One-jets of groupoids and higher Atiyah–Lie connections

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What is our goal?

To describe higher connections on higher principal bundles in the setting of simplicial model of higher geometry

Outline

- What is the playground? Tales about model categories and simplicial smooth manifolds
- From the parallel transport to the Atiyah's sequence
- Lie functor in the game
- Higher connections and higher gauge transformations

PART I. THE PLAYGROUND

Model categories and simplicial manifolds

- Appropriate higher category is the category of higher smooth stacks $\mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd})$ over the site Mfd of the smooth manifolds.
- We present this by a model category of simplicial presheaves on smooth manifolds $\mathsf{sPsh}(\mathsf{Mfd})$ with the local projective model structure.
- Most convenient for doing geometry are geometric smooth stacks – smooth stacks represented (in some way) by some higher smooth groupoid.
- Higher smooth groupoids are presented by Kan-fibrant simplicial smooth manifolds.

What are Kan-fibrant simplicial smooth manifolds?

Definition: The **simplicial smooth manifold** is a simplicial object in the category Mfd of the smooth manifolds, i.e. a contravariant functor $\mathcal{K}\colon \Delta^{op} \to \mathsf{Mfd}$.

Definition: A simplicial smooth manifold is called **Kan-fibrant simpicial smooth manifold** when it fullfils *Kan property*:

> The natural morphism $\mathsf{hom}(\Delta^n, \mathcal{K}) \to \mathsf{hom}(\Lambda^n_i, \mathcal{K})$ is a submersion $\forall 0 \leq i \leq n \in \mathbb{N}_0$.

Definition: **Simplicial smooth morphisms** are natural transformations between simplicial smooth manifolds which have all components being smooth functions.

These ingredients combined give a category of **Kan-fibrant simplicial smooth manifolds** which embeds into the category sPsh(Mfd) as a category of fibrant objects. It is sometimes referred to as the category of higher local Lie groupoids.

PART II. PARALLEL TRANSPORT AND ATIYAH

Parallel transport and Atiyah's sequence

According to the standard literature, a higher connection on a G-principal bundle $g : X \to \mathbf{BG}$ is a (π_0 of a) lift of the form

$$\begin{array}{ccc} X & \xrightarrow{\nabla} & \mathbf{BG}_{\mathrm{conn}} \\ & {\scriptstyle g}\searrow \quad \swarrow & \\ & \mathbf{BG} & \end{array}$$

in the category $\mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd})$.

Remark: Whenever this diagram exists in the category of $(n-1)$-stacks we say that it defines *n-connection*.

Parallel transport and Atiyah's sequence

Definition: We say that a connection $\nabla : X \to \mathbf{B}G_{\mathsf{conn}}$ admits n-parallel transport if for all maps $\psi : M \to X$ (where $\dim M = n$) the associated curvature form to $\nabla \circ \psi : M \to \mathbf{B}G_{\mathsf{conn}}$ vanishes.

Example (trivial): Any 1-connection admits 1-parallel transport since all 2-forms for $\dim M = 1$ vanishes.

Example: For 2-connections this condition (on admitting 2-parallel transport) accounts to have vanishing 2-form so called *fake curvature*

$$\mathrm{d}A + [A \wedge A] - t_*(B) = 0.$$

Claim: The class of n-connections which admit n-parallel transport is bijective to the class of morphisms $\{\mathcal{P}_n(X) \to \mathbf{BG}\}$ whenever $\mathcal{P}_n(X)$ called *path n-groupoid* of X is defined.

Classical fact: The class of flat n-connections is always bijective to the class of morphisms $\{\Pi_n(X) \to \mathbf{BG}\}$ where Π_n is the *fundamental n-groupoid* of X, which is defined as the nth truncation of ∞-fundamental groupoid:

$$\Pi_\infty := \mathrm{colim}\Big(\dots \underline{hom}(\Delta^2, X) \Rrightarrow \underline{hom}(\Delta^1, X) \rightrightarrows \underline{hom}(\Delta^0, X) \Big).$$

Functors of parallel transport $\nabla : \mathcal{P}_n(X) \to \mathbf{B}G$ are not convenient for presentation in 1-categories since $\mathbf{B}G$ is not geometric stack and thus not all morphisms with this target can be presented.

Parallel transport and Atiyah's sequence

The situation is not so bad, because every morphism $\nabla : \mathcal{P}(X) \to \mathbf{B}G$ admits a (1-connected, 1 truncated) decomposition

$$\begin{array}{ccc} \mathcal{P}(X) & \xrightarrow{\quad\nabla\quad} & \mathbf{B}G \\ & {\scriptstyle \nabla'}\searrow \quad \nearrow & \\ & \mathrm{im}_1(\nabla) & \end{array}$$

with $\mathrm{im}_1(\nabla) := \mathrm{colim}(\check{C}(\nabla))$ being geometric stack. The symbol $\check{C}(\nabla)$ stands for the Čech nerve of the mapping ∇. This object is in fact 1-groupoid object in $\mathrm{Sh}_{(\infty,1)}$. We call this *Atiyah–Lie groupoid*.

Parallel transport and Atiyah's sequence

In this way we can represent any connection (admitting parallel transport) by some ∇'. Moreover, $\mathrm{im}_1(\nabla)$ has a Kan-fibrant simplicial presentation, hence ∇' can be seen as a morphism in a 1-category (of simplicial presheaves on manifolds $\mathrm{sPsh}(\mathrm{Mfd})$).

Remark: In our work we simplify the situation and specialise on the case of flat connections.

Parallel transport and Atiyah's sequence

The idea to encode ordinary connections into (infinitesimal) morphism between the fundamental groupoid (or something locally equivalent) and the Atiyah–Lie groupoid goes back to work of M.Atiyah (57'). This is the famous exact sequence of Lie algebroids

$$0 \to P \times_G \mathfrak{g} \to TP/\mathsf{G} \xrightarrow{\rho} TM \to 0.$$

or the exact sequence of Lie groupoids respectively

$$0 \to P \times_{\mathsf{G}} \mathsf{G} \to P \times_{\mathsf{G}} P \xrightarrow{r} \mathsf{Pair}(M) \to 0$$

Parallel transport and Atiyah's sequence

We can recast and generalise this result by *presenting* higher analogues of this sequence of groupoids and by *differentiating* these Kan simplicial manifolds by methods of 1-jet (simplicial presentation of the Lie functor) developed by P.Ševera (06').

Claim: Presentation of 1-connected morphism ∇' for ordinary 1-categorical setting reproduces exactly the section of the Atiyah sequence of groupoids.

PART III. LIE FUNCTOR

Lie functor

The second Lie theorem says that ordinary Lie functor is fully-faithful. It is believed that the second Lie theorem survives the transition to higher category theory. This is why we want to investigate connections in the realm of (higher) algebroids rather than (higher) groupoids.

Lie functor

This functor presented as a 1-functor acting on the category of higher local Lie groupoids is called *one-jet* functor.

Theorem [Ševera '06]: Let us have $\mathscr{K}$ Kan-fibrant (finitely truncated) simplicial manifold. Then the **one-jet functor** $\mathrm{hom}(\check{\mathrm{C}}(\bullet), \mathscr{K}) : \mathrm{SSM}^{op} \to \mathrm{Set}$ is a representable presheaf on the category SSM of surjective submersions $Y \to X$ in SMfd.

Observation: If we moreover restrict this functor on the subcategory SSM_1 of surjective submersions of type $\mathbb{R}^{0|1} \times X \to X$, it is representable as the functor from the category of super-manifolds SMfd, moreover naturally furnished with the action of a monoid $\underline{\mathrm{hom}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$.

Hence, as a result we obtain super-manifold furnished with some homological vector field (giving the bracket structure) and the Euler vector field (defining gradation), in other words NQ-manifold (as a special case of dg manifold) aka L_∞-algebroid. This is exactly the algebroid associated with the Kan-fibrant simplicial manifold $\mathscr{K}$.

PART IV. HIGHER CONNECTIONS

Higher connections and gauge transformations

In the very general point of view, we want to investigate the following set

$$\pi_0 \ \mathrm{hom}^{@}_{\mathsf{dgMfd}}(J^1\mathcal{P}(M), J^1\underline{\mathrm{hom}}_{\mathsf{sPsh(Mfd)}}((\Delta^1)^{\times\bullet}, \mathsf{im}_1(\nabla)),$$

where π_0 is the zeroth homotopy group, $J^1(\bullet)$ stands for the one-jet functor, $\underline{\mathrm{hom}}_{\mathsf{sPsh(Mfd)}}((\Delta^1)^{\times n}, \bullet)$ is nth iterated path object remembering homotopies of the nth order, by the symbol @ we just indicate that we are interested in sections.

Higher connections and gauge transformations

For $n = 0$ it boils down to the one-jet of known ∇'.

Indeed:

$$\hom^{@}_{\mathsf{dgMfd}}(J^1\mathcal{P}(M), J^1\underline{\hom}_{\mathsf{sPsh(Mfd)}}((\Delta^1)^{\times 0}, \mathsf{im}_1(\nabla)) \cong \hom^{@}_{\mathsf{dgMfd}}(J^1\mathcal{P}(M), J^1\mathsf{im}_1(\nabla)),$$

what is nothing but the space of higher connections forgetting on their gauge equivalence classes.

Higher connections and gauge transformations

For $n = 1$ it gives the homotopy between two connections. We claim this is nothing but the first order gauge transformations, what follows from the more general result in the deformation theory saying that nth homotopy controls $n$fold gauge transformations.

RESULTS AND OUTLOOK

Results and outlook

- We have computed a local form of the NQ manifold structure of the higher Atiyah algebroid

  Example for the 2-truncated case:

  $$(T_{\sigma_1^{M_\bullet}(m_0)}M_1)\oplus Lie(G_0)[1]\oplus(\ker(T\mathsf{f}_1^{M_2}\restriction_{(\sigma_2^{M_\bullet})(m_0)})_{(T\tau)\circ(T\mathsf{f}_0^{M_2})}\times_{T\mathsf{f}_0^{G_1}\restriction_{e_1}}Lie(G_1))[2])$$

- We have showed that for simplicially constant bases (ordinary smooth manifolds) we obtain usual local description of higher gauge fields (Kalb-Ramond field, C-field, etc.)

Results and outlook

- We have computed the local relationship between $\mathbb{N}$-graded structures of $J^1\underline{\hom}_{\mathsf{sPsh(Mfd)}}((\Delta^1),\mathcal{K})$ and $J^1\mathcal{K}$ for general Kan-fibrant simplicial manifold $\mathcal{K}$:

$$\bigoplus_{q=0}^{P}\Big(\big(\bigcap_{i=1}^{q-1}\ker T_\bullet \mathsf{f}^{\mathcal{K}_{q+1}}_{\mathcal{A}(0,i)}\big)\times\big(\bigoplus_{j=1}^{q-1}\bigcap_{i=0}^{q-1}\ker T_\bullet \mathsf{f}^{\mathcal{K}_{q+1}}_{\mathcal{A}(j,i)}\big)\times\big(\bigcap_{i=0}^{q}\ker T_\bullet \mathsf{f}^{\mathcal{K}_{q+1}}_{\mathcal{A}(q,i)}\big)\Big).$$

Results and outlook

- One goal is to generalise our previous results to global description – this might bring some novel topological effects
- Introduce curved connections by considering higher path groupoid instead of the fundamental groupoid
- Understanding the structure of higher connections for higher basis, what necessarily brings corrections to the concept of ordinary connections

References

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THANK YOU!