The Differential Geometry of $SO^*(2n)$- and $SO^*(2n)Sp(1)$-Manifolds

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Definition

The **Quaternion algebra** *is the associative non-commutative algebra* $\mathbb{H}$*, with basis* $1, i, j, k$ *such that*

$$i^2 = j^2 = k^2 = ijk = -1$$

The elements of $\mathbb{H}$ *are called quaternions. Given a quaternion* q*, the projections to* $< 1 >$ *and* $< i, j, k >$*, are called the real and imaginary part of* q*. We define the conjugation* $\bar{q} = \Re(q) - \Im(q)$*. The quaternion algebra is a normed division algebra, with* $1, i, j, k$ *an orthonormal basis and inverse given by* $q^{-1} = \frac{\bar{q}}{|q|^2}$*.*

Definition

A **Quaternionic vector space** *is a real vector space V, equipped with a scalar action $V \times \mathbb{H} \to V$ such that the induced map $\mathbb{H} \to \mathrm{End}(V)$ is a homomorphism. Quaternionic vector spaces are non-canonically isomorphic to $\mathbb{H}^n \simeq \mathbb{R}^{4n}$. Note that while $\mathbb{H}^n$ is an $\mathbb{H}$-bimodule, one typically restricts to either left- or right quaternionic vector spaces. The conjugation provides an isomorphism between these points of view.*

The image $Q \subset \mathrm{End}(V)$ is called a **linear quaternionic structure**.

Let's consider maps $h : V \times V \to \mathbb{H}$ for a right quaternionic vector space V.

Definition

We say that h is **sesquilinear** *if*

$$h(xp, yq) = \bar{p}h(x, y)q,$$

There is an involution on the space of such maps, given by $h'(x, y) = \overline{h(y, x)}$ (switch arguments and conjugate). The positive and negative eigenspaces of this involution are called **Hermitian** *and* **skew-Hermitian***, respectively.*

Thus, a skew-Hermitian form is a sesquilinear form which satisfies

$$h(x, y) = -\overline{h(y, x)}$$

A sesquilinear form h is *non-degenerate* if $h(x,\cdot) = 0 \Leftrightarrow x = 0$

Definition

A **linear hypercomplex skew-Hermitian structure** *on a quaternionic right vector space V is a non-degenerate skew-Hermitian form on V.*

More generally,

Definition

A **linear quaternion skew-Hermitian structure** *on a real vector space V is an endomorphism-valued bilinear form $h : V \times V \to \mathrm{End}(V)$ such that*

1. *The image $Q \subset \mathrm{End}(V)$ of h is a linear quaternionic structure on V.*
2. *For any admissible basis $I, J, K \subset Q$, h is a linear hypercomplex skew-Hermitian structure.*

Proposition

We can also reformulate as follows:

- *h is equivalent to the pair (Q, ω), where $\Re(h(x,y)) = \omega(x,y)Id$*
- *Real valued 2-forms ω giving rise to such h can be characterised as non-degenerate $Sp(1)$-invariant 2-forms. Here $Sp(1) = \exp(\Im(Q))$.*

Definition

Let V be a vector space equipped with a linear quaternionic structure Q. Then $\omega \in \Lambda^2 V^*$ is called **scalar** *if it is non-degenerate and $Sp(1)$-invariant;*

$$\forall J \in S(Q), \omega(Jx, Jy) = \omega(x,y)$$

We define the Lie groups $SO^*(2n)$ and $Sp(1)SO^*(2n)$ as the respective automorphism groups of our two structures. $\mathfrak{so}^*(2n)$ for even and odd quaternionic dimension $n = 2m$ or $n = 2m + 1$:

$$\begin{pmatrix} A & C \\ B & -A^* \end{pmatrix},$$

for $A, B, C \in \mathfrak{gl}(m, \mathbb{H})$, $B^* = B, C^* = C$.

$$\begin{pmatrix} A & Y & C \\ X & uj & jY^* \\ B & -X^*j & -A^* \end{pmatrix},$$

for $A, B, C \in \mathfrak{gl}(m, \mathbb{H})$, $B^* = B, C^* = C$, $X, Y \in \mathbb{H}^n$, and $u \in \mathbb{R}$ This is the embedding into $\mathfrak{gl}(n, \mathbb{H})$. We may also embed $\mathfrak{so}^*(2n)$ into $\mathfrak{sp}(2n, \mathbb{R}) \simeq \mathfrak{sp}(\omega)$. Moreover, $\mathfrak{so}^*(2n)$ is a real form of $\mathfrak{so}(2n, \mathbb{C})$.

We denote

- E is the first fundamental representation of $SO^*(2n)$, i.e. $\mathbb{H}^n$ considered as a complex module.
- H is the first fundamental representation of $Sp(1)$. In particular, it is equivalent to $\mathbb{H}$ as a complex module (for a choice of complex structure).
- $[EH]$ is a real form inside the complex tensor product $E \otimes_{\mathbb{C}} H$. In particular, $[EH] \simeq \mathbb{H}^n$ as a real module with invariant linear quaternionic structure.

We introduce the notation K for the complex $SO^*(2n)$-module with highest weight $\pi_1 + \pi_2$.
Now, for example, we may write

$$\Lambda^2[EH] = [\Lambda^2 E S^2 H] \oplus [S^2 E \Lambda^2 H]$$

From now on we let $n > 1$, and M is connected.

Definition

An **Almost Quaternionic Structure** *on a manifold M is a smooth algebra sub-bundle (1 is included) $Q \subset \mathrm{End}(TM)$ modelled on $\mathbb{H}$, i.e. such that $\forall x \in M$, $\exists$ an isomorphism $\phi_x : \mathbb{H} \to Q_x$. Thus the dimension of M is $4n$. Equivalently an AQS is a G-structure for $G = Sp(1)GL(n, \mathbb{H}) \subset \mathrm{End}(\mathbb{R}^{4n})$.*

Remark

If $\varphi : Q \simeq M \times \mathbb{H}$ is a trivialization of the algebra bundle, then φ is called an almost hypercomplex structure. In that case, the tangent spaces $T_x M$ are naturally quaternionic vector spaces.

Definition

An **almost quaternion skew-Hermitian structure** *on a manifold can be equivalently defined as either*

- *An almost quaternionic manifold* (M, Q) *equipped with a section* $\omega \in \Omega^2(M)^{Sp(1)}$ *of the bundle of scalar 2-forms.*
- *A manifold* (M, h) *equipped with a smooth tensor field* h *such that for every point* $x \in M$, h_x *is a linear quaternion skew-Hermitian structure on* T_xM.

We have a globally defined 4-tensor Φ, given by

$$\Phi := g_I \odot g_I + g_J \odot g_J + g_K \odot g_K = \mathrm{Sym}(g_I \otimes g_I + g_J \otimes g_J + g_K \otimes g_K) \in \Gamma\left(S^4 T^* M\right),$$

where g_A are contractions of A and ω, $\mathrm{Sym} : \mathcal{T}^4 T^* M \to S^4 T^* M$ denotes the operator of complete symmetrization at the bundle level, and $\{I, J, K\}$ is an arbitrary local admissible frame of Q. We call Φ the **fundamental 4-tensor (field)** associated to the almost quaternionic skew-Hermitian structure (Q, ω).

Proposition

A $4n$-dimensional connected smooth manifold M admits a $SO^*(2n)Sp(1)$-structure if and only if it admits a symmetric 4-tensor Φ which is pointwise equivalent to the above.

This can be used to define $SO^*(2n)Sp(1)$-structures in an alternative way, via a global symmetric 4-tensor.

Definition

Let $\pi : \mathcal{P} \to M$ be a G-structure on M. A linear connection ∇ is called **adapted** *to $\mathcal{P} \subset \mathcal{F}$, or simply a G-connection, when the corresponding connection on the frame bundle $\mathcal{F}$ of M reduces to $\mathcal{P}$.*

The **first prolongation** of $\mathfrak{g}$ is defined by

$$\mathfrak{g}^{(1)} := (V^*\otimes\mathfrak{g})\cap(S^2V^*\otimes V) = \{\alpha \in V^*\otimes\mathfrak{g} : \alpha(x)y = \alpha(y)x,\ \forall\, x, y \in V\}$$

Note that for any Lie subalgebra $\mathfrak{g} \subset \mathrm{End}(V)$ we may consider the G-equivariant map

$$\delta : V^* \otimes \mathfrak{g} \to \Lambda^2 V^* \otimes V\,, \quad \delta(\alpha)(x, y) := \alpha(x)y - \alpha(y)x\,,$$

with $\alpha \in V^* \otimes \mathfrak{g}$ and $x, y \in V$. This is the **Spencer operator of alternation**, which is actually one of the boundary maps of the Spencer complex of $\mathfrak{g} \subset \mathrm{End}(V)$, also called **Spencer differential**. It fits into the following exact sequence

$$0 \longrightarrow \ker\delta \cong \mathfrak{g}^{(1)} \longrightarrow V^*\otimes\mathfrak{g} \cong \mathrm{Hom}(V, \mathfrak{g}) \xrightarrow{\delta} \Lambda^2 V^*\otimes V \longrightarrow \mathcal{H}(\mathfrak{g}) \longrightarrow 0$$

where we denote by $\mathcal{H}(\mathfrak{g}) \equiv \mathcal{H}^{0,2}(\mathfrak{g})$ the following Spencer cohomology of $\mathfrak{g}$:

$$\mathcal{H}(\mathfrak{g}) := \mathrm{Hom}(\Lambda^2 V, V)/\mathrm{Im}(\delta) = \Lambda^2 V^* \otimes V/\mathrm{Im}(\delta)\,.$$

Proposition

$$\begin{aligned}
\Lambda^2[EH]^* \otimes [EH] &\cong [(\Lambda^3 E \oplus K \oplus E) \otimes S^3 H]^* , \\
&\oplus [(\Lambda^3 E \oplus 2K \oplus 3E \oplus S^3_0 E) \otimes H]^* , \\
\delta([EH]^* \otimes \mathfrak{so}^*(2n)) &\cong [(\Lambda^3 E \oplus K \oplus E) \otimes H]^* , \\
\delta([EH]^* \otimes \mathfrak{sp}(1)) &\cong [E \otimes (S^3 H \oplus H)]^* .
\end{aligned}$$

The intrinsic torsion of a $SO^*(2n)Sp(1)$-structure is

$$\mathcal{H}(\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)) = [KS^3 H] \oplus [\Lambda^3 E S^3 H] \oplus [KH] \oplus [EH] \oplus [S^3_0 EH]$$

Normalization condition

Consider the traces $\mathrm{Tr}_i : \Lambda^2[EH]^* \otimes [EH] \to [EH]^*$ for $i = 1, \dots, 4$:

1. $\mathrm{Tr}_1(A)(X) := \mathrm{Tr}(A(\cdot\,, X))$;
2. $\mathrm{Tr}_2(A)(X) := \mathrm{Tr}(A(X\,,\cdot))$;
3. $\mathrm{Tr}_3(A)(X) := \mathrm{Tr}(A_X^T)$, where A_X^T is the symplectic transpose of $A_X := A(X, \cdot)$;
4. $\mathrm{Tr}_4(A)(X) := \mathrm{Tr}(\mathcal{J} A(\mathcal{J} X\,,\cdot))$, for $\mathcal{J} \in S(Q)$.

Proposition

Let $\mathcal{D}(\mathfrak{sp}(1) \oplus \mathfrak{so}^*(2n)) = \ker(2\mathrm{Tr}_1 + \mathrm{Tr}_3) \cap \ker(\mathrm{Tr}_1 - \mathrm{Tr}_4)$. *Then* $\mathcal{D}(\mathfrak{sp}(1) \oplus \mathfrak{so}^*(2n)) \simeq [KS^3H] \oplus [\Lambda^3 ES^3 H] \oplus [KH] \oplus [EH] \oplus [S^3_0 EH]$ *is transversal to the image of* δ, *and isomorphic to* $H^{0,2}(\mathfrak{sp}(1) \oplus \mathfrak{so}^*(2n))$. *Hence it is a normalization condition.*

Following Alekseevsky, Marchiafava we may start with an arbitrary Oproiu connection ∇ then obtain a unique quaternionic connection preserving vol:

$$\nabla^{(Q,\mathrm{vol})} = \nabla + \frac{1}{4(n+1)} S^{\theta}$$

Where θ is defined by $\nabla \mathrm{vol} = \theta \otimes \mathrm{vol}$ and S^{θ} is an equivariant map from T^*M to $TM \otimes S^2 T^*M$.

Proposition

Let (M, Q) be an almost quaternionic manifold equipped with the volume form $\mathrm{vol} \in \Omega^{4n}(M)$. Then there exists a unique Oproiu connection $\nabla^{Q,\mathrm{vol}}$ f or which $\nabla^{Q,\mathrm{vol}}\mathrm{vol} = 0$. We will call this the **unimodular Oproiu connection**.

Proposition

We have a forgetful functor from almost quaternion Skew-Hermitian manifolds to unimodular quaternionic manifolds. This is given by taking $\mathrm{vol} = \omega^{2n}$.

The unimodular Oproiu connection is not adapted, but we can modify it:

Theorem

The assignment $(Q, \omega) \mapsto (Q, \omega, \nabla^{Q,\omega})$, *where*

$$\nabla^{Q,\omega} = \nabla^{Q,\mathrm{vol}} + A,$$

and A *is defined by* $\omega(A(X, Y), Z) = \frac{1}{2}(\nabla^{Q,\mathrm{vol}}_X \omega)(Y, Z)$, *is a functorial assignment of adapted connections.*

Proposition

We have the prolongations $\mathfrak{g}^{(1)} = \{0\}$ *for* $\mathfrak{g} = \mathfrak{so}^*(2n)$ *and* $\mathfrak{g} = \mathfrak{sp}(1) \oplus \mathfrak{so}^*(2n)$

Theorem

The connection $\nabla^{Q,\omega}$ *is the unique minimal adapted connection with respect to the normalization condition above.*

We consider the torsion of $\nabla^{Q,\omega}$

Corollary

- *The torsion component $S^3H(\Lambda^3E + K)$ coincides with the intrinsic torsion of the almost-quaternionic structure Q.*
- *The torsion component $H(E + K) \oplus S^3H\Lambda^3E$ is the intrinsic torsion of the almost-symplectic structure ω, branched.*
- *The torsion component $H\,S^3_0E$ is the "compatibility" torsion of (Q,ω).*

There is a corresponding statement for the almost hypercomplex case, but with many more torsion components.

Theorem

The symmetric space $SO^*(2n+2)/SO^*(2n)U(1)$ and the pseudo-Wolf spaces

$$SU(2+p,q)/(SU(2)SU(p,q)U(1)), \qquad SL(n+1,\mathbb{H})/(GL(1,\mathbb{H})SL(n,\mathbb{H}))$$

are the only (up to covering) homogeneous spaces K/L with K semisimple, admitting invariant torsion-free quaternionic skew-Hermitian structures (Q,ω). In particular, the corresponding canonical connections on these symmetric spaces provide the associated minimal quaternionic skew-Hermitian connection $\nabla^{Q,\omega}$.

Let P be the connected subgroup of $SO^*(2n+2)$ which stabilizes an isotropic (with respect to ω) quaternionic line in $\mathbb{H}^{n+1}$. Then, the homogeneous space $N = SO^*(2n+2)/P$ admits an invariant contact structure, and we denote by $\mathcal{D}$ the corresponding contact distribution.

Proposition

Let $(Q, \omega, \nabla^{Q,\omega})$ be a smooth torsion-free $SO^*(2n)Sp(1)$-structure with special symplectic holonomy, i.e., $T^{Q,\omega} = 0$ and $\mathrm{Hol}(\nabla^{Q,\omega}) = SO^*(2n)Sp(1)$. Then $(Q, \omega, \nabla^{Q,\omega})$ is analytic, and locally equivalent to a symplectic reduction $\mathbb{T}\backslash U$ by a one-parameter subgroup $\mathbb{T} \subset SO^*(2n+2)$ with Lie algebra $\mathfrak{t}$, such that the corresponding right-invariant vector fields are transversal to the contact distribution $\mathcal{D}$ everywhere on U. Here $U \subset N$ is a sufficiently small open subset of N. In particular, the moduli space of such structures is n-dimensional, where n represents the quaternionic dimension of the symplectic reduction.

This result follows from a theorem of Cahen and Schwachhöfer about special symplectic connections in general.

- Structures on quaternionic twistor space
- Natural Spin bundles

Question

Even if the quaternionic structure Q and the scalar two-form ω are both integrable, the quaternion skew-Hermitian structure could have non-trivial intrinsic torsion. But does there exist examples which are not torsion-free?