Relative tractor bundles for Lagrangean contact structures

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Definition

Let M be a $(2n+1)$-dimensional $\mathscr{C}^\infty$-manifold endowed with a contact structure (M, H). We write $TM = H \oplus Q$ for $Q = TM/H$.

Definition

A splitting of the contact subbundle into a direct sum of two rank-n subbundles $E \oplus F = H$ such that

$$\mathcal{L}|_{E\times E} = 0 \quad \text{and} \quad \mathcal{L}|_{F\times F} = 0,$$

is called a *Lagrangean contact structure.*

Remark

$\Rightarrow$ there exists of a canonical nondegenerate pairing $E \otimes F \to Q$,
$\Rightarrow$ $[\xi_1, \xi_2] \in \Gamma(H)$ for $\xi_i \in \Gamma(E)$ and $[\eta_1, \eta_2] \in \Gamma(H)$ $\eta_i \in \Gamma(F)$.

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Analogues

Lagrangian subbundles in symplectic geometry
The study of splittings of the contact distribution into a pair of $\mathcal{L}$-isotropic subbundles can be can be thought of as the contact analogue of the study of Lagrangian (maximally isotropic) subbundles in symplectic geometry.

Almost-CR structures
A nondegenerate almost-CR structure of hypersurface type consists of a contact manifold endowed with a complex structure on the contact subbundle, which implies that the complexification of it splits into two subbundles of complex rank n.

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Parabolic interpretation

Lie algebra $\mathfrak{sl}(n+2) =: \mathfrak{g}$ together with two nested parabolic subalgebras $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$, where $\mathfrak{q} = \mathfrak{p} \cap \tilde{\mathfrak{q}}$ (we choose of one side of the fibration).

decomposition of $\mathfrak{g}$ wrt. $\mathfrak{q}$

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$$\underbrace{\left\{\begin{pmatrix} \mathfrak{q}_0 & \mathfrak{q}_1^E & \mathfrak{q}_2 \\ \mathfrak{q}_{-1}^E & \mathfrak{q}_0 & \mathfrak{q}_1^F \\ \mathfrak{q}_{-2} & \mathfrak{q}_{-1}^F & \mathfrak{q}_0 \end{pmatrix}\right\}}_{\mathfrak{g}} \supset \underbrace{\left\{\begin{pmatrix} \mathfrak{q}_0 & \mathfrak{q}_1^E & \mathfrak{q}_2 \\ 0 & \mathfrak{q}_0 & \mathfrak{q}_1^F \\ 0 & \mathfrak{q}_{-1}^F & \mathfrak{q}_0 \end{pmatrix}\right\}}_{\mathfrak{p}} \supset \underbrace{\left\{\begin{pmatrix} \mathfrak{q}_0 & \mathfrak{q}_1^E & \mathfrak{q}_2 \\ 0 & \mathfrak{q}_0 & \mathfrak{q}_1^F \\ 0 & 0 & \mathfrak{q}_0 \end{pmatrix}\right\}}_{\mathfrak{q}}$$

$$\mathfrak{q}_+ := \mathfrak{q}_1^E \oplus \mathfrak{q}_1^F \oplus \mathfrak{q}_2 \qquad \mathfrak{p}_+ := \mathfrak{q}_1^E \oplus \mathfrak{q}_2$$

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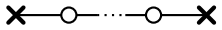

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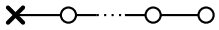

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Parabolic interpretation

Remarks

- A restriction of the Lie bracket induces an isomorphism $\mathfrak{q}^F_{-1} \cong \mathscr{L}(\mathfrak{q}^E_{-1}, \mathfrak{q}_{-2})$.
- The bracket $\mathfrak{q}_{-1} \times \mathfrak{q}_{-1} \to \mathfrak{q}_{-2}$ is nondegenerate, and thus, the grading is indeed contact.
- Viewing it as a symplectic form on $\mathfrak{g}_{-1}$, the subspaces $\mathfrak{g}^E_{-1}$ and $\mathfrak{g}^F_{-1}$ of $\mathfrak{g}_{-1}$ are Lagrangian.

Finally, Lie group $PSL(n+2) =: G$ together with two nested subgroups $Q \subset P \subset G$.

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Homogeneous model

$$\begin{array}{ccccc} & & F_{1,n+1}(\mathbb{R}) = G/Q & & \\ & \swarrow & & \searrow & \\ G/P = \mathbb{R}P^{n+1} & & & & \mathbb{R}P^{(n+1)*} = G/\widetilde{P} \end{array}$$

- The homogeneous model G/Q is the flag manifold $F_{1,n+1}(\mathbb{R})$ of lines in hyperplanes in $\mathbb{R}^{n+2}$.
- Mapping such a flag to its line makes $F_{1,n+1}$ into a fibre bundle over $\mathbb{R}P^{n+1}$ with fibre $\mathbb{R}P^{n*}$.
- Projecting to the hyperplane shows that $F_{1,n+1}$ is a fibre bundle over $\mathbb{R}P^{(n+1)*}$ with fibre $\mathbb{R}P^{n}$.

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- Projective structures on $(n+1)$-dimensional manifolds are equivalent to normal parabolic geometries of type (G, P).
- Those projective structures carry a canonical Lagrangean contact structure. Let N be an $(n+1)$-dim manifold, and

$$M := \mathcal{P}(T^*N) \xrightarrow{\pi} N.$$

M carries a canonical contact structure $H \subset TM$, and we obtain $H = E \oplus F$ by taking

$$F := \ker(T\pi), \quad E \leftrightsquigarrow [\nabla].$$

- In this setting, F is always integrable and the corresponding foliation of M is the foliation by the fibres.
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Relative setting

We know that natural bundles are induces by representations of Q. For our purpose, we need to introduce an important subclass called *relative* natural bundles.

Definition

Let $Q \subset P \subset G$ be nested parabolic subgroups and let $\mathbb{V}$ be a representation of Q. For the corresponding natural vector bundle $\mathcal{V} = \mathcal{G} \times_Q \mathbb{V}$ on parabolic geometries of type (G, Q),

- $\mathcal{V}$ is called a *relative natural bundle* if the subgroup $P_+ \subset Q$ acts trivially on $\mathbb{V}$;
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Relative bundles

Definition

$$\begin{array}{ll} \mathcal{A}_\rho M := \mathcal{G} \times_Q (\mathfrak{p}/\mathfrak{p}_+) & \text{relative adjoint bundle} \\ T_\rho M := \mathcal{G} \times_Q (\mathfrak{p}/\mathfrak{q}) & \text{relative tangent bundle} \\ T^*_\rho M := \mathcal{G} \times_Q (\mathfrak{q}_+/\mathfrak{p}_+) & \text{relative cotangent bundle} \end{array}$$

Definition

As $\mathfrak{g}/\mathfrak{p}$ is a completely reducible representation of P,

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Basic tangent bundle

Lemma

$\mathcal{V} = TM/F$ and $\mathcal{V} \cong \pi^*(TN)$.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{A}^0 M & \cdots & \mathcal{A}^0 M & & & & \\
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0 & \longrightarrow & \mathcal{A}_{\mathfrak{p}} M & \longrightarrow & \mathcal{A} M & \longrightarrow & \mathcal{V} & \longrightarrow & 0 \\
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Important remarks

- The bundle $\mathcal{V}$ is induced by an irreducible representation of P and one can construct all relative tractor bundles from it by tensorial constructions.
- Consequently, all relative tractor connections come from the one on $\mathcal{V}$.

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Relative tractor connection

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$\mathcal{V}$ is endowed with a relative tractor connection

$$\nabla^{\mathcal{V}}_{\mathrm{proj}_{T_\rho M}(m_\rho)}\sigma = D^\rho_{m_\rho}\sigma + m_\rho \bullet \sigma$$

for $m_\rho \in \Gamma(\mathcal{A}_\rho M)$, $\sigma \in \Gamma(TM/F)$, and D^ρ the *relative fundamental derivative.*

However, $\mathcal{V} \cong \mathcal{A}M/\mathcal{A}_\mathfrak{p}M$, where $\mathcal{A}_\mathfrak{p}M = \mathcal{G}\times_Q\mathfrak{p}$ (!!!)

Lemma †

For $\xi \in \Gamma(F)$, $s \in \Gamma(\mathcal{A}M)$, and $\nabla^{\mathcal{A}M}$ denoting the (usual) tractor connection on $\mathcal{A}M$

$$\nabla^{\mathcal{V}}_\xi \Pi(s) = \Pi(\nabla^{\mathcal{A}M}_\xi s).$$

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$$\nabla^{\mathcal{V}}_\xi \Pi(s) = \Pi(\nabla^{\mathcal{A}M}_\xi s).$$

Sketch of the proof †

- Does the RHS depend on $\Pi(s)$?
  - D^ρ is compatible with all natural bundle maps coming from Q-equivariant maps between the inducing representations;
  - $\bullet : \mathcal{A}M \times E \to E$ coincides with the algebraic bracket $\{\ ,\ \} : \mathcal{A}M \times \mathcal{A}M \to \mathcal{A}M$ if $E = \mathcal{A}M$;
  - so, indeed $\nabla_\xi^{\mathcal{A}M} s' \in \ker(\Pi)$ for $s' \in \Gamma(\mathcal{A}_\mathfrak{p} M)$.
- Does the equality hold?
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Bott connection

Theorem ‡

For $\xi \in \Gamma(F)$, $\sigma \in \Gamma(TM/F)$ and $\widetilde{\sigma}$ denoting any of its lifts to $\mathfrak{X}(M)$

$$\nabla^{\mathcal{V}}_{\xi}\sigma = \Pi([\xi, \widetilde{\sigma}] + \kappa(\xi, \widetilde{\sigma})).$$

In particular, when the bundle F is involutive

$$\nabla^{\mathcal{V}}_{\xi}\sigma = \Pi([\xi, \widetilde{\sigma}]).$$

Remark

The partial connection $\nabla^{\mathcal{V}}$ given by the Lie bracket is sometimes referred to as *Bott connection*.

Sketch of the proof ‡

- For $s \in \Gamma(\mathcal{A}M)$ and $m_{\mathfrak{p}} \in \Gamma(\mathcal{A}_{\mathfrak{p}}M)$ the projection $\Pi(D_s m_{\mathfrak{p}})$ vanishes identically, so we can write

$$\nabla^{\mathcal{V}}_{\xi}\sigma = \Pi(D_{m_{\mathfrak{p}}}s - D_s m_{\mathfrak{p}} + \{m_{\mathfrak{p}}, s\}).$$

- This coincides with the Lie bracket on $\mathcal{A}M$ induced from $\mathfrak{X}(\mathcal{G})^Q$, so $\nabla^{\mathcal{V}}_{\xi}\sigma = \Pi\left([m_{\mathfrak{p}}, s] + \kappa(\xi, \widetilde{\sigma})\right)$.
- $\Pi\left(\kappa(\xi, \widetilde{\sigma})\right)$ is the $\Lambda^2 F \otimes E$ component of the torsion.
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- For $s \in \Gamma(\mathcal{A}M)$ and $m_\mathfrak{p} \in \Gamma(\mathcal{A}_\mathfrak{p} M)$ the projection $\Pi(D_s m_\mathfrak{p})$ vanishes identically, so we can write

$$\nabla_\xi^\mathcal{V} \sigma = \Pi(D_{m_\mathfrak{p}} s - D_s m_\mathfrak{p} + \{m_\mathfrak{p}, s\}).$$

- This coincides with the Lie bracket on $\mathcal{A}M$ induced from $\mathfrak{X}(\mathcal{G})^Q$, so $\nabla_\xi^\mathcal{V} \sigma = \Pi\left([m_\mathfrak{p}, s] + \kappa(\xi, \widetilde{\sigma})\right)$.
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Distinguished connection

Definition

Given a contact form θ on M, the following formulae define partial connections ∇^E on the bundle E and ∇^Q on the line bundle Q (in F-directions)

$$\mathcal{L}(\nabla^E_{\eta_1}\xi, \eta_2) = -d\theta([\eta_1, \xi], \eta_2)q(r) \text{ and } \nabla^Q_{\eta_1} q(\psi) = [\eta_1 \cdot \theta(\psi)]q(r),$$

where r is the Reeb vector field, $\eta_1, \eta_2 \in \Gamma(F)$, $\xi \in \Gamma(E)$, and $\psi \in \Gamma(TM)$.

Relative tractor connection

Theorem

Given a contact form θ on M, the following slot-wise formula characterises a partial tractor connection $\nabla^{\mathcal{V}M}$ on $\mathcal{V}M$ in F-directions

$$\nabla_\xi (t)_\theta := \begin{pmatrix} \nabla^Q_\xi \rho + \mathcal{L}(\xi, \mu) \\ \nabla^E_\xi \mu - \theta(\rho)[\xi, r]_E \end{pmatrix},$$

where $\xi \in \Gamma(F)$, $\rho \in \Gamma(Q)$, and $\mu \in \Gamma(E)$.

Remarks

- The formula can be used as an alternative definition.
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