Recursive formulas for L_∞ homotopy transfer to non-minimal models

(work in progress)

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Incarnations of L_∞ algebras

- L_∞ algebras are a “homotopical” generalization of Lie algebras.
- Many different incarnations: multibrackets, free dg-coalgebra, free dg-algebra, derived brackets, ...
- Free dg-coalgebra: simple axioms, straightforward generalization of Chevalley-Eilenberg algebra, “large” underlying vector space.
- Multibrackets: direct generalization of (super-)Lie algebras, more complicated higher Jacobi identities, “small” underlying vector space.
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Generating function notation

- Start with ($\mathbb{R}$- or $\mathbb{C}$-) vector space V graded by (parity $\mathbb{Z}_2$, (ghost)# $\mathbb{Z}$).
  - Formally, we allow ourselves to extend the scalars of V to polynomials ($\mathbb{R}$ or $\mathbb{C}$)$[\epsilon_1, \epsilon_2, \ldots]$ in odd variables ϵ_1, ϵ_2, ... Multiplication by an ϵ swaps V_{odd} and V_{even}.
- **Poloarization:**
  - Adopt sign conventions for graded multilinear maps $[-, \cdots, -] \colon S^n V \to V$ that are compatible with extension of scalars.
  - **Obs.** Any graded multilinear $[-, \ldots, -] \colon S^n V \to V$ is determined by its values $[A, \ldots, A]$, for **even** A.
  - $S(V)_{\text{even}} = \operatorname{span}\langle A^n \mid A \in V_{\text{even}} \rangle$.
  - For **odd** B use $A = \epsilon B$, where $|\epsilon| = |B|$. **Ex.:** $\underbrace{\epsilon}_{\text{odd}} [\underbrace{B}_{\text{odd}}, -] := (-)^{|\epsilon|} [\underbrace{\epsilon B}_{\text{even}}, -]$.
- A sequence of n-ary brackets, $[]_0 = 0$, $[A]_1 = \mathbf{s}A$, $[A, B]_2$, ..., can be collected as the graded components of the functional

$$[-] \colon S(V) \to V, \quad [-] = []_0 \oplus [-]_1 \oplus [-,-]_2 \oplus [-,-,-]_3 \oplus \cdots$$

- Hence, to fix the total bracket $[-]$ it is sufficient to specify the values of $[e^A]$, where for even A

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Compact L_∞ axioms

- L_∞ algebra on V, $(V, [-],$ **"higher Jacobi"**$)$.
  $\iff$ Free unital, differential, graded commutative algebra on V^*
  $(S(V^*), \mathrm{D}, \mathbf{D^2 = 0})$, with $[-] = (\mathrm{D}|_{V^*})^*$.
- Normalize the pairing between $S(V)$ and $S(V^*)$ so that $\langle e^z, e^A \rangle = e^{\langle z, A \rangle}$, or $\langle z^n, A^n \rangle = n! \langle z, A \rangle^n$, where $z \in V^*$ and $A \in V$ are naturally paired.
- **Lem.** The higher Jacobi identities are equivalently expressed as

$$\mathrm{D}^2 = 0 \iff [e^A[e^A]] = 0.$$

In terms of n-ary brackets

$$\mathbf{s}^2 A = 0, \quad 2[A\mathbf{s}A] + \mathbf{s}[A^2] = 0, \quad 3[A^2\mathbf{s}A] + 3[A[A^2]] + \mathbf{s}[A^3] = 0,$$

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Grading conventions, Example

- **Convention:**
  - $|[-]_n| = |[-]| =$ (parity odd, ghost # 1), *à la* BV-BRST.
  - Front bracket is odd:

$$[\epsilon \cdots] = (-)^{|\epsilon|}[\cdots], \quad [\cdots \epsilon, \cdots] = [\cdots, \epsilon \cdots], \quad [\cdots \epsilon] = [\cdots]\epsilon.$$

  - Maurer-Cartan elements $A \in V$ $\left([e^A] = 0\right)$ live in (ghost) degree 0.
  - Other conventions correspond to parity or degree shifts.

- **Canonical example:** (Super-)dg-Lie algebra

$$(\mathfrak{g}, \mathbf{s}, [-,-]) \rightsquigarrow (\mathfrak{g}[\mathrm{odd}, -1], [-] = \mathbf{s} + [-,-])$$

canonically gives rise to an L_∞ algebra.

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L_∞ morphisms

- **Lem.** A dg-algebra homomorphism $\Lambda\colon S(W^*) \to S(V^*)$, translates to an L_∞ algebra homomorphism $\lambda = (\Lambda|_{V^*})^*\colon S(V) \to W$ satisfying

$$\Lambda \mathrm{D}_W = \mathrm{D}_V \Lambda \iff [e^{\lambda(e^A)}]_W = \lambda(e^A [e^A]_V).$$

- In terms of n-ary brackets and functionals

$$\mathbf{s}_W \lambda(A) = \lambda(\mathbf{s}_V A),$$

$$\frac{1}{2}(\mathbf{s}_W \lambda(A^2) + [\lambda(A)^2]_W) = \frac{1}{2}\lambda(2A\mathbf{s}_V A + [A^2]_V),$$

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The general formula is

$$\frac{1}{n!}[B_n(\lambda(A), \ldots, \lambda(A^n))]_W = \frac{1}{n!}\sum_{k=1}^{n} \frac{n!}{(n-k)!k!}\lambda(A^{n-k}[A^k]_V),$$

where $B_1(x_1, \ldots, x_n) = x_n + \cdots + x_1^n$ are the complete (exponential) Bell polynomials, defined by the generating function

$$\exp\left(\sum_{k=1}^{\infty} x_k \frac{t^k}{k!}\right) = \sum_{n=0}^{\infty} B_n(x_1, \ldots, x_n)\frac{t^n}{n!}.$$

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L_∞ homotopy transfer

- **Folklore:** Consider homotopy equivalent dg-vector spaces $\lambda_1 \colon (V, \mathbf{s}) \xrightarrow{\sim} (V', \mathbf{s}')$ and an L_∞ algebra structure $(V', [-]' = \mathbf{s}' + \cdots)$. Then there exists unique up to homotopy L_∞ algebra structure on V and and isomorphism $\lambda = \lambda_1 + \cdots \colon (V, [-] = \mathbf{s} + \cdots) \to (V', [-]')$.
- Closest precise statement known to me: Thm 10.3.{1,5}, Loday-Vallette ('12). Uses highly abstract language of operads, ∞-operads, (co-)bar constructions, etc.
- **Q:** In what sense the general homotopy transfer theorem provides explicit formulas?

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Homotopy transfer to minimal model

- **Lem.** *Given a homotopy equivalence* $\lambda_1\colon (V, \mathbf{s} = 0) \xrightarrow{\sim} (V', \mathbf{s}')$ *of dg-vector spaces, it extends to an* L_∞ *algebra morphism* $\lambda\colon (V, [-]) \to (V', [-]')$, *with* $\lambda|_V = \lambda_1$, $[-]_1 = \mathbf{s}$, $[-]'_1 = \mathbf{s}'$.

  Proof: Proceed by induction on arity. Induction hypothesis implies

  $$\lambda(e^A[e^A]) - [e^{\lambda(e^A)}]' = \lambda_1([A^n]) - \mathbf{s}'\lambda_n(A^n) + \mathrm{err(l.o.t)} = \mathrm{err(l.o.t)}$$

  The cohomology class of (err) fixes $[-]_n$, then we solve for λ_n. Consistency of (err) terms and inductive check of higher Jacobi are guaranteed by generating function identities.

- The output is similar to the minimal model brackets in the papers of Jurčo *et al* (e.g., review '20) and other references.

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Homotopy transfer to minimal model

- **Lem.** *Given a homotopy equivalence* $\lambda_1 \colon (V, \mathbf{s} = 0) \xrightarrow{\sim} (V', \mathbf{s}')$ *of dg-vector spaces, it extends to an* L_∞ *algebra morphism* $\lambda \colon (V, [-]) \to (V', [-]')$, *with* $\lambda|_V = \lambda_1$, $[-]_1 = \mathbf{s}$, $[-]'_1 = \mathbf{s}'$.

  Proof: Proceed by induction on arity. Induction hypothesis implies

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- **Hyp.** There exists an integer q such that $V|_{\#>q} = 0$, the graded vector space V is trivial for degrees $> q$.
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  Proof: Select $\lambda = \lambda_1$ and any extension $[-]$ of $\mathbf{s}$, so that $[-]_n|_{\#>nq} = 0$ and systematically correct it in a double induction on degree and arity

total#:			$\cdots$	$q-2$	$q-1$	q
$[-]_2$	$\cdots$	$*$	$*$	$*$	$*$	$*$
$[-]_3$	$\cdots$	$*$	$*$	$*$	$*$	$*$
$[-]_4$		$\leftarrow$	$\circledast$	$*$	$*$	$*$
$\vdots$				$\downarrow$		

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Thank you for your attention!

References

G. Barnich, R. Fulp, T. Lada, and J. Stasheff, "The sh Lie structure of Poisson brackets in field theory," *Communications in Mathematical Physics* **191** (1998) 585–601, `arXiv:hep-th/9702176`.

J.-L. Loday and B. Vallette, *Algebraic Operads*, vol. 346 of *Grundlehren der mathematischen Wissenschaften*.
Springer, Berlin, 2012.

C. Saemann, B. Jurčo, H. Kim, T. Macrelli, and M. Wolf, "Perturbative quantum field theory and homotopy algebras," in *Proceedings of Corfu Summer Institute 2019 "School and Workshops on Elementary Particle Physics and Gravity" — PoS(CORFU2019)*.
2020.
`arXiv:2002.11168`.