Homogeneous 2-nondegenerate CR manifolds of hypersurface type and their modified CR symbols

David Sykes

Masaryk University

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Hypersurface-type CR Structures

Let M be a real hypersurface of $\mathbb{C}^{n+1}$ with odd dimension $2n+1$

$D = TM \cap iTM$ denotes the maximal complex subbundled of TM

$J : D \to D$ denotes multiplication by i

$H \subset \mathbb{C} \otimes_{\mathbb{R}} TM$ denotes the i-eigenspace of J

Definition

(M, H) is a Cauchy–Riemann (CR) manifold of hypersurface-type

Local Invariants of Generalized Levi Forms

The *Levi form* $\mathcal{L}$ is a field of Hermitian forms given by

$$\mathcal{L}(X_p, Y_p) := \frac{i}{2}\left[X, \overline{Y}\right]_p \pmod{H_p \oplus \overline{H}_p} \quad \forall X, Y \in \Gamma(H)$$

taking values in $\mathbb{C} \otimes_{\mathbb{R}} T_pM / \left(H_p \oplus \overline{H}_p\right) \cong \mathbb{C}$.
Let K denote the *Levi kernel*, i.e., the kernel of $\mathcal{L}$.

For $v \in K_p$ we define the antilinear operator $\mathrm{ad}_v : H_p/K_p \to H_p/K_p$ by taking $V \in \Gamma(K)$ such that $V_p = v$ and setting

$$\mathrm{ad}_v(X_p + K_p) := [V, \overline{X}]_p \mod K_p \oplus \overline{H}_p \qquad \forall X \in \Gamma(H).$$

H is *2-nondegenerate* if $K \neq 0$ and $\mathrm{ad}_v \neq 0$ for all $v \in K \setminus \{0\}$.

Truncated Levi-Tanaka Algebras

For $p \in M$, set

$$\mathfrak{g}_{-2}(p) := \mathfrak{g}_{-2,0}(p) := \mathbb{C}T_pM/(H_p \oplus \overline{H}_p), \quad \mathfrak{g}_{-1,1}(p) := H_p/K_p,$$

and

$$\mathfrak{g}_{-1,-1}(p) := \overline{H}_p/\overline{K}_p.$$

If H is 2-nondegenerate then, with $\mathfrak{g}_{-1}(p) := \mathfrak{g}_{-1,1}(p) \oplus \mathfrak{g}_{-1,-1}(p)$,

$$\mathfrak{g}_{-}(p) := \mathfrak{g}_{-2}(p) \oplus \mathfrak{g}_{-1}(p)$$

has the structure of a Heisenberg algebra with nontrivial Lie brackets given by

$$[v, \overline{w}] := i\mathcal{L}(v, w) \qquad \forall\, v, w \in \mathfrak{g}_{-1,1}(p).$$

Derivations of the Heisenberg Algebra

For $v \in K_p$, the map ad_v determines an element

$$\widetilde{\mathrm{ad}_v} \in \mathfrak{csp}(\mathfrak{g}_{-1}(p)) \cong \mathfrak{der}(\mathfrak{g}_-(p))$$

defined by

$$\widetilde{\mathrm{ad}_v}(x) := \begin{cases} 0 & \text{if } x \in H_p/K_p \\ \mathrm{ad}_v(\bar{x}) & \text{if } x \in \overline{H}_p/\overline{K}_p. \end{cases}$$

Define

$$\mathfrak{g}_{0,2}(p) := \left\{ \widetilde{\mathrm{ad}_v} \in \mathfrak{der}(\mathfrak{g}_-(p)) \,\middle|\, v \in K_p \right\},$$

$$\mathfrak{g}_{0,-2}(p) := \overline{\mathfrak{g}_{0,2}(p)},$$

and

$$\mathfrak{g}_{0,0}(p) := \left\{ v \in \mathfrak{der}\big(\mathfrak{g}_-(p)\big) \,\middle|\, [v, \mathfrak{g}_{i,j}(p)] \subset \mathfrak{g}_{i,j}(p)\, \forall (i,j) \right\}.$$

CR Symbols of 2-nondegenerate Structures

The *CR symbol* of a 2-nondegenerate, hypersurface-type structure H at $p \in M$, introduced in [1, Porter and Zelenko], is the space

$$\mathfrak{g}^0(p) := \mathfrak{g}_-(p) \oplus \mathfrak{g}_{0,0}(p) \oplus \mathfrak{g}_{0,-2}(p) \oplus \mathfrak{g}_{0,2}(p)$$

together with the involution induced on $\mathfrak{g}_-(p)$ induced by conjugation on $\mathbb{C}T_pM$

The CR symbol $\mathfrak{g}^0(p)$ is *regular* if it is a Lie subalgebra of $\mathfrak{g}_-(p) \rtimes \mathfrak{der}\big(\mathfrak{g}_-(p)\big)$.
In general $[\mathfrak{g}_{0,2}, \mathfrak{g}_{0,-2}]$ is not a subspace of $\mathfrak{g}^0$.

Adapted Frame Bundle

Fix a model abstract CR symbol $\mathfrak{g}^0 = \mathfrak{g}_- \oplus \mathfrak{g}_{0,-2} \oplus \mathfrak{g}_{0,0} \oplus \mathfrak{g}_{0,2}$.
$\mathrm{pr} : P^0 \to M$ is the bundle whose fiber $\mathrm{pr}^{-1}(p)$ over $p \in M$ is comprised of all *adapted frames at p*, which are the Lie algebra isomorphisms ψ satisfying

- $\psi : \mathfrak{g}_- \to \mathfrak{g}_-(p)$
- $\psi([y_1, y_2]) = [\psi(y_1), \psi(y_2)] \quad \forall\, y_1, y_2 \in \mathfrak{g}_-$
- $\psi(\mathfrak{g}_{i,j}) = \mathfrak{g}_{i,j}(p) \quad \forall\, (i,j) \in \{(-1, \pm 1), (-2, 0)\}$
- $\psi^{-1} \circ \mathfrak{g}_{0,\pm 2}(p) \circ \psi = \mathfrak{g}_{0,\pm 2}$
- ψ commutes with the CR symbols' involutions, i.e.,

$$\begin{array}{ccc} \mathfrak{g}_- & \xrightarrow{\quad\psi\quad} & \mathfrak{g}_-(p) \\ \Big\downarrow{\scriptstyle\iota} & \circlearrowright & \Big\downarrow{\scriptstyle\iota} \\ \mathfrak{g}_- & \xrightarrow{\quad\psi\quad} & \mathfrak{g}_-(p) \end{array}$$

The Levi Leaf Space

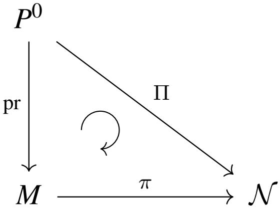
$K \oplus \overline{K}$ is integrable and induces *the Levi foliation* of M.

Define the *Levi leaf space* $\mathcal{N}$ to be the leaf space of this foliation, with natural projection denoted by

$$\pi : M \to \mathcal{N}.$$

and label

$$\Pi := \pi \circ \mathrm{pr}.$$



Modified CR Symbols

Fix $\psi_0 \in P^0$. Let $\psi : (-\varepsilon, \varepsilon) \to P^0_{\Pi(\psi_0)}$ be a curve in $P^0_{\Pi(\psi_0)} = \Pi^{-1}\big(\Pi(\psi_0)\big)$ with

$$\psi(0) = \psi_0,$$

and define $\theta_0 : T_{\psi_0} P^0_{\Pi(\psi_0)} \to \mathfrak{csp}(\mathfrak{g}_{-1})$ by

$$\theta_0\big(\psi'(0)\big) := \psi_0^{-1} \circ \psi'(0) = \left.\frac{d}{dt}\right|_{t=0} (\pi_* \circ \psi_0)^{-1} \circ (\pi_* \circ \psi(t)) .$$

The *modified CR symbol* of the structure H at $\psi_0 \in P^0$ is

$$\mathfrak{g}^{0,\mathrm{mod}}(\psi_0) := \mathfrak{g}_- \oplus \mathfrak{g}_0^{\mathrm{mod}}(\psi_0)$$

where

$$\mathfrak{g}_0^{\mathrm{mod}}(\psi_0) := \mathrm{span}_{\mathbb{C}} \left(\theta_0 \left(T_{\psi_0} P^0_{\Pi(\psi_0)} \right) \right) .$$

Reduced Modified CR Symbols (of homogeneous structures)

For a modified symbol $\mathfrak{g}^{0,\mathrm{mod}} \subset \mathfrak{g}_- \rtimes \mathfrak{csp}(\mathfrak{g}_{-1})$, define its reduction

$$\mathfrak{g}^{0,\mathrm{red}} = \mathfrak{g}_- \oplus \mathfrak{g}_0^{\mathrm{red}} \subset \mathfrak{g}^{0,\mathrm{mod}}$$

by

$$\mathfrak{g}_0^{\mathrm{red}} := V_s \subset V_{s-1} \subset \cdots \subset V_1 = \mathfrak{g}_0^{\mathrm{mod}}$$

where

$$V_{j+1} := V_j \cap N_{\mathfrak{csp}(\mathfrak{g}_{-1})}(V_j) \qquad \forall j \in \{1, \ldots, s\}$$

and $V_s = V_{s+1}$.

We call $\mathfrak{g}^{0,\mathrm{red}}$ a *reduced modified CR symbol (RMS)* of $\mathfrak{g}^{0,\mathrm{mod}}$.

Homogeneity criterion

Proposition

If (M, H) is homogeneous with a reduced modified symbol $\mathfrak{g}^{0,\mathrm{red}} = \mathfrak{g}_- \oplus \mathfrak{g}_0^{\mathrm{red}}$ then $\mathfrak{g}_0^{\mathrm{red}}$ is a subalgebra in $\mathfrak{csp}(\mathfrak{g}_{-1})$ with decomposition $\mathfrak{g}_0^{\mathrm{red}} = \mathfrak{g}_{0,-}^{\mathrm{red}} \oplus \mathfrak{g}_{0,0}^{\mathrm{red}} \oplus \mathfrak{g}_{0,+}^{\mathrm{red}}$ satisfying

- $\iota(\mathfrak{g}_0^{\mathrm{red}}) = \mathfrak{g}_0^{\mathrm{red}}$,
- $\iota(\mathfrak{g}_{0,-}^{\mathrm{red}}) = \mathfrak{g}_{0,+}^{\mathrm{red}}$,
- $[v, \mathfrak{g}_{-1,1}] \not\subset \mathfrak{g}_{-1,1} \qquad \forall\, v \in \mathfrak{g}_{0,-}^{\mathrm{red}}$
- $[v, \mathfrak{g}_{-1,-1}] \subset \mathfrak{g}_{-1,-1} \qquad \forall\, v \in \mathfrak{g}_{0,-}^{\mathrm{red}}$
- $[v, \mathfrak{g}_{-1,i}] \subset \mathfrak{g}_{-1,i} \qquad \forall\, v \in \mathfrak{g}_{0,0}^{\mathrm{red}},\ i \in \{-1, 1\}$.

An *abstract reduced modified CR symbol (ARMS)* is any subalgebra $\mathfrak{g}^{0,\mathrm{red}} \subset \mathfrak{g}_- \rtimes \mathfrak{csp}(\mathfrak{g}_{-1})$ with a structure as above.

Symmetry Bounds

The *universal Tanaka prolongation* $\mathfrak{u} = \bigoplus_{j \geq -2} \mathfrak{u}_j$ of $\mathfrak{g}^{0,\mathrm{red}}$ is given by

$$\mathfrak{u}_{-2} = \mathfrak{g}_{-2}, \quad \mathfrak{u}_{-1} = \mathfrak{g}_{-1}, \quad \text{and} \quad \mathfrak{u}_0 = \mathfrak{g}_0^{\mathrm{red}},$$

and, $\forall j \geq 0$,

$$\mathfrak{u}_j := \left\{ \varphi \in \bigoplus_{k=1}^{2} \hom\left(\mathfrak{u}_{-k}, \mathfrak{u}_{j-k}\right) \,\middle|\, \varphi([v,w]) = [\varphi(v), w] + [v, \varphi(w)] \right\}.$$

Theorem (D. S. and I. Zelenko)

If (M,H) is a homogeneous CR manifold with a reduced modified symbol $\mathfrak{g}^{0,\mathrm{red}}$ then

$$\dim \mathrm{Aut}(M,H) \leq \dim_{\mathbb{C}} \mathfrak{u}.$$

Symmetry Bounds

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Theorem ([4, D. S. and I. Zelenko])

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$$\dim \mathrm{Aut}(M,H) \leq \dim_{\mathbb{C}} \mathfrak{u}.$$

Symmetry Bounds for $\mathrm{Rank}(K) = 1$

Let (M, H) be homogeneous with CR symbol $\mathfrak{g}^0$, RMS $\mathfrak{g}^{0,\mathrm{red}}$, and $\dim(M) = 2n + 1$.

Theorem ([1, C. Porter and I. Zelenko])

If $\mathfrak{g}^0$ *is regular and* $\mathrm{rank}(K) = 1$ *then*

$$\dim \mathrm{Aut}(M, H) \leq n^2 + 7.$$

Theorem ([5, D. S. and I. Zelenko])

If $\mathfrak{g}^0$ *is not regular and* $\mathrm{rank}(K) = 1$ *then*

$$\dim \mathrm{Aut}(M, H) < (n-1)^2 + 7.$$

Matrix Representations of ARMS (with $\mathrm{rank}(K) = 1$)

Let $(e_0, e_1, \ldots, e_{2n-2})$ be a basis of $\mathfrak{g}_-$ with

$$\mathfrak{g}_{-2} = \mathrm{span}\{e_0\}, \quad \mathfrak{g}_{-1,1} = \mathrm{span}\{e_1, \ldots, e_{n-1}\},$$

$$\iota(e_0) = e_0, \quad \text{and} \quad \iota(e_j) = e_{j+n-1} \qquad \forall j \in \{1, \ldots, n-1\}.$$

Let $H_{\mathcal{L}}$ be the nondegenerate Hermitian matrix satisfying

$$[e_j, \iota(e_k)] = i(H_{\mathcal{L}})_{j,k} e_0 \qquad \forall j, k \in \{1, \ldots, n-1\}.$$

We have

$$\mathfrak{csp}(\mathfrak{g}_{-1}) = \left\{ \begin{pmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{pmatrix} + cI \;\middle|\; \begin{array}{l} X_{2,2} = -H_{\mathcal{L}}^{-1} X_{1,1}^T H_{\ell}, \\ X_{2,1} = H_{\mathcal{L}}^{-1} X_{2,1}^T \overline{H_{\mathcal{L}}}, \\ X_{1,2} = \overline{H_{\mathcal{L}}}^{-1} X_{1,2}^T H_{\mathcal{L}}, \text{ and} \\ c \in \mathbb{C} \end{array} \right\}.$$

Matrix Representations of ARMS (with $\mathrm{rank}(K) = 1$)

An ARMS $\mathfrak{g}^{0,\mathrm{red}}$ is represented by a tuple $(H_{\mathcal{L}}, C, \Omega, \mathscr{A})$:

C and Ω are $(n-1) \times (n-1)$ matrices, and $\mathscr{A}$ a matrix algebra

$$\mathscr{A} \subset \left\{ \alpha \;\middle|\; \begin{array}{l} \alpha C H_\ell^{-1} + C H_\ell^{-1} \alpha^T = \eta C H_\ell^{-1} \text{ and} \\ \alpha^T H_\ell \overline{C} + H_\ell \overline{C} \alpha = \eta' H_\ell \overline{C} \text{ for some } \eta, \eta' \in \mathbb{C} \end{array} \right\}.$$

such that

$$\mathfrak{g}^{\mathrm{red}}_{0,+} = \left\langle \begin{pmatrix} \Omega & C \\ 0 & -H_\ell^{-1} \Omega^T H_\ell \end{pmatrix} \right\rangle, \quad \mathfrak{g}^{\mathrm{red}}_{0,-} = \left\langle \begin{pmatrix} -\overline{H_\ell}^{-1} \Omega^* \overline{H_\ell} & 0 \\ \overline{C} & \overline{\Omega} \end{pmatrix} \right\rangle$$

and

$$\mathfrak{g}^{\mathrm{red}}_{0,0} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & -H_\ell^{-1} \alpha^T H_\ell \end{pmatrix} \;\middle|\; \alpha \in \mathscr{A} \right\} \cup \langle I \rangle.$$

Note $H_{\mathcal{L}} C = (H_{\mathcal{L}} C)^T$.

ARMS of Flat Structures in $\mathbb{C}^4$

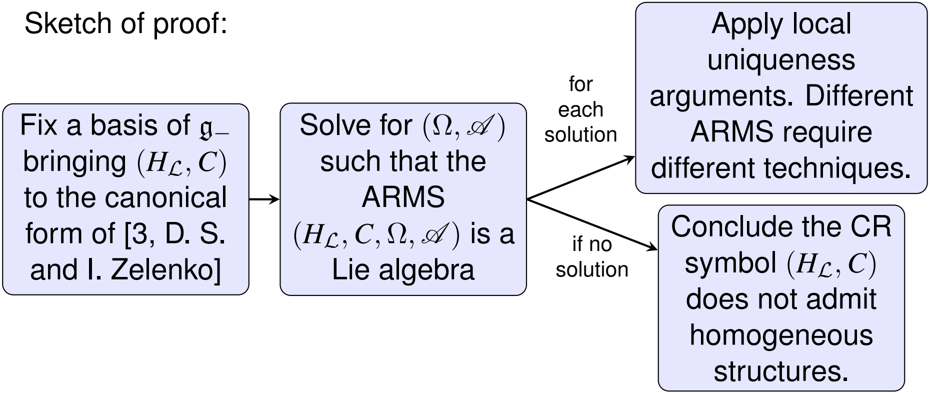
	$H_{\mathcal{L}}$	C	Ω	max. symmetry group dim.
i	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$	8
ii	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$	8
iii	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ \sqrt{\frac{3}{4}} & 0 \end{pmatrix}$	9
iv	$\begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & \eta \end{pmatrix}$	0	10 if $\eta = 0$ 15 if $\eta = 1$
v	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	0	15
vi	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	0	16

Here $\epsilon \in \{-1, 1\}$ and $\eta \in \{0, 1\}$

Theorem ([2, D. S.])

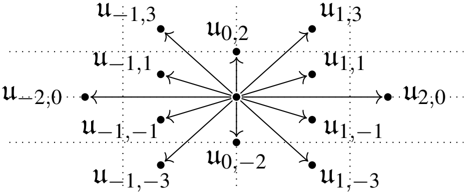
- *Every homogeneous 2-nondegenerate hypersurface in $\mathbb{C}^4$ has a RMS from the previous table.*
- *The maximally symmetric homogeneous structure with given RMS from the previous table is unique.*

Sketch of proof:



Symmetry Group and Prolongation Comparisons

- For flat structures of ARMS of type i and ii, $\dim\big(\mathrm{Aut}(M,H)\big) = \dim_{\mathbb{C}}(\mathfrak{u}) = 8$.
- For flat structures of ARMS of type iv with $\eta = 0$, $\dim\big(\mathrm{Aut}(M,H)\big) = 10$ and $\dim_{\mathbb{C}}(\mathfrak{u}) = \infty$.
- For flat structures of ARMS of type iii, $\dim\big(\mathrm{Aut}(M,H)\big) = 9$ and $\mathfrak{u}$ is the 14-d. exceptional Lie algebra $\mathfrak{g}_2$ having the root space decomposition



with $\mathfrak{u}_{-2,0} = \langle e_0\rangle$, $\mathfrak{u}_{-1,-3} = \langle e_4\rangle$, $\mathfrak{u}_{-1,-1} = \langle e_3\rangle$, $\mathfrak{u}_{-1,1} = \langle e_1\rangle$, $\mathfrak{u}_{-1,3} = \langle e_2\rangle$, $\mathfrak{u}_{0,-2} = \mathfrak{g}^{\mathrm{red}}_{0,-}$, $\mathfrak{u}_{0,2} = \mathfrak{g}^{\mathrm{red}}_{0,+}$, and $\mathfrak{u}_{0,0} = \mathfrak{g}^{\mathrm{red}}_{0,0}$.

References

[1] Curtis Porter and Igor Zelenko. Absolute parallelism for 2-nondegenerate CR structures via bigraded Tanaka prolongation. *Journal für die reine und angewandte Mathematik*, in press. doi: 10.1515/crelle-2021-0012.

[2] David Sykes (in preparation). Homogeneous 2-nondegenerate CR manifolds of hypersurface type in low dimensions.

[3] David Sykes and Igor Zelenko. A canonical form for pairs consisting of a hermitian form and a self-adjoint antilinear operator. *Linear Algebra and its Applications*, 590:32–61, 2020.

[4] David Sykes and Igor Zelenko. On geometry of 2-nondegenerate CR structures of hypersurface type and flag structures on leaf spaces of Levi foliations. *arXiv e-prints*, art. arXiv:2010.02770, October 2020.

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Thank you very much!

Extending and Linking ARMS

Given an ARMS $\mathfrak{g}^{0,\mathrm{red}} = \{H_\ell, C, \Omega, \mathscr{A}\}$, its *2-d. extensions* have the form $\widetilde{\mathfrak{g}^{0,\mathrm{red}}} = \{\widetilde{H_\ell}, \widetilde{C}, \widetilde{\Omega}, \widetilde{\mathscr{A}}\}$ with

$$\widetilde{H_\ell} = \left(\begin{array}{c:c} H_\ell & 0 \\ \hdashline 0 & \epsilon \end{array}\right), \widetilde{C} = \left(\begin{array}{c:c} C & 0 \\ \hdashline 0 & 0 \end{array}\right), \widetilde{\Omega} = \left(\begin{array}{c:c} \Omega & 0 \\ \hdashline 0 & 0 \end{array}\right),$$

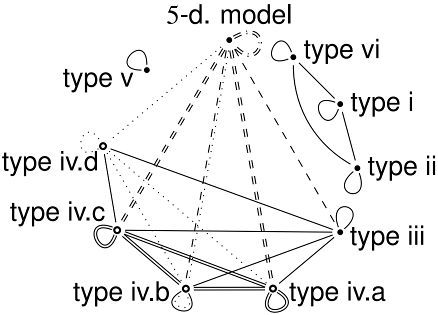
and given another ARMS $\hat{\mathfrak{g}}^{0,\mathrm{red}} = \{\hat{H}_\ell, \hat{C}, \hat{\Omega}, \hat{\mathscr{A}}\}$, the *links of $\hat{\mathfrak{g}}^{0,\mathrm{red}}$ and $\mathfrak{g}^{0,\mathrm{red}}$* are ARMS of the form $\widetilde{\mathfrak{g}^{0,\mathrm{red}}} = \{\widetilde{H_\ell}, \widetilde{C}, \widetilde{\Omega}, \widetilde{\mathscr{A}}\}$ with

$$\widetilde{H_\ell} = \left(\begin{array}{c:c} H_\ell & 0 \\ \hdashline 0 & \epsilon\hat{H}_\ell \end{array}\right), \quad \widetilde{C} = \left(\begin{array}{c:c} C & 0 \\ \hdashline 0 & \hat{C} \end{array}\right), \quad \widetilde{\Omega} = \left(\begin{array}{c:c} \Omega & 0 \\ \hdashline 0 & \hat{\Omega} \end{array}\right)$$

where $\epsilon = \pm$.

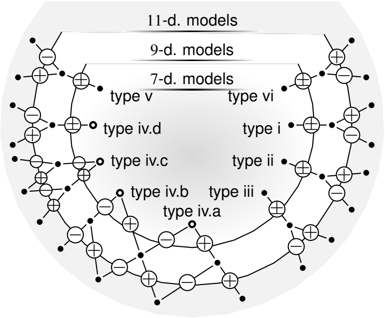
Extending and Linking ARMS of Flat Structures in $\mathbb{C}^3$ and $\mathbb{C}^4$

Linking ARMS



6 9-d. structures and
24 11-d. structures

Extending ARMS



11 9-d. structures and
19 11-d. structures

Hypersurface realizations of maximally symmetric models

The hypersurfaces

$$\left\{ (z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \;\middle|\; \Im(z_0 + z_1^2\overline{z_n}) = z_1\overline{z_2} + \overline{z_1}z_2 + \sum_{i=3}^{n-1} \varepsilon_i z_i \overline{z_i} \right\}$$

with $\varepsilon_i = \pm 1$ are 2-nondegenerate. Their Levi form's signature is determined by $\{\varepsilon_i\}$ and their algebras of infinitesimal symmetries attain the upper bound $\frac{1}{4}(\dim M - 1)^2 + 7 = n^2 + 7$.[1]

[1] reference: [1, C. Porter and I. Zelenko]