Symplectic twistor operators form complexes for Weyl-flat connections

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Paper Source and Motivation

- "Twistor ops. in symplectic spin geom." Adv. Appl. Cliff. Analysis, Springer, in print; contains the proof done on the white-board, preprint `https://www2.karlin.mff.cuni.cz/~krysl/Twist.pdf`
- Complexes of Dolbeault operators, or their complex conjugates
- Twistor operators in spin geometry for manifolds with $Spin(p,q)$-structure. [Penrose] for dimension four.
- Relations between curvature and integrability. Representation theory - because the geometrically defined sequences are associated by submodules to the principal bundles that capture the symmetry.
- Complexes of infinite rank bundles, cohomology questions: image with or without completion $\mathrm{Ker} D^i/\mathrm{Im} D^{i-1}$ or $\mathrm{Ker} D^i/\overline{\mathrm{Im} D^{i-1}}$? (Not touched here.)
- **Our aim:** Symplectic twistor sequences form complexes if their inducing connection is Weyl-flat.

Symplectic Vector Spaces

- (V, ω) real symplectic vector space of dimension $2n$
- $Sp(2n, \mathbb{R})$ symplectic group, $\pi_1(Sp(2n, \mathbb{R})) = \mathbb{Z}$
- There exists a unique connected Lie group that covers $Sp(2n, \mathbb{R})$ twice
- the metaplectic group, denoted by $\widetilde{G}$
- Let us choose a complex structure on (V, ω) such that $g(u, v) = \omega(Ju, v)$ is positive definite (adapted cplx str.), and a maximal ω-isotropic subspace $L \subseteq V$

Symplectic Spinors

- There is a distinguished faithful unitary representation of $\widetilde{G}$.
- We call it the symplectic spinor representation (oscillator, metaplectic) and denote it by (ρ, E), $\rho : \widetilde{G} \to U(E)$, where $E = L^2(L)$ is the Hilbert space of square integrable functions on the Euclidean vector space (L, g), (Berezin, [Shale], I. Segal, [Weil]). U denotes unitary operators. Realizable by minimal left ideals in

$$sCliff = T(V)/\langle v \otimes w - w \otimes v - \omega(v, w)1 | \, v, w \in V \rangle$$

- There is no faithful representation on a finite dimensional vector space.
- $E = E_+ \oplus E_-$, even and odd square integrable functions on the maximal isotropic L modulo ae., it is a decomposition into irreducibles

Symplectic spinor valued exterior forms

Notation:

- The double cover $\lambda : \widetilde{G} \to Sp(2n, \mathbb{R}) \simeq \lambda^* : \widetilde{G} \to \mathrm{Aut}(V^*)$ is a representation
- $\lambda^i : \widetilde{G} \to \mathrm{Aut}(\bigwedge^i V^*)$
- $\rho^i_\pm : \widetilde{G} \to \mathrm{Aut}(\bigwedge^i V^* \otimes E_\pm)$
- **Theorem** [Krysl, Lie Theory]: For each i, there are irreducible modules $E^{ij}_\pm$, $j = 0, \ldots, k_i = n - |n - i|$, such that

$$\bigwedge^i V^* \otimes E_\pm = E^i_\pm = \oplus_{j=0}^{k_i} E^{ij}_\pm.$$

- We set $E^{ij} = E^{ij}_+ \oplus E^{ij}_-$.

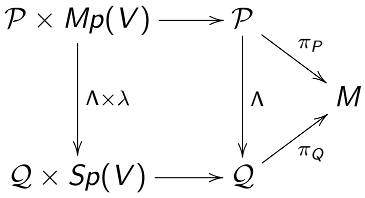
Decomposition - Continuation, Diagram

Representations $E^{ij}_{\pm}$ described by highest weight (of the $\mathfrak{g}$-structure on C^∞-vectors) with respect to Cartan subalgebra and positive roots, $2n = 6$.

E^0	E^1	E^2	E^3	E^4	E^5	E^6
E^{00}	E^{10}	E^{20}	E^{30}	E^{40}	E^{50}	E^{60}
	E^{11}	E^{21}	E^{31}	E^{41}	E^{51}	
		E^{22}	E^{32}	E^{42}		
			E^{33}			

Metaplectic structure

- $\mathcal{Q} = \{A : V \to T_m M | \, \omega(Av, Aw) = \omega_0(v, w), v, w \in V, m \in M\}$ bundle of symplectic frames, $p_Q : \mathcal{Q} \to M$
- $(\mathcal{P}, \Lambda)$ metaplectic structure if $p_P : \mathcal{P} \to M$ is principal $\widetilde{G}$-bundle and
- $\Lambda : \mathcal{P} \to \mathcal{Q}$ is a morphism of fibre bundles and for each $g \in \widetilde{G} : \Lambda(Ag) = \Lambda(A)\lambda(g)$
- Last equation: the diagram commutes



- Existence by [Forger, Hess]

Symplectic Connections

- **Definition:** Let (M, ω) be a symplectic manifold. A covariant derivative on TM is called symplectic if it preserves the symplectic form ($\nabla\omega = 0$). It is called *Fedosov* if it symplectic and torsion free.
- The affine space of Fedosov connections is in an affine bijection with $(\Gamma(S^3M), \text{zero section of } S^3M)$. See [Tondeur].

Symplectic Twistor Operators

- Let us set $E^{ij_i} = 0$ if $i \notin \{0, \ldots, 2n\}$ or $j_i \notin \{0, \ldots k_{n,i}\}$
- $p^{ij} : E^i \to E^{ij}$ well defined since E^i are multiplicity-free. The same symbol - when it acts on bundles and bundle sections
- **Definition:** For any (i,j) we set $T^{ij}_{\pm} = p^{i+1,j\pm1} \circ \nabla^{ij} : \Gamma(\mathcal{E}^{ij}) \to \Gamma(\mathcal{E}^{i,j\pm1})$ and call it the (i,j)th symplectic twistor operator.
- Parallel to Penrose's twistor operators in four dimensional (Lorentzian) spin geometry. They act one step down or up, and right in the decomposition diagram above, either up $(-)$, or down $(+)$.
- We would like to investigate $T^{i+k+1,j+k\pm1}_{\pm} \circ T^{i+k,j+k}_{\pm} = p^{i+k+2,j+k\pm2} \circ \nabla^{i+k+1,j+k\pm1} \circ p^{i+k+1,j+k\pm1} \circ \nabla^{i+k,j+k}$

Curvature Tensors

- Classical curvature tensor
  $R^{\nabla}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$
- $\sigma(X,Y) = \mathrm{Tr}(Z \mapsto R^{\nabla}(Z,X)Y)$, $\sigma_{ij} = {R^k}_{ijk} = {R^k}_{ikj}$, coordinates with respect to a local symplectic frame $(e_i)_{i=1}^{2n}$
- $\hat{\sigma}_{ijkl} = \frac{1}{2n+2}(\omega_{il}\sigma_{jk} - \omega_{ik}\sigma_{jl} + \omega_{jl}\sigma_{ik} - \omega jk\sigma_{il} + 2\sigma_{ij}\omega_{kl})$
- $W = R^{\nabla} - \hat{\sigma}$ (See [Vaisman].)
- **Definition:** A Fedosov connection is called *Weyl-flat* (or Ricci-type) if $W = 0$.

Curvature of Symplectic Spinor Derivative - General Principles

- symplectic spinor bundle $\mathcal{E}^i = \mathcal{P} \times_\rho E^i$, $\mathcal{E}^{ij} = \mathcal{P} \times_\rho E^{ij}$, $\mathcal{E} = \mathcal{E}^0$ symplectic spinors (sections are symplectic spinor fields, of Kostant)
- ∇ Fedosov connection $\longrightarrow$ associated cov. derivative ∇^0 on $\mathcal{E}$
- ∇^i exterior cov. derivative on $\mathcal{E}^i$, and ∇^{ij} on $\mathcal{E}^{ij}$
- i-th curvature $R^i = \nabla^{i+1}\nabla^i$. Total curvature $R = \sum_{i=0}^{2n-2} R^i$

Curvature for Weyl-Flat Connection

- $(\epsilon^i)_{i=1}^{2n}$ the dual frame to a local symplectic frame $(e_i)_{i=1}^{2n}$, $\epsilon^i \in \Gamma(U, TM)$, $U \subseteq M$ open
- $(e_i.s)(x^1, \ldots, x^n) = \imath x^i s(x)$, $e_{i+n}.s = \frac{\partial s}{\partial x^i}$, $1 \leq i \leq n$. See [Habermann, K., Habermann, L.]. So-called symplectic spinor multiplication.
- $F^+(\alpha \otimes s) = \frac{1}{2} \sum_{i=1}^{n} \epsilon^i \wedge \alpha \otimes e_i \cdot s$, $\alpha \otimes s \in \bigwedge^i V \otimes E$
- $E^+ = 2\{F^+, F^+\}$, anti-commutator. (See [Krysl, Monats] for a super-algebra setting.)
- $\Sigma^\sigma(\alpha \otimes s) = \sum_{i,j=1}^{2n} \sigma^i{}_j \epsilon^j \wedge \alpha \otimes e_i \cdot s$
- $\Theta^\sigma(\alpha \otimes s) = \sum_{i,j=1}^{2n} \alpha \otimes \sigma^{ij} e_i \cdot e_j \cdot s$
- **Lemma** (curvature, [SK, Monats]). If ∇ is a symplectic Weyl-flat connection, then $R = \frac{1}{n+2}(E^+\Theta^\sigma + 2F^+\Sigma^\sigma)$.

Parts of the Curvature in Diagram Decomposition

- F^+:

$$\underline{E^{i,j}} \xrightarrow{F^+} E^{i+1,j}$$

- Σ^σ:

$$\begin{array}{ccc} E^{i,j-1} & & E^{i+1,j-1} \\ & \nearrow & \\ \underline{E^{i,j}} & \longrightarrow & E^{i+1,j} \\ & \searrow & \\ E^{i,j+1} & & E^{i+1,j+1} \end{array}$$

- E^+:

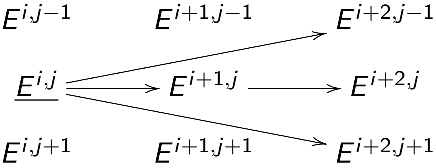
$$E^{i,j} \xrightarrow{\quad F^+ \quad} E^{i+2,j}$$

- Θ^σ:

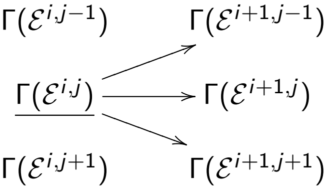
$$\begin{array}{c} E^{i,j-1} \\ \uparrow \\ \underline{E^{i,j}} \\ \downarrow \\ E^{i,j+1} \end{array}$$

Curvature and Connection

- Curvature $R = \frac{1}{n+2}(E^{+}\Theta^{\sigma} + 2F^{+}\Sigma^{\sigma})$:



- Cov. derivative ∇^{ij} :



Cov. derivative's target are right also if the connection has torsion and is pre-symplectic only.

The Theorem and its Proof

- **Theorem:** Let (M, ω) be symplectic manifold admitting a metaplectic structure and let ∇ a Weyl-flat Fedosov connection on (M, ω). Then for all pairs of integers (i, j), the sequences $(\Gamma(\mathcal{E}^{i+k, j\pm k}), T_{\pm}^{i+k, j\pm k})_{k\in\mathbb{Z}}$ form complexes.
  Proof. White-board or [Krysl, AACA]. □
- Parallel to: Let (M, J) be a complex manifold. Then for each couple of integers (i, j), the sequence $(\Gamma(E^{i,j+k}), \overline{\partial}_{i,j+k})_{k\in\mathbb{Z}}$ of holomorphic/antiholomorphic differential forms and Dolbeault differentials is a complex. Recall: Complex manifold := almost complex structure J is integrable (is complex), i.e. (Nirenberg–Newlander thm.), Nijenhuis tensor $N(X, Y) = [X, Y] + J[JX, Y] + [X, JY] - [JX, JY]$ of (M, J) is zero for all $X, Y \in \Gamma(TM)$: J complex $\Rightarrow N = 0$ is easy; opposite difficult.

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