Differential Calculi on Quantum Principal Bundles over Projective Bases

Thomas Weber (University of Eastern Piedmont)

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Srni

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Collaboration with Paolo Aschieri, Rita Fioresi and Emanuele Latini

First Order Differential Calculi

A associative unital algebra (in general noncommutative) over
$\mathbb{k}$ commutative unital ring (e.g. $\mathbb{R}, \mathbb{C}, \mathbb{R}[[\hbar]], \mathbb{C}[[\hbar]]$).

Definition

We call (Γ, d) a first order differential calculus (FODC) on A, if

1. Γ is A-bimodule;
2. $\mathrm{d}\colon A \to \Gamma$ is $\mathbb{k}$-linear s.t.

$$\mathrm{d}(ab) = (\mathrm{d}a)b + a\mathrm{d}b \qquad \text{(Leibniz rule)}$$

   holds for all $a, b \in A$;
3. $\Gamma = A\mathrm{d}A := \mathrm{span}_{\mathbb{k}}\{\sum a\mathrm{d}b \mid a, b \in A\}$;

Example

i.) $A = \mathscr{C}^\infty(M)$, $\Gamma = \Gamma^\infty(T^*M)$, $\mathrm{d}\colon A \to \Gamma$ de Rham differential $\mathrm{d}f|_U = \frac{\partial f}{\partial x^i}\mathrm{d}x^i$ in local chart (U, x).

ii.) $A = \mathbb{C}_q\mathbb{S}^1 := \mathbb{C}[t, t^{-1}]$, $q \in \mathbb{C}^\times$ not root of unity $\Gamma = A\mathrm{d}t$, $\mathrm{d}t \cdot f(t) := f(qt)\mathrm{d}t$, and $\mathrm{d}f|_t := \frac{f(qt)-f(t)}{t(q-1)}\mathrm{d}t$, for f rational function in t.

Universal First Order Differential Calculus

Theorem (Woronowicz '89)

For any A associative unital algebra $\exists$ FODC (Γ_u, d_u) on A, where

$$\Gamma_u := \ker \mu_A = \left\{ \sum_i a^i \otimes b^i \mid \sum_i a^i b^i = 0 \right\} \subseteq A \otimes A$$

and $\mathrm{d}_u a = 1 \otimes a - a \otimes 1$ for all $a \in A$.
Moreover, if (Γ, d) is FODC on A, $\exists\, \mathcal{N} \subseteq \Gamma_u$ A-subbimodule s.t.
$\pi \colon \Gamma_u \to \Gamma_u / \mathcal{N} \cong \Gamma$ and $\mathrm{d} = \pi \circ \mathrm{d}_u$.

Proof.

$\pi \colon \Gamma_u \ni \sum_i (a^i \otimes b^i) \mapsto \sum_i a^i \mathrm{d} b^i \in \Gamma$ is surjective, because

$$\pi\left(\sum_i (a^i \otimes b^i - a^i b^i \otimes 1) \right) = \sum_i a^i \mathrm{d} b^i .$$

Then take $\mathcal{N} := \ker \pi$. □

Covariant Differential Calculi

(H, Δ, ϵ, S) Hopf algebra
(A, δ_A) right H-comodule algebra with coaction $\delta_A \colon A \to A \otimes H$.
We write $\delta_A(a) =: a_0 \otimes a_1$.

Definition

A FODC (Γ, d) on A is right H-covariant if

$$a\mathrm{d}a' \mapsto a_0 \mathrm{d}a'_0 \otimes a_1 a'_1 \tag{1}$$

for $a, a' \in A$ extends to a well-defined $\Bbbk$-linear map $\Gamma \to \Gamma \otimes H$.

Proposition

A FODC (Γ, d) on A is right H-covariant if and only if

- (Γ, Δ_Γ) *is a right H-covariant A-bimodule:* $\Delta_\Gamma(a \cdot \omega \cdot a') = \delta_A(a) \cdot \Delta_\Gamma(\omega) \cdot \delta_A(a')$
- d *is right H-colinear:* $\Delta_\Gamma \circ \mathrm{d} = (\mathrm{d} \otimes \mathrm{id}_H) \circ \delta_A$

Then Δ_Γ is determined by (1).

Lemma

Let (Γ, d) be a right H-covariant FODC on a right H-comodule algebra A.

i.) *An injective right H-comodule algebra map $\iota\colon A' \hookrightarrow A$ induces a right H-covariant FODC $(\Gamma_\iota, \mathrm{d}_\iota)$ on A', where*

$$\Gamma_\iota := \iota(A')\mathrm{d}\iota(A') \subseteq \Gamma$$

and $\mathrm{d}_\iota\colon A' \ni a' \mapsto \mathrm{d}\iota(a') \in \Gamma_\iota$.

ii.) *A surjective right H-comodule algebra map $\pi\colon A \to A'$ induces a right H-covariant FODC $(\Gamma_\pi, \mathrm{d}_\pi)$ on A', where*

$$\Gamma_\pi := \Gamma/\Gamma_I$$

is the A-bimodule quotient with $\Gamma_I := I\mathrm{d}A + A\mathrm{d}I$, where $I := \ker\pi \subseteq A$ and $\mathrm{d}_\pi\colon A' \ni \pi(a) \mapsto [\mathrm{d}a] \in \Gamma_\pi$.

iii.) *If ι is a section of π we have an isomorphism $(\Gamma_\iota, \mathrm{d}_\iota) \cong (\Gamma_\pi, \mathrm{d}_\pi)$ of right H-covariant FODC.*

We call $(\Gamma_\iota, \mathrm{d}_\iota)$ the pullback calculus, while we call $(\Gamma_\pi, \mathrm{d}_\pi)$ the quotient calculus.

Smash Product Calculus

For $(B, \triangleright)$ an H-module algebra the smash product $B\#H$ is an algebra with product $(b\#h)(b'\#h') := b(h_1 \triangleright b')\#h_2h'$ and unit $1\#1$.

A FODC (Γ_B, d_B) on an H-module algebra B is said to be an H-module FODC if Γ_B is an H-equivariant B-bimodule $[h \triangleright (b \cdot \omega \cdot b') = (h_1 \triangleright b) \cdot (h_2 \triangleright \omega) \cdot (h_3 \triangleright b')]$ and d_B is left H-linear $[h \triangleright \mathrm{d}_B(\omega) = \mathrm{d}_B(h \triangleright \omega)]$.

Theorem (Pflaum-Schauenburg '94)

Given an H-module FODC (Γ_B, d_B) on B and a bicovariant FODC (Γ_H, d_H) on H there is a right H-covariant FODC $(\Gamma_\#, \mathrm{d}_\#)$ on $B\#H$, where

$$\Gamma_\# := \Gamma_B \otimes H \oplus B \otimes \Gamma_H$$

is a $B\#H$-bimodule via

$$\begin{aligned}(b\#h) \cdot (\omega^B \otimes h') \cdot (b' \otimes h'') &= b(h_1 \triangleright \omega^B)(h_2 h'_1 \triangleright b') \otimes h_3 h'_2 h'',\\ (b\#h) \cdot (b' \otimes \omega^H) \cdot (b'' \otimes h') &= b(h_1 \triangleright b')(h_2 \omega^H_{-1} \triangleright b'') \otimes h_3 \omega^H_0 h'\end{aligned}$$

and

$$\mathrm{d}_\#(b\#h) := \mathrm{d}_B b \otimes h + b \otimes \mathrm{d}_H h.$$

Quantum Principal Bundles

$\mathrm{pr}\colon E \to M$ surjective morphisms of algebraic varieties, P affine group with associated Hopf algebra H.

Theorem (Pflaum '94)

pr *is P-principal bundle if and only if $\mathcal{F}(U) := \mathcal{O}_E(\mathrm{pr}^{-1}(U))$ defines a sheaf of right H-comodule algebras such that on an open cover $\{U_i\}$ of M*

1. $\mathcal{F}(U_i)^{\mathrm{co}H} \cong \mathcal{O}_M(U_i)$
2. $\mathcal{F}(U_i) \cong \mathcal{F}(U_i)^{\mathrm{co}H} \otimes H$

$(M, \mathcal{O}_M)$ quantum ringed space, H Hopf algebra.

Definition (Aschieri-Fioresi-Latini '21)

Sheaf $\mathcal{F}$ of right H-comodule algebras is (locally cleft) quantum principal bundle over M if there is open cover $\{U_i\}$ of M such that 1. and 2. hold.

Condition 2. says that $\mathcal{F}(U_i)^{\mathrm{co}H} \subseteq \mathcal{F}(U_i)$ is a cleft Hopf-Galois extension, i.e. there is a convolution invertible right H-colinear map $j\colon H \to \mathcal{F}(U_i)$.

If j is algebra map (locally trivial QPB) then $\mathcal{F}(U_i) \cong \mathcal{F}(U_i)^{\mathrm{co}H} \# H$ as comodule algebras, where $h \triangleright b := j(h_1) b j^{-1}(h_2)$.

Construction

G complex semisimple algebraic group, P parabolic subgroup.
$\Rightarrow G/P$ is projective variety and $G \to G/P$ principal bundle.
Let $\mathcal{O}_q(G)$, $\mathcal{O}_q(P)$ be Hopf algebra quantizations of $\mathcal{O}(G)$, $\mathcal{O}(P)$.

$s \in \mathcal{O}_q(G)$ is quantum section if $(\mathrm{id} \otimes \pi)\Delta(s) = s \otimes \pi(s)$ and $s = t \mod (q-1)$ for a classical section t (see Ciccoli-Fioresi-Gavarini '08). We write $\Delta(s) = s^i \otimes s_i$.
$\Rightarrow \{s_i\}$ determine an algebra $\mathcal{O}_q(G/P)$ and an open cover $\{U_i\}$ of $M = G/P$.

Define $U_I := U_{i_1} \cap \ldots \cap U_{i_r}$ for $I = (i_1, \ldots, i_r)$.

Theorem (Aschieri-Fioresi-Latini '21)

1. $U_I \mapsto \mathcal{O}_M(U_I) := \mathbb{C}_q[s_{k_1} s_{i_1}^{-1}, \ldots, s_{k_r} s_{i_1}^{-1}]$ *for* $1 \leq k_j \leq n$ *defines a sheaf* $\mathcal{O}_M$ *of algebras on* $M = G/P$.
2. $U_I \mapsto \mathcal{F}_G(U_i) := \mathcal{O}_q(G)\{s_i^r \mid r \leq 0\}$ *defines a sheaf* $\mathcal{F}_G$ *of right H-comodule algebras.*
3. $\mathcal{F}_G(U_i)^{\mathrm{co}\mathcal{O}_q(P)} = \mathcal{O}_M(U_i)$, *i.e.* $\mathcal{F}_G$ *is a QPB over* M, *possibly non-cleft.*

Example $\mathcal{O}_q(\mathrm{SL}_2)$ over $\mathcal{O}_q(\mathbf{P}^1(\mathbb{C}))$

$A := \mathcal{O}_q(\mathrm{SL}_2)$ free commutative algebra over $\mathbb{C}_q := \mathbb{C}[q, q^{-1}]$ generated by $\alpha, \beta, \gamma, \delta$ modulo

$$\begin{aligned}\alpha\beta =& q^{-1}\beta\alpha,\ \alpha\gamma = q^{-1}\gamma\alpha,\ \beta\delta = q^{-1}\delta\beta,\ \gamma\delta = q^{-1}\delta\gamma,\\ \beta\gamma =& \gamma\beta,\ \alpha\delta - \delta\alpha = (q^{-1} - q)\beta\gamma,\ \alpha\delta - q^{-1}\beta\gamma = 1\end{aligned}$$

with algebra $H := \mathcal{O}_q(P) := \mathbb{C}_q < t, t^{-1}, p > /(tp - q^{-1}pt)$ on parabolic subgroup P.

They are Hopf algebras with $\Delta \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \otimes \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and Hopf algebra projection $\pi \colon \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \to \begin{pmatrix} t & p \\ 0 & t^{-1} \end{pmatrix}$.

Example $\mathcal{O}_q(\mathrm{SL}_2)$ over $\mathcal{O}_q(\mathbf{P}^1(\mathbb{C}))$

Consider the topology $\{\emptyset, U_1, U_2, U_{12}, \mathbf{P}^1(\mathbb{C})\}$ on $\mathbf{P}^1(\mathbb{C})$.
We define the sheaves

$$\mathcal{F}(\emptyset) := \{0\},\ \mathcal{F}(U_1) := A[\alpha^{-1}],\ \mathcal{F}(U_2) := A[\gamma^{-1}],$$
$$\mathcal{F}(U_{12}) := (A[\alpha^{-1}])[\gamma^{-1}],\ \mathcal{F}(\mathbf{P}^1(\mathbb{C})) := A$$

of right H-comodule algebras and

$$\mathcal{O}_{\mathbf{P}^1(\mathbb{C})}(\emptyset) := \{0\},\ \mathcal{O}_{\mathbf{P}^1(\mathbb{C})}(U_1) := \mathbb{C}_q[\alpha^{-1}\gamma] = \mathbb{C}_q[u],$$
$$\mathcal{O}_{\mathbf{P}^1(\mathbb{C})}(U_2) := A[\gamma^{-1}\alpha] = \mathbb{C}_q[v],$$
$$\mathcal{O}_{\mathbf{P}^1(\mathbb{C})}(U_{12}) := \mathbb{C}_q[u, u^{-1}],\ \mathcal{O}_{\mathbf{P}^1(\mathbb{C})}(\mathbf{P}^1(\mathbb{C})) := \mathbb{C}_q$$

of algebras with restriction morphism $r_{2,12}\colon v \mapsto u^{-1}$.

$\Rightarrow \mathcal{F}$ is QPB over $\mathcal{O}_{\mathbf{P}^1(\mathbb{C})}$ with cleaving maps $j_1\colon t^{\pm} \mapsto \alpha^{\pm}$, $p \mapsto \beta$ and $j_2\colon t^{\pm} \mapsto \gamma^{\pm}$, $p \mapsto \delta$.

Here $\alpha \in \mathcal{O}_q(\mathrm{SL}_2)$ is the quantum section.

Calculi on Quantum Principal Bundles

Definition

A right H-covariant FODC on sheaf $\mathcal{F}$ of right H-comodule algebras is a sheaf Υ of $\mathcal{F}$-bimodules together with a morphism $\mathrm{d}\colon \mathcal{F} \to \Upsilon$ of sheaves of right H-comodules, such that on an open cover $\{U_i\}$ of M

1. $\mathrm{d}_i(fg) = (\mathrm{d}_i f)g + f\mathrm{d}_i g$ for all $f, g \in \mathcal{F}(U_i)$
2. $\Upsilon(U_i) = \mathcal{F}(U_i)\mathrm{d}_i\mathcal{F}(U_i)$

hold, where $\mathrm{d}_i := \mathrm{d}|_{U_i}\colon \mathcal{F}(U_i) \to \Upsilon(U_i)$.

Theorem (Aschieri-Fioresi-Latini-W '21)

Let (Γ, d) be a right $\mathcal{O}_q(P)$-covariant FODC on the Hopf algebra $\mathcal{O}_q(G)$ and $\mathcal{F}_G$ as before. Then

1. *there is a right $\mathcal{O}_q(P)$-covariant FODC $(\Upsilon_G, \mathrm{d}_G)$ on the sheaf $\mathcal{F}_G$, where $(\Upsilon_G(U_I), \mathrm{d}_I)$ are the localizations of (Γ, d).*
2. *$(\Upsilon_G, \mathrm{d}_G)$ induces a FODC $(\Upsilon_M, \mathrm{d}_M)$ on the sheaf $\mathcal{O}_M$ and a right covariant FODC (Γ_H, d_H) on the Hopf algebra $\mathcal{O}_q(P)$.*
3. *If the QPB $\mathcal{F}_G$ is locally trivial we can recover $(\Upsilon_G, \mathrm{d}_G)$ on local charts via the smash product construction of $(\Upsilon_M(U_i), (\mathrm{d}_M)_i)$ and (Γ_H, d_H).*

Ore Extension of Calculi

Let (Γ, d) be a right H-covarinat FODC on A and $a \in A$ be an Ore element such that $\delta(a) \in A \otimes H$ is invertible. We define the $A[a^{-1}]$-bimodule

$$\Gamma_a := A[a^{-1}]\,\Gamma\,A[a^{-1}] := A[a^{-1}] \otimes_A \Gamma \otimes_A A[a^{-1}]$$

and the $\mathbb{k}$-linear map

$$\mathrm{d}_a : A[a^{-1}] \to \Gamma_a, \qquad \mathrm{d}_a(f) = \begin{cases} \mathrm{d}f & f \in A \\ -a^{-1}\,\mathrm{d}a\,a^{-1} & f = a^{-1} \end{cases},$$

where we extend d_a to $A[a^{-1}]$ by the Leibniz rule. Then (Γ_a, d_a) is a right H-covariant FODC on $A[a^{-1}]$.

Example $\mathcal{O}_q(\mathrm{SL}_2)$ over $\mathcal{O}_q(\mathbf{P}^1(\mathbb{C}))$

There is a bicovariant 4-dim FODC (Γ, d) on $\mathcal{O}_q(\mathrm{SL}_2)$ (see Schmüdgen-Schüler '95). We consider the bicovariant quotient calculus (Γ_H, d_H) on $H = \mathcal{O}_q(P)$ induced by the projection $\pi \colon A \to H$.

- The induced calculi on $B_i := A_i^{\mathrm{co}H}$ give back the Chu-Ho-Zumino calculus: on $B_1 = \mathbb{C}_q[u]$ we have $\Gamma_1 := B_1 \mathrm{d}_1 B_1 = \mathrm{span}_{\mathbb{C}_q}\{u^k \mathrm{d}u^\ell \mid k, \ell \in \mathbb{N}_0\}$ and
$$(\mathrm{d}u) \cdot u = q^2 u \mathrm{d}u$$
and on $B_2 = \mathbb{C}_q[v]$ we have $\Gamma_2 := B_2 \mathrm{d}_2 B_2 = \mathrm{span}_{\mathbb{C}_q}\{v^k \mathrm{d}v^\ell \mid k, \ell \in \mathbb{N}_0\}$ and
$$(\mathrm{d}v) \cdot v = q^{-2} v \mathrm{d}v.$$
- The (trivial) calculus on $\Upsilon_M(\mathbf{P}^1(\mathbb{C}))$ is obtained by gluing
$$\mathcal{O}_M(U_1 \cup U_2) := \{(\omega_1, \omega_2) \in \Gamma_1 \times \Gamma_2 \mid r_{1,12}(\omega_1) = r_{2,12}(\omega_2)\} = \{0\}.$$

Similarly for the total space.

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Thank you for your attention!