $\alpha = \sigma(v)$.

Sketch of BV-proof IV; $\alpha(u) \neq 0$

2. Consider polar decomposition (r, θ): $B \cong (0,1) \times S^{2n+1}$. By uniqueness in Step 1 we have

$$\underline{\Phi}_{p,q}(\sigma(v)) = f(r)(\theta^* \sigma(v))$$

   for $f\colon (0,1) \to \mathbb{R}$ smooth.
3. The property $\Delta \circ \underline{\Phi}_{p,q} = 0$ yields a hypergeometric ODE on $f(r)$. Its solution converges for $r \to 1$.

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