Holomorphic relative Hopf modules over the irreducible quantum flag manifolds

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GAČR project 20-17488Y

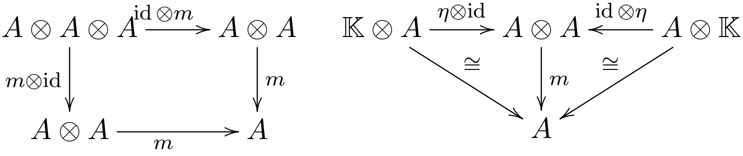
42th Winter School Geometry and Physics
Srni, Czech Republic
17 Jan 2022

The talk is based on

F. Díaz García, A. Krutov, R. Ó Buachalla, P. Somberg, K. R. Strung (2021), Holomorphic relative Hopf modules over the irreducible quantum flag manifolds.
Lett. Math. Phys., **111**(1), 24 p.
arXiv:2005.09652 [math.QA]

Hopf Algebras

An associative algebra over $\mathbb{K}$ is a 3-tuple (A, m, η)



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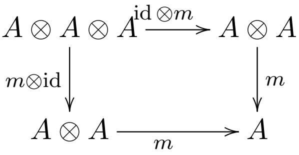
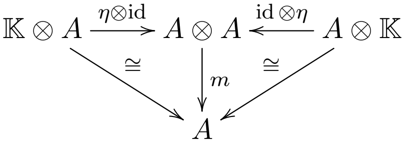
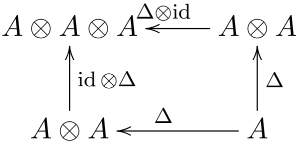
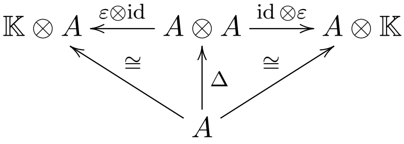




A coassociative coalgebra over $\mathbb{K}$ is a 3-tuple (A, Δ, ε)





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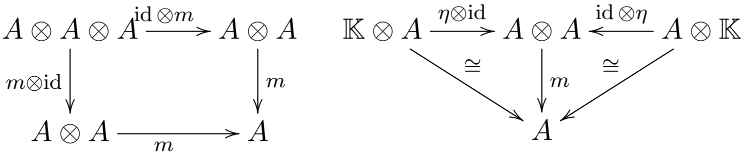

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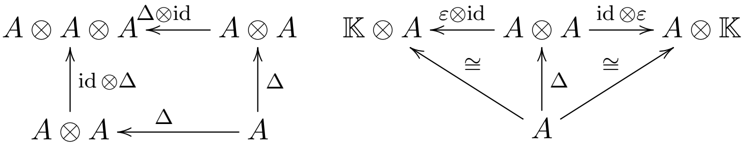

A Hopf algebra over $\mathbb{K}$ is a 6-tuple $(A, m, \eta, \Delta, \varepsilon, S)$, $S\colon A \to A$

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \varepsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta$$

Example

Let G be a group, $A = \mathbb{K}G$. For $g \in G$, we have

$$\Delta g = g \otimes g, \qquad \varepsilon(g) = 1, \qquad S(g) = g^{-1}.$$

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$$U(\mathfrak{g}) \qquad \longleftrightarrow \qquad \mathcal{O}(G)$$

Notation

Let H be a Hopf algebra

- $\varepsilon\colon H \to \mathbb{C}$
- $\Delta\colon H \to H \otimes H$, $\Delta h = \sum_i x_i \otimes y_i = h_{(1)} \otimes h_{(2)}$
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- $\Delta_R\colon V \to V \otimes H$ denotes a right H-coation of V

$$\Delta_R v = \sum v_i \otimes h_i = v_{(0)} \otimes v_{(1)}, \qquad v_i \in V, h_i \in H$$

- $\Delta_L\colon V \to H \otimes V$ denotes a left H-coaction of V

$$\Delta_L v = \sum h_i \otimes v_i = v_{(-1)} \otimes v_{(0)}, \qquad v_i \in V, h_i \in H$$

Let $H^+ := H \cap \ker \varepsilon$ and $h^+ = h - \varepsilon(h)1$ for $h \in H$.

A differential calculus over an algebra B is a dg-algebra $(\Omega^\bullet \simeq \bigoplus_{k\in\mathbb{Z}_{\geq 0}} \Omega^k, \mathrm{d})$ which is generated in degree 0 as a dg-algebra and such that $\Omega^0 = B$.

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A differential $*$-calculus is a DC equipped with a conjugate linear involutive map $*\colon \Omega^\bullet \to \Omega^\bullet$ satisfying

$$\mathrm{d}(\omega^*) = (\mathrm{d}\omega)^*,$$
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Note that if $(\Omega^\bullet, \mathrm{d})$ is a differential $*$-calculus over B, then B is a $*$-algebra.

Covariant DC

Let A be a Hopf algebra. A left A-comodule algebra B is a left A-comodule which is also an algebra, such that the comodule structure map

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A differential calculus $(\Omega^\bullet(B), \mathrm{d})$ over a right A-comdule algebra B is *covariant* if there exists a map $\Delta_L \colon \Omega^\bullet(B) \to A \otimes \Omega^\bullet(B)$ such that

$$\Delta_L(\mathrm{d}\omega) = (\mathrm{id} \otimes \mathrm{d}) \circ \Delta_L(\omega), \qquad \text{for all } \omega \in \Omega^\bullet(B).$$

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The associated *quantum homogeneous space* is defined to be the space of coinvariant elements

$$B = A^{\mathrm{co}(H)} := \Big\{ b \in A \mid \Delta_R b = b \otimes 1 \Big\}. \tag{2}$$

Connections

For $\Omega^\bullet$ a DC over an algebra B and $\mathcal{F}$ a finitely generated projective left B-module, a *connection* on $\mathcal{F}$ is a $\mathbb{C}$-linear map

$$\nabla\colon \mathcal{F} \to \Omega^1 \otimes_B \mathcal{F}$$

satisfying

$$\nabla(bf) = \mathrm{d}b \otimes f + b\nabla f, \qquad \text{for all } b \in B,\ f \in \mathcal{F}.$$

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Any connection can be extended to $\nabla \colon \Omega^\bullet \otimes_B \mathcal{F} \to \Omega^\bullet \otimes_B \mathcal{F}$

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The *curvature* of a connection is a left B-module map $\nabla^2\colon \mathcal{F} \to \Omega^2 \otimes_B \mathcal{F}$. A connection is said to be *flat* if $\nabla^2 = 0$. Since $\nabla^2(\omega \otimes f) = \omega \wedge \nabla^2(f)$, a connection is flat if and only if the pair $(\Omega^\bullet \otimes_B \mathcal{F}, \nabla)$ is a complex.

Complex stucture

A *complex structure* $\Omega^{(\bullet,\bullet)}$, for a differential $*$-calculus $(\Omega^\bullet, \mathrm{d})$, is an $\mathbb{N}_0^2$-algebra grading $\bigoplus_{(a,b)\in\mathbb{N}_0^2} \Omega^{(a,b)}$ for $\Omega^\bullet$ such that, for all $(a,b) \in \mathbb{N}_0^2$:

1. $\Omega^k = \bigoplus_{a+b=k} \Omega^{(a,b)}$,
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An element of $\Omega^{(a,b)}$ is called an (a,b)*-form*. For $\mathrm{proj}_{\Omega^{(a+1,b)}}$, and $\mathrm{proj}_{\Omega^{(a,b+1)}}$, the projections from Ω^{a+b+1} to $\Omega^{(a+1,b)}$, and $\Omega^{(a,b+1)}$ respectively, we write

$$\partial|_{\Omega^{(a,b)}} := \mathrm{proj}_{\Omega^{(a+1,b)}} \circ \mathrm{d}, \qquad \overline{\partial}|_{\Omega^{(a,b)}} := \mathrm{proj}_{\Omega^{(a,b+1)}} \circ \mathrm{d}.$$

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Thus $\left(\bigoplus_{(a,b)\in\mathbb{N}_0^2} \Omega^{(a,b)}, \partial, \overline{\partial}\right)$ is a double complex. Both ∂ and $\bar{\partial}$ satisfy the graded Leibniz rule. Moreover,

$$\partial(\omega^*) = \left(\bar{\partial}\omega\right)^*, \qquad \bar{\partial}(\omega^*) = \left(\partial\omega\right)^*, \qquad \text{for all } \omega \in \Omega^\bullet.$$

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In the classical setting the kernel of the holomorphic structure map coincides with the space of holomorphic sections of a holomorphic vector bundle. This motivates us to call

$$H^0_{\bar{\partial}}(\mathcal{F}) = \ker\left(\bar{\partial}_\mathcal{F} : \mathcal{F} \to \Omega^{(0,1)} \otimes_B \mathcal{F}\right),$$

the *space of holomorphic sections* of $(\mathcal{F}, \bar{\partial}_\mathcal{F})$.

Induced homogeneous vector bundles

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Let ${}^A_B\mathrm{mod}_0$ be the category whose objects are finite-dimensional left A-comodules $\Delta_L : \mathcal{F} \to A \otimes \mathcal{F}$, endowed with a B-bimodule structure, such that

1. $\Delta_L(bf) = \Delta_L(b)\Delta_L(f)$, for all $f \in \mathcal{F}, b \in B$,
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and whose morphisms are left A-comodule, B-bimodule, maps.

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Set $A\square_H V := \ker(\Delta_R \otimes \text{id} - \text{id} \otimes \Delta_L : A \otimes V \to A \otimes H \otimes V)$.

$$\Phi : {}^A_B\text{mod}_0 \to {}^H\text{mod}, \qquad \mathcal{F} \mapsto \mathcal{F}/B^+\mathcal{F},$$

$$\Psi : {}^H\text{mod} \to {}^A_B\text{mod}_0, \qquad V \mapsto A\square_H V,$$

where the left H-comodule structure of $\Phi(\mathcal{F})$ is given by $(\pi \otimes \text{id}) \circ \Delta_L$, and the B-module, and left A-comodule, structures of $\Psi(V)$ are defined on the first tensor factor.

Relative Hopf modules

A FODC $(\Omega^1(B), \mathrm{d})$ over $B = A^{\mathrm{co}(H)}$ is left-covariant if there exist a left A-coaction $\Delta_L \colon \Omega^1(B) \to A \otimes \Omega^1(B)$ giving $\Omega^1(B)$ the structure of an object in ${}^{A}_{B}\mathrm{mod}$ and such that d is a left A-comodule map.

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A complex structure $\Omega^{(\bullet,\bullet)}$ for $\Omega^\bullet$ is said to be *covariant* if the $\mathbb{N}_0^2$-decomposition of $\Omega^\bullet$ is a decomposition in the category ${}^{A}_{B}\mathrm{mod}$.

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Definition

A *holomorphic relative Hopf module* is a pair $(\mathcal{F}, \bar{\partial}_{\mathcal{F}})$ where $\mathcal{F} \in {}^A_B\mathrm{mod}$, $\bar{\partial}_{\mathcal{F}} : \mathcal{F} \to \Omega^{(0,1)} \otimes_B \mathcal{F}$ is a covariant $(0,1)$-connection, and $(\mathcal{F}, \bar{\partial}_{\mathcal{F}})$ is a holomorphic left B-module.

Drinfeld–Jimbo Quantum Groups I

Let $\mathfrak{g}$ be a complex simple finite-dimensional Lie algebra with Cartan matrix $A = (a_{ij})$.

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The *quantised enveloping algebra* $U_q(\mathfrak{g})$ is the noncommutative associative algebra generated by the elements E_i, F_i, K_i, and K_i^{-1}, for $i = 1, \ldots, r$, subject to the relations

$$K_i E_j = q_i^{a_{ij}} E_j K_i, \;\; K_i F_j = q_i^{-a_{ij}} F_j K_i, \;\; K_i K_j = K_j K_i, ,$$

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \;\; E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

along with the *quantum Serre relations*

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{q_i} E_i^{1-a_{ij}-r} E_j E_i^r = 0, \qquad \text{for } i \neq j,$$

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Drinfeld–Jimbo Quantum Groups II

A Hopf algebra structure is defined on $U_q(\mathfrak{g})$ by

$$\Delta(K_i) = K_i \otimes K_i,$$
$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,$$
$$S(E_i) = -E_i K_i^{-1}, \quad S(F_i) = -K_i F_i, \quad S(K_i) = K_i^{-1},$$
$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1.$$

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$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,$$
$$S(E_i) = -E_i K_i^{-1}, \quad S(F_i) = -K_i F_i, \quad S(K_i) = K_i^{-1},$$
$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1.$$

A Hopf $*$-algebra structure, called the *compact real form* of $U_q(\mathfrak{g})$, is defined by

$$K_i^* := K_i, \qquad E_i^* := K_i F_i, \qquad F_i^* := E_i K_i^{-1}.$$

Let $\mathcal{P}$ be the weight lattice of $\mathfrak{g}$, and $\mathcal{P}^+$ its set of dominant integral weights.

Drinfeld–Jimbo Quantum Groups III

Let $\mathcal{P}$ be the weight lattice of $\mathfrak{g}$, and $\mathcal{P}^+$ its set of dominant integral weights. For each $\mu \in \mathcal{P}^+$ there exists an irreducible finite-dimensional $U_q(\mathfrak{g})$-module V_μ uniquely defined by the existence of a vector $v_\mu \in V_\mu$, which we call a *highest weight vector*, satisfying

$$E_i \triangleright v_\mu = 0, \qquad K_i \triangleright v_\mu = q^{(\mu,\alpha_i)} v_\mu \qquad \text{for all } i = 1, \ldots, r.$$

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$$K_i \triangleright v = q^{(\mathrm{wt}(v),\alpha_i)} v, \qquad \text{for all } i = 1, \ldots, r.$$

Quantised Coordinate Algebras $\mathcal{O}_q(G)$

Let V be a finite-dimensional $U_q(\mathfrak{g})$-module, $v \in V$, and $f \in V^*$, the linear dual of V. Consider the function

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We call $\mathcal{O}_q(G)$ the *quantum coordinate algebra of* G, where G is the unique connected, simply connected, complex algebraic group having $\mathfrak{g}$ as its complex Lie algebra.

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From the Hopf algebra embedding $\iota : U_q(\mathfrak{l}_S) \hookrightarrow U_q(\mathfrak{g})$, we get the dual Hopf algebra map $\iota^\circ : U_q(\mathfrak{g})^\circ \to U_q(\mathfrak{l}_S)^\circ$. We have

$$\pi_S := \iota^\circ|_{\mathcal{O}_q(G)} : \mathcal{O}_q(G) \to U_q(\mathfrak{l}_S)^\circ,$$

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$$\pi : \mathcal{O}_q(G) \to \mathcal{O}_q(L_S), \qquad (3)$$

is called the *quantum flag manifold associated* to S and denoted by

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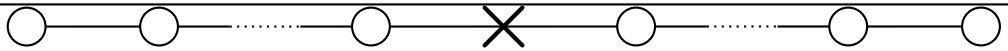
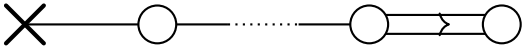
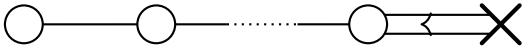
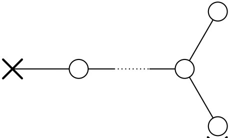
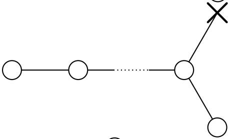
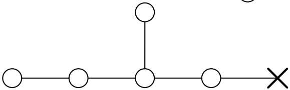
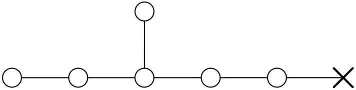
The pair $(\mathcal{O}_q(G/L_S), \Delta_L)$ is a left $\mathcal{O}_q(G)$-comodule algebra.

Irreducible Quantum Flag Manifolds

A_n		$\mathcal{O}_q(\mathrm{Gr}_{k,n+1})$
B_n		$\mathcal{O}_q(\mathrm{Q}_{2n+1})$
C_n		$\mathcal{O}_q(\mathrm{L}_n)$
D_n		$\mathcal{O}_q(\mathrm{Q}_{2n})$
D_n		$\mathcal{O}_q(\mathrm{S}_n)$
E_6		$\mathcal{O}_q(\mathbb{OP}^2)$
E_7		$\mathcal{O}_q(\mathrm{F})$

Theorem (Heckenberger–Kolb'2004)

Over any irreducible quantum flag manifold $\mathcal{O}_q(G/L_S)$, there exists a unique finite-dimensional left $\mathcal{O}_q(G)$-covariant differential $*$-calculus $\Omega_q^\bullet(G/L_S) \in {}^{\mathcal{O}_q(G)}_{\mathcal{O}_q(G/L_S)}\mathrm{mod}_0$, of classical dimension, that is to say, satisfying

$$\dim \Phi\Big(\Omega_q^k(G/L_S)\Big) = \binom{2M}{k}, \qquad \textit{for all } k = 0, \ldots, 2M,$$

where M is the complex dimension of the corresponding classical manifold.

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Proposition

1. $\Omega_q^\bullet(G/L_S)$ *admits precisely two left $\mathcal{O}_q(G)$-covariant complex structures, each of which is opposite to the other,*
2. *for each complex structure $\Omega^{(1,0)}$ and $\Omega^{(0,1)}$ are simple objects in ${}^{\mathcal{O}_q(G)}_{\mathcal{O}_q(G/L_S)}\mathrm{mod}_0$.*

Main Theorem

Theorem (Díaz García, K., Ó Buachalla, Somberg, Strung, 2021)

Let $\mathcal{O}_q(G/L_S)$ be an irreducible quantum flag manifold endowed with its Heckenberger–Kolb calculus, and $\mathcal{F} \in {}^{\mathcal{O}_q(G)}_{\mathcal{O}_q(G/L_S)}\mathrm{mod}_0$. It holds that

1. *$\mathcal{F}$ admits a left $\mathcal{O}_q(G)$-covariant connection $\nabla : \mathcal{F} \to \Omega^1_q(G/L_S) \otimes_{\mathcal{O}_q(G/L_S)} \mathcal{F}$, and this is the unique such connection if $\mathcal{F}$ is simple,*
2. *$\bar{\partial}_{\mathcal{F}} := \mathrm{proj}^{(0,1)} \circ \nabla$ is a left $\mathcal{O}_q(G)$-covariant holomorphic structure for $\mathcal{F}$, and this is the unique such holomorphic structure if $\mathcal{F}$ is simple.*

Further discussion

Theorem (Matassa, 2021)

For quantum projective spaces the corresponding connection coincides with the Levi–Civita connection for q-deformed analogues of the Fubini-Study metric.

Thank you