# Virasoro Hamiltonian spaces

# &

# 2D - 1D duality

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based on joint works with

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Plan:

## Lecture 1

- Physics motivation: JT gravity
- Mathematic motivation: moment maps

## Lecture 2

- Coadjoint orbits & central extensions
- Virasoro coadjoint orbits

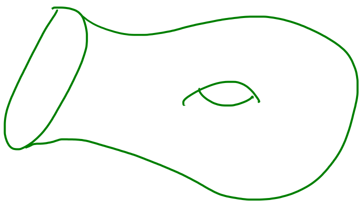
## Lecture 3

- Moduli of hyperbolic metrics
- Mathematics of 2D-1D duality

- Physics motivation:

  JT gravity & 2D-1D duality

2D gravity: $S_\Sigma(g)$



$\Sigma$ = a surface

$g$ = a (Riemannian) metric on $\Sigma$

- Einstein gravity

$$S(g) = \int_\Sigma R_g \, d^2x + \text{bdry terms}$$

$$= \chi(\Sigma)$$

$\uparrow$

Gauss-Bonnet

- Polyakov gravity

$$S(g) = \int_\Sigma R_g \frac{1}{\Delta_g} R_g \, d^2x$$

- Jackiw-Teitelboim (JT) gravity

$$S(g,\phi) = \int_\Sigma \phi\,(R_g + 1)\, d^2x \; +$$

$+$ bdry terms

dilaton field

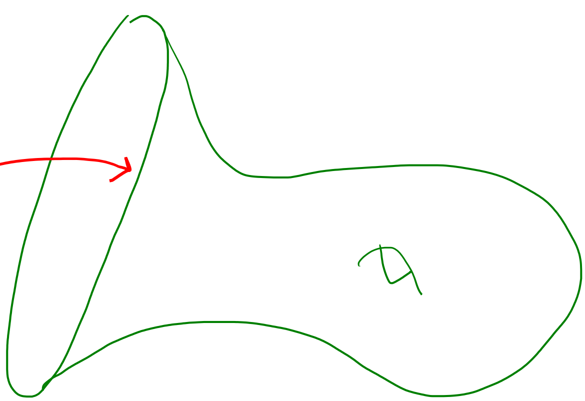
$= -\lambda$

"cosmological constant"

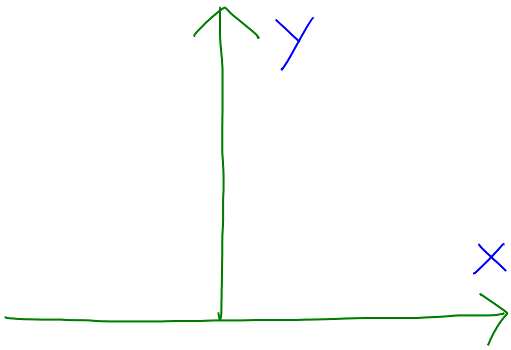
Euler-Lagrange equations

$\Rightarrow \quad R_g = -1 \quad \Rightarrow \quad g$ is hyperbolic

$\infty$ boundary



Recall: the standard hyperbolic metric on $\mathbb{H}$




$$\boxed{g = \frac{dx^2 + dy^2}{y^2}}$$

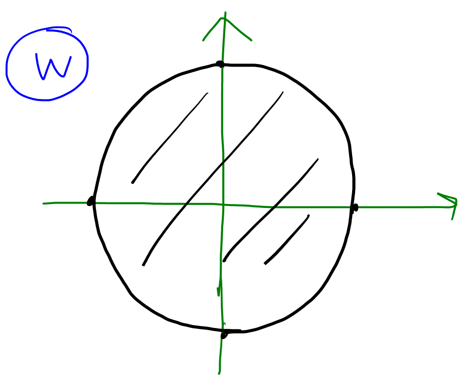
$$z = x + iy$$

Isometries of $g$ = Möbius transformations

$$z \longmapsto \frac{az+b}{cz+d}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{R})$$

- Poincaré disc:




$$g = \frac{4\, dw\, d\bar{w}}{(1-|w|^2)^2}$$

Digression: wonderful compactification of $PSL(2,\mathbb{R})$

$$p: SL(2,\mathbb{R}) \longrightarrow S^3$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \frac{1}{r} A$$

$$r = (a^2+b^2+c^2+d^2)^{1/2}$$

Fact: $p$ is injective

$$\overline{SL(2,\mathbb{R})} = \overline{\operatorname{Im}(p)}$$

Add boundary:

$$\left\{ (x_1\ x_2) \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} ; \ x, y \in \mathbb{R}^2, \ \|x\| = \|y\| = 1 \right\}$$

$$\cong$$

$$S^1 \times S^1 / \mathbb{Z}_2$$

# 2D – 1D duality

[Saad-Shenker-Stanford]

- $\Sigma = \text{disc}$ $\longleftrightarrow$ Schwarzian theory on $S^1$

JT gravity

$$S(f) = \int_{S^1} \left( f'(x)^2 + \frac{1}{2} \mathcal{S}(f) \right) dx$$

$$\mathcal{S}(f) = \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2$$

Schwarzian derivative

$$\underbrace{\{ f : \mathbb{R} \to \mathbb{R} \,,\ f'(x) > 0 \,,\ f(x+2\pi) = f(x) + 2\pi \}}_{\displaystyle \begin{array}{c} \| \\ \mathrm{Diff}^+(S^1) \end{array}}$$

Properties of $\mathcal{S}(f)$:

- $\mathcal{S}(f) = 0 \iff f(x) = \dfrac{ax+b}{cx+d}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{R}) \; \nabla !$$

- $\mathcal{S}(f \circ g) =$

$$= (\mathcal{S}(f)\circ g)\, g'^2 + \mathcal{S}(g)$$

the composition formula

# Mathematics motivation:

## moment map theory

$G$ = compact connected Lie group

$\mathfrak{g}$ = Lie$(G)$

Def: $G \circlearrowright (M, \omega \in \Omega^2(M), \mu : M \to \mathfrak{g}^*)$ is a Hamiltonian $G$-space if

① $(M, \omega)$ is symplectic

- $d\omega = 0$
- $\omega$ non-degenerate

$$\omega = \frac{1}{2} \sum_{i,j} \omega_{ij} \, dx_i \wedge dx_j$$

$$\omega^\flat : TM \xrightarrow{\cong} T^*M$$

② Moment map condition:

$$\boxed{\omega(X_M, \cdot) = -d\langle \mu, X \rangle}$$

$X \in \mathfrak{g}$, $X_M \in \mathfrak{X}(M)$

③ Equivariance:

$$\mu(g.m) = \underbrace{Ad_g^*}_{\text{coadjoint action}}(\mu(m))$$

## $\Rightarrow$ Darboux Theorem

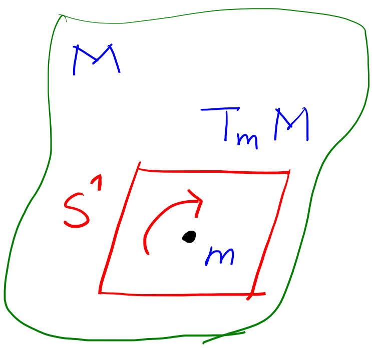
- Locally: $$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$$

$$\dim M = 2n$$

$$z_i = p_i + i q_i \Rightarrow \boxed{\omega = \frac{i}{2} \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i}$$

- Equivariant version:

$G = S^1$, $m \in M^{S^1}$ isolated fixed point




$\Rightarrow S^1 \circlearrowright T_m M$ linear action

$$\Rightarrow T_m M \cong \bigoplus_{i=1}^{n} \mathbb{C}_{k_i \neq 0}$$

$\psi$ $z_i$

weights at $m$

$$S^1 \ni e^{i\theta} : z_i \longmapsto \exp(i k_i \theta) z_i$$

$$\boxed{\mu(z) = \mu(m) + \frac{1}{2} \sum_{i=1}^{n} k_i |z_i|^2}$$

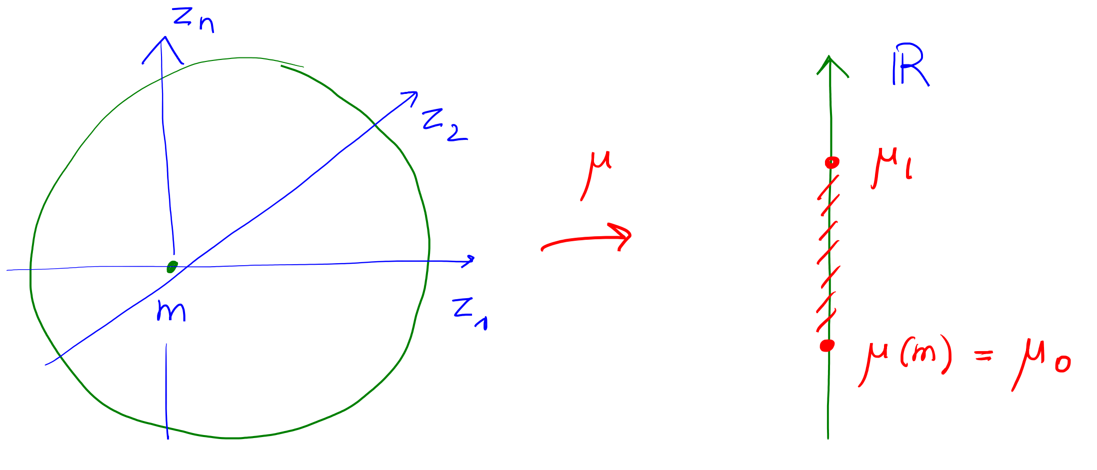
**Theorem** [Karshon–Tolman]

Assume : • $k_i > 0 \quad \forall\, i = 1, \dots, n$

• $(\mu_0, \mu_1)$ are regular values

$$\Rightarrow \quad \mu^{-1}\left([\mu_0, \mu_1)\right) \cong$$

$$\left( \{ (z_1, \dots, z_n) \in \mathbb{C}^n \,;\, \mu(z) < \mu_1 \},\ \omega = \frac{1}{2} \sum_{i=1}^n dz_i \wedge d\bar{z}_i \,,\ \mu(z) = \mu(m) + \frac{1}{2} \sum_{i=1}^n k_i |z_i|^2 \right)$$




Remark : $\mu_1 = +\infty \Rightarrow$ global Darboux chart

$\Rightarrow$ **REDUCTION** [Marsden - Weinstein]

$\xi \in \mathfrak{g}^*$, $G_\xi = \{g \in G ; Ad^*_g \xi = \xi\}$

coadjoint stabiliser

Assume : $G_\xi \circlearrowright \mu^{-1}(\xi)$ free action

$$\begin{tikzcd} \mu^{-1}(\xi) \arrow[r, hook, "i_\xi"] \arrow[d, "\pi_\xi"] & M \\ M_\xi = \mu^{-1}(\xi)/G_\xi & \end{tikzcd}$$

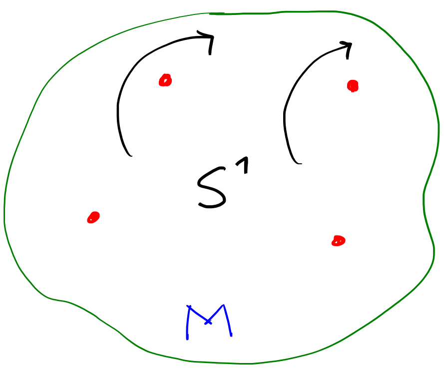
$\Rightarrow \exists! \ \omega_\xi \in \Omega^2(M_\xi)$ s.t.

$$\boxed{\pi_\xi^* \omega_\xi = i_\xi^* \omega}$$

$\Rightarrow$ $\omega_\xi$ is symplectic

## ⇒ Duistermaat-Heckman localization

Assume:
- $M$ = compact
- $G = S^1$, $M^{S^1}$ finite set



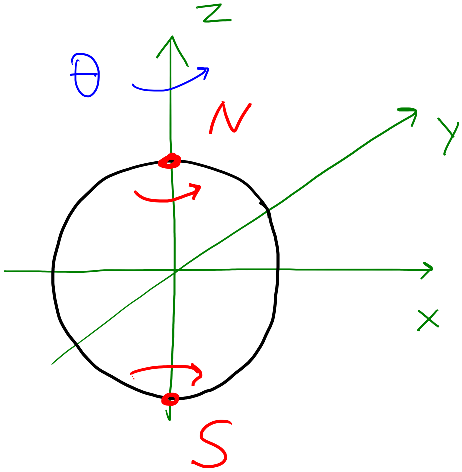

$$\Rightarrow \quad I(t) = \int_M e^{t\mu(m)} \frac{\omega^n}{n!}$$

Liouville volume

$$= \left(\frac{2\pi}{t}\right)^n \sum_{m \in M^{S^1}} \frac{e^{t\mu(m)}}{\prod_{i=1}^{n} \underbrace{k_i(m)}}$$

weights at $m$

Example : $S^1 \circlearrowright S^2$




$$\omega = dz \wedge d\theta$$

$$\mu(z,\theta) = z$$

$$I(t) = \int e^{t\mu}\,\omega$$

$$= \frac{2\pi}{t}\left[\underbrace{\frac{e^{tr}}{(+1)}}_{N} + \underbrace{\frac{e^{-tr}}{(-1)}}_{S}\right]$$

$$k_N = +1 \qquad k_S = -1$$

Example: a Karshon-Tolman space

$$M_1 = +\infty$$

$$\left. I(t) = \int_M e^{t\mu(m)} \frac{\omega^n}{n!} \right\} \rightarrow$$ Gaussian integral !

$$= \left(\frac{2\pi}{t}\right)^n \frac{e^{t\mu(m)}}{\prod_{i=1}^{n} k_i}$$

(converges for $t < 0$)

$\Rightarrow$ Convexity

Assume : • $M$ = compact

$\mu(M)/Ad^*_G$ convex, polyhedral

$\Rightarrow$ $[Q, R] = 0$

# $\infty$ - dimensional HAMILTONIAN GEOMETRY

- $G$ = compact connected, 1-connected Lie group

  $\mathfrak{g} = \mathrm{Lie}(G)$

  $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ invariant scalar product

  $LG = \mathrm{Maps}(S^1, G)$ loop group

$$\boxed{1 \longrightarrow S^1 \longrightarrow \widehat{LG} \longrightarrow LG \longrightarrow 1}$$

Theory of HAMILTONIAN $\widehat{LG}$-actions

[MEINRENKEN - Woodward '97]

- Today : Bott - Virasoro group

$$\begin{tikzcd}
1 \arrow[r] & S^1 \arrow[r] & \widehat{\mathrm{Diff}^+(S^1)} \arrow[r] & \mathrm{Diff}^+(S^1) \arrow[r] & 1 \\
 & & \hat{\mathcal{G}} \arrow[u] & \widetilde{\mathrm{Diff}^+(S^1)} = \mathcal{G} \arrow[u] &
\end{tikzcd}$$

- $\mathcal{G} = \left\{ f : \mathbb{R} \to \mathbb{R} \ ; \ f'(x) \ , \ f(x+2\pi) = f(x) + 2\pi \right\}$

Things to do :

- central extensions & coadjoint action
- Examples : - coadjoint orbits 1D
  - hyperbolic metrics 2D

# Central extensions & coadjoint action

$$1 \to S^1 \to \hat{H} \to H \to 1$$

$$\begin{tikzcd} \hat{H} \arrow[r] & H \arrow[l, bend left, "\sigma"] \end{tikzcd}$$

[Assume $\hat{H} = H \times S^1$]

$$\sigma(h_1)\,\sigma(h_2) = \exp\big(i\gamma(h_1,h_2)\big)\,\sigma(h_1 h_2)$$

Associativity $\Rightarrow$ cocycle condition

$$\boxed{\gamma(h_1,h_2) + \gamma(h_1h_2,h_3) = \gamma(h_1,h_2h_3) + \gamma(h_2,h_3)}$$

$$\mathfrak{h} = \mathrm{Lie}(H), \quad \hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{R}$$

$$\hat{H} \curvearrowright \hat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{R}$$

FACT : $\mathrm{Ad}^*_{(h,z)}(\xi + c) =$

$$= \big(\mathrm{Ad}^*_h(\xi) + c\,\beta(h)\big) + c$$

$$\beta : H \to \mathfrak{h}^* \Leftrightarrow \text{knowledge of } \gamma$$

Digression: coadjoint orbits

$$\xi \in \mathfrak{g}^*, \quad \mathcal{O}_\xi = \{ Ad_g^*(\xi) \,;\, g \in G \}$$

Theorem:

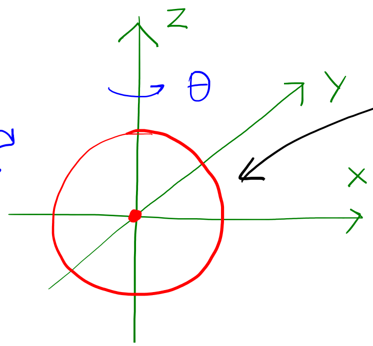
$$G \circlearrowright \left( \mathcal{O}_\xi, \ \exists!\ \omega \in \Omega^2(\mathcal{O}_\xi), \mu : \mathcal{O}_\xi \overset{id}{\hookrightarrow} \mathfrak{g}^* \right)$$

is a Hamiltonian $G$-space

Example:

$$G = SU(2), \quad \mathfrak{g}^* \cong \mathbb{R}^3$$

$SU(2) \to SO(3)$ $\circlearrowright$




$S^2_r$

$$\omega = dz \wedge d\theta$$

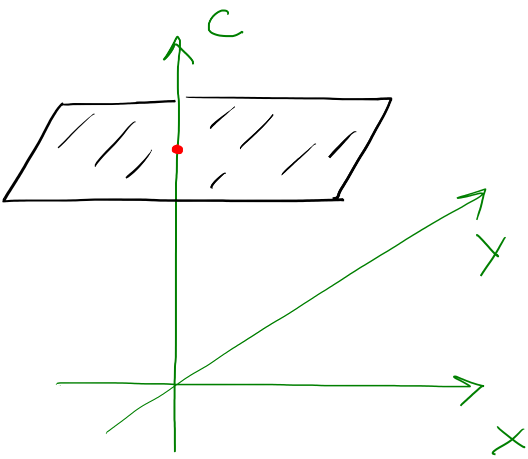
Example : Heisenberg group

$H = \left(\mathbb{R}^2 , \ (a,b) \longmapsto a+b\right)$

- $\gamma(a,b) = a_1 b_2 - a_2 b_1$

- $$\begin{array}{ccccc} \beta : & H & \longrightarrow & \mathfrak{g}^* \\ & \cong & & \cong \\ & \mathbb{R}^2 & & \mathbb{R}^2 \\ & \rotatebox[origin=c]{-90}{$\in$} & & \rotatebox[origin=c]{-90}{$\in$} \\ & (a_1, a_2) & \longmapsto & (a_2, -a_1) \end{array}$$

- Coadjoint orbits :




$\mathcal{O}_c = \{(x,y,c) \, ; \, x,y \in \mathbb{R}\}$

$\omega_c = c \, dx \wedge dy$

# Bott-Virasoro group

- $G = \{ f : \mathbb{R} \to \mathbb{R} \,;\, f'(x) \,,\, f(x+2\pi) = f(x) + 2\pi \}$
- $\mathfrak{g} = \mathfrak{X}(S^1) \ni u = u(x) \frac{\partial}{\partial x}$

  vector fields

- $\mathfrak{g}^* = \{ T = T(x)\, dx^2 \}$

  quadratic differentials

$$\langle T, u \rangle = \int_{S^1} T(x)\, u(x)\, dx$$

- $(\mathrm{Ad}_f^* T)(x) = T(f(x))\, f'(x)^2$
- $$\boxed{\gamma(f,g) = \int_{S^1} \log f'(g(x))\, (\log g'(x))'\, dx}$$

check the cocycle condition !

- $\beta : f \longmapsto \frac{1}{2} \mathcal{S}(f)\, dx^2$

$$= \frac{1}{2} \left( \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2 \right) dx^2$$

- Coadjoint action:

$$f : T \longmapsto T^f(x) = T(f(x))\, f'(x)^2 + \frac{c}{2} \mathcal{S}(f)$$

check that this is an action

- Coadjoint orbits:

$$O_T = \{ T^f(x) \, ; \, f \in \mathcal{G} \}$$

- $\mathcal{G}_c^* = \{ c \frac{d^2}{dx^2} + T(x) \}$

HILL operators

[HILL: Motion of the Moon 1886]

**DEF:** $(\mathfrak{g} \circlearrowright M,\ \omega \in \Omega^2(M),\ \mu : M \to \mathfrak{g}_c^*)$

is a Virasoro Hamiltonian space

at level $c$ if

1. $(M, \omega)$ is symplectic

2. $\omega(u_M, \cdot) = -d\langle \mu, u \rangle$

$$\mu(m) = c\frac{d^2}{dx^2} + T(x)$$

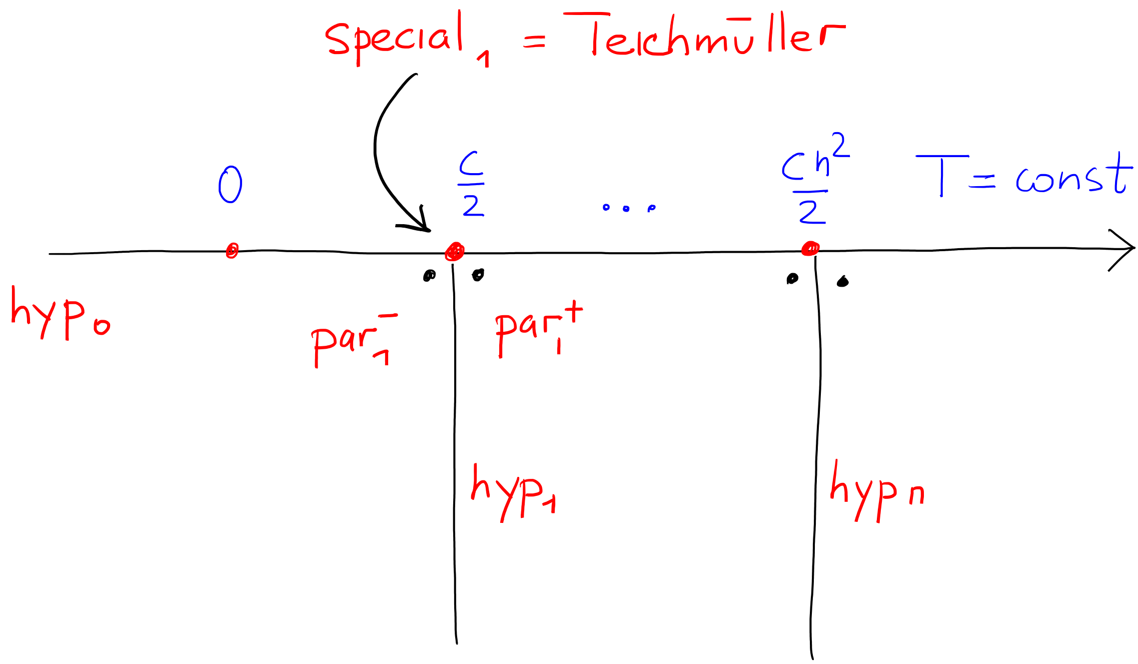
$$\langle \mu(m), u \rangle = \int_{S^1} T(x)\, u(x)\, dx$$

3. Equivariance:

$$\mu(f.m) = c\frac{d^2}{dx^2} + T^f(x)$$

# Classification of coadjoint orbits

[LAZUTKIN – PANKRATOVA, SEGAL, KIRILLOV, WITTEN, ...]




| | | |
|---|---|---|
| hyp$_0$ | $T < 0$ | $\mathrm{Diff}^+(S^1)/S^1$ |
| ell | $T > 0$ | $\mathrm{Diff}^+(S^1)/S^1$ |
| special$_1$ | $T = \frac{c}{2}$ | $\mathrm{Diff}^+(S^1)/PSL(2,\mathbb{R})$ |
| hyp$_1$ | $T \neq$ const | $\mathrm{Diff}^+(S^1)/\mathbb{R}_+$ |

Digression:

$G = SU(2)$

Coadjoint orbits: $r=0$ $\qquad\qquad$ $r \neq 0$

| | |
|---|---|
| $r = 0$ | $SU(2)/SU(2) = pt$ |
| $r \neq 0$ | $SU(2)/S^1 \cong S^2$ |

$$\begin{array}{ccccl} \mathcal{D} = & SU(2) & \times & [0, +\infty) & \\ & \Psi & & \Psi & \\ & g & & r & \longmapsto \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix} = \xi_r \end{array}$$

$$\begin{tikzcd}[column sep=small] SU(2) \circlearrowright & \mathcal{D} \arrow[dl, "\mu_{SU(2)}"'] \arrow[dr, "\mu_{S^1}"] & \circlearrowleft S^1 \\ g\xi_r g^{-1} \in \mathbb{R}^3 & & \mathbb{R} \ni r \end{tikzcd}$$

- $M$ = Hamiltonian $SU(2)$ - space

$$\begin{array}{c} N = SU(2) \backslash (M \times_{\mathbb{R}^3} \mathcal{D}) \circlearrowleft S^1 \\ \downarrow \\ \mathbb{R} \end{array}$$

- $N$ = Hamiltonian $S^1$- space

$$M = (\mathcal{D} \times_{\mathbb{R}} N)/S^1$$

# Developing maps

$$\mathcal{D} = \{ \psi : \mathbb{R} \to \mathbb{RP}^1 \, ; \; \psi'(x) > 0$$
$$\psi(x + 2\pi) = h . \psi . (x) \, , \; h \in PSL(2, \mathbb{R}) \}$$

$$\mathrm{Diff}^+(S^1) \circlearrowright \mathcal{D} \circlearrowleft PSL(2, \mathbb{R})$$

$$\begin{tikzcd} & \mathcal{D} \arrow[dl, "T"'] \arrow[dr, "\tilde{h}"] & \\ \mathfrak{g}^*_c & & \widetilde{SL(2,\mathbb{R})} \end{tikzcd}$$

- $\mathrm{Diff}^+(S^1) \circlearrowright \mathcal{D} \qquad f : \psi(x) \longmapsto \psi(f(x))$

  stabilizers : $1$ or $\mathbb{Z}$

- $PSL(2, \mathbb{R}) \circlearrowright \mathcal{D} \qquad$ free ation ?

  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \; \psi = [u_1 : u_2] \longmapsto [a u_1 + b u_2 : c u_1 + d u_2]$

- $T : \mathcal{D} \longrightarrow \{ c \frac{d^2}{dx^2} + T(x) \}$

$$\psi(x) = [u_1(x) : u_2(x)]$$

$$\left( c \frac{d^2}{dx^2} + T(x) \right) u_{1,2} = 0 \; , \quad u_1 u_2' - u_1' u_2 = 1$$

$\exists !$

- $M$ = Virasoro Hamiltonian space

$$N = (M \times_{\mathfrak{g}^*_c} \mathcal{D}) \circlearrowleft PSL(2,\mathbb{R})$$

$$\searrow \widetilde{SL(2,\mathbb{R})}$$

Fact:

{ Virasoro coadjoint orbits }

$\updownarrow$ 1 - 1

{ Conjugacy classes in $\widetilde{SL(2,\mathbb{R})}$ such that for $\tilde{h} \in \mathcal{C}$ $\exists\ x \in \mathbb{R}$ with $\tilde{h}(x) > x$ }

# Stanford - Witten integrals

- $\mathcal{O}_{T=\mathrm{const},\, \neq \frac{cn^2}{2}} \cong \mathrm{Diff}^+(S^1)/S^1$
- $S^1 \subset \mathrm{Diff}^+(S^1)$ rigid rotations

  $f(x) = x + \theta$

Facts:
- $\mathcal{O}_T^{S^1} = \{T\}$
- $T \leq \frac{c}{2} \Rightarrow \mu(\mathcal{O}_T) = (-\infty, \mu(T)]$

Karshon - Tolman spaces ?

Formal Duistermaat - Heckman integral:

$$\int_M e^{t\mu(m)} \, d\nu = \frac{e^{t\mu(T)}}{\prod_{k=1}^{\infty} \left(\frac{t}{2\pi} k\right)}$$

$\zeta$ - function regularization $= \boxed{\left(\frac{t}{2\pi}\right)^{1/2} e^{2\pi T t}}$

**Theorem** [ACY]

Global $S^1$-equivariant Darboux charts on $\mathcal{O}_T$ for $T<0$:

Let $\lambda = \sqrt{-\frac{2T}{c}}$

$$u(x) = \lambda f(x) + \log f'(x)$$

$$\Rightarrow \quad \omega = \frac{c}{2} \int \delta u \wedge \delta u' \, dx$$

$$= i c \sum_{n=1}^{\infty} n \, d\bar{u}_n \wedge du_n$$

$$\Rightarrow \quad \mu = 2\pi T - 2\pi \sum_{n=1}^{\infty} n^2 |u_n|^2$$

Questions:

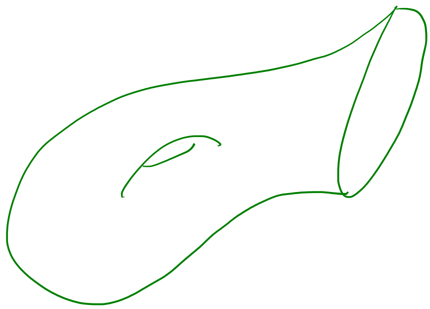
- elliptic orbits $0<T<\frac{c}{2}$ ?
- Teichmüller $T=\frac{c}{2}$ ?

## Goals:

- Show that Teichmüller spaces and moduli spaces of metrics are Virasoro Hamiltonian spaces
- Mathematics of 2D-1D duality
- Duistermaat-Heckman integrals for Virasoro coadjoint orbits

# Inspiration [MW] :

$G$ = compact Lie group
$\mathfrak{g}$ = $\mathrm{Lie}(G)$ matrix Lie algebra



$\Sigma$ = oriented 2-mfld
with $\partial\Sigma \cong S^1$

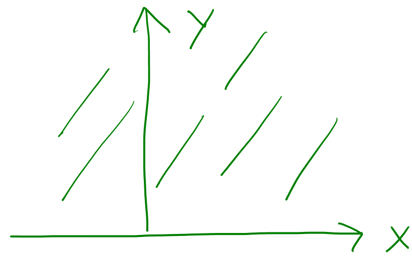
$$M(\Sigma, G) = \frac{\{A \in \Omega^1(\Sigma, \mathfrak{g})\,;\, F_A = 0\}}{\left\{A \sim A^g = gAg^{-1} + dgg^{-1}\,;\, g : \Sigma \to G \atop \underline{g|_{\partial\Sigma} = e}\right\}}$$

- $LG \cong \mathrm{Map}(\Sigma; G)/\mathrm{Map}(\Sigma, \partial\Sigma; G) \subset M(\Sigma, G)$
- $\mu : [A] \longmapsto A|_{\partial\Sigma} \in \Omega^1(S^1, \mathfrak{g})$
- $\omega = \int_\Sigma \mathrm{Tr}\, \delta A \wedge \delta A$ Atiyah-Bott form

$\delta$ = de Rham on $\Omega^1(\Sigma, \mathfrak{g})$

# Examples of hyperbolic metrics:

① $\mathbb{H}$




$$g = \frac{dx^2 + dy^2}{y^2}$$

is a 0-metric

0-vector fieds: $y\frac{\partial}{\partial x}$, $y\frac{\partial}{\partial y}$

0-1-forms: $\frac{dx}{y}$, $\frac{dy}{y}$

$$g = \left(\frac{dx}{y}\right)^2 + \left(\frac{dy}{y}\right)^2$$

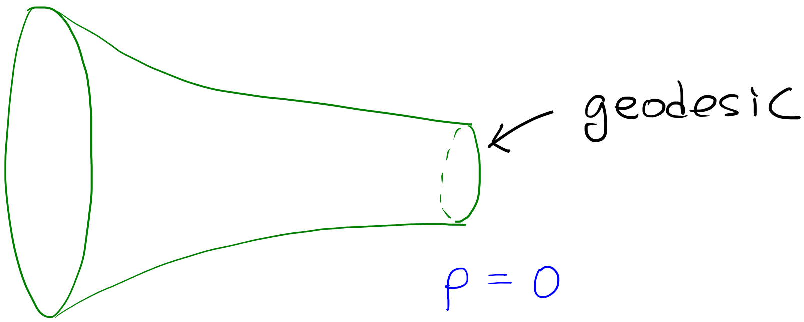
② Poincaré disc

$$g = d\rho^2 + \sinh^2(\rho)\, d\theta^2$$

$$x = \frac{\theta}{2}, \quad y = e^{-\rho}$$

$$\Rightarrow \quad g = \frac{dx^2}{y^2}(1-y^2)^2 + \frac{dy^2}{y^2}$$

## ③ Hyperbolic ends




$$g = d\rho^2 + b^2 \cosh^2(\rho)\, d\theta$$

length of geodesic $= 2\pi b$

$$x = \frac{b\theta}{2}, \quad y = e^{-\rho}$$

$$\Rightarrow \quad g = \frac{dx^2}{y^2}\left(1+y^2\right)^2 + \frac{dy^2}{y^2}$$

**DEF:** $\overline{\Sigma}$ = oriented 2-manifold with $\partial\overline{\Sigma} \cong S^1$

$$\mathcal{T}(\Sigma) = \frac{\{ g = \text{hyperbolic } 0\text{-metrics on } \Sigma \}}{\mathrm{Diff}^+_0(\overline{\Sigma}; \partial\overline{\Sigma})}$$

$\uparrow$ connected component fixing $\partial\overline{\Sigma}$

$$M(\Sigma) = \frac{\{ g = \text{hyperbolic } 0\text{-metrics on } \Sigma \}}{\mathrm{Diff}^+(\overline{\Sigma}; \partial\overline{\Sigma})}$$

**Theorem** [AM]

$\mathcal{T}(\Sigma)$ and $M(\Sigma)$ are Virasoro HAMILTONIAN SPACES

- Things to do:

$\mathcal{G} \circlearrowright$ , $\mu : \mathcal{T}(\Sigma), M(\Sigma) \to \mathcal{G}^*_1$ , $\omega$

**Digression: Cartan moving frame**

$\alpha_1, \alpha_2$ = orthonormal coframe for $g$

$\Rightarrow$ $\exists!\ \delta \in \Omega^1(\Sigma)$ spin-connection

$$d\alpha_1 = -\delta \wedge \alpha_2 \,, \quad d\alpha_2 = \delta \wedge \alpha_1$$

**Facts:**

- $$A = \frac{1}{2}\begin{pmatrix} \alpha_2 & \alpha_1 + \delta \\ \alpha_1 - \delta & -\alpha_2 \end{pmatrix} \text{ flat}$$

  $$\Updownarrow$$

  $g$ is hyperbolic

- $\alpha_1, \alpha_2, \delta$ = 0 - 1-forms

  $$A \in \Omega^1_0(\Sigma, sl(2))$$

## Examples :

① $\mathbb{H}$

$$\alpha_1 = \frac{dx}{y}, \quad \alpha_2 = \frac{dy}{y}$$

$$A = \begin{pmatrix} \frac{1}{2}\frac{dy}{y} & \frac{dx}{y} \\ 0 & -\frac{1}{2}\frac{dy}{y} \end{pmatrix}$$

② Poincaré disc

$$\alpha_1 = \frac{dx}{y}(1-y^2), \quad \alpha_2 = \frac{dy}{y}$$

$$A = \begin{pmatrix} \frac{1}{2}\frac{dy}{y} & \frac{dx}{y} \\ y\,dx & -\frac{1}{2}\frac{dy}{y} \end{pmatrix}$$

③ Hyberbolic end

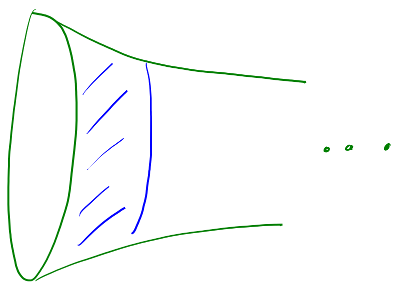
$$\alpha_1 = \frac{dx}{y}(1+y^2), \quad \alpha_2 = \frac{dy}{y}$$

$$A = \begin{pmatrix} \frac{1}{2}\frac{dy}{y} & \frac{dx}{y} \\ -y\,dx & -\frac{1}{2}\frac{dy}{y} \end{pmatrix}$$

- $\mathcal{G} = \mathrm{Diff}_0^+(\overline{\Sigma}) / \mathrm{Diff}_0^+(\overline{\Sigma}; \partial\overline{\Sigma}) \circlearrowright \mathcal{T}(\Sigma)$

- Moment map

Construction 1 :



$A = dh\, h^{-1}$

- choice of coframe:

$h(x) \longmapsto a(x)\, h(x)$

- choice of initial value : $h(x) \mapsto h(x)\, b$

- quasi-periodic

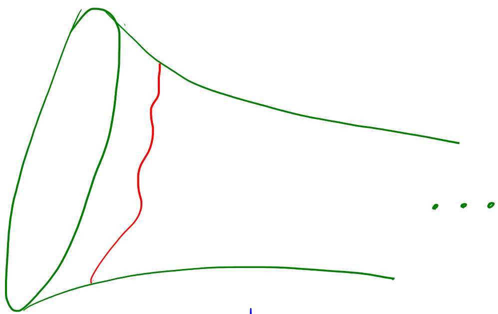
$h(x+2\pi, y) = h(x,y)\, h_0$

- $h(x,y)$ extends to $\overline{h} : \overline{\Sigma} \to \overline{PSL(2,\mathbb{R})}$

$$\overline{h}\big|_{\partial\overline{\Sigma}} : S^1 \longrightarrow \partial\, \overline{PSL(2,\mathbb{R})} = \mathbb{R}P^1 \times \mathbb{R}P^1 \xrightarrow{\pi_2} \mathbb{R}P^1$$

Recall : $\mathcal{D}/PSL(2,\mathbb{R}) \cong \boxed{\mathcal{G}_1^*}$

# Construction 2 :




- $g = a(x)^2 \dfrac{dx^2}{y^2} + \ldots$

- geodesic curvature of $y = \text{const}$

$$\exists \lim_{y \to 0} \frac{k_g(x,y) - 1}{y^2} = \kappa_g(x)$$

- $$\boxed{T(x) = a(x)\kappa_g(x) + \frac{1}{2}\left(\frac{a''}{a} - \frac{3}{2}\left(\frac{a'}{a}\right)^2\right)}$$

2-form $\omega$

$$\omega = \int_{\Sigma} \mathrm{Tr}\, \delta A \wedge \delta A$$

Atiyah-Bott 2-form

Claim: for $g(x,y) = \begin{pmatrix} y^{-1/2} & 0 \\ 0 & y^{1/2} \end{pmatrix}$

$A^g = g A g^{-1} + dg\, g^{-1}$ is regular

Exampes:

① Poincaré disc

$$A^g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} dx$$

② Hyperbolic end

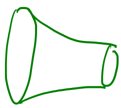
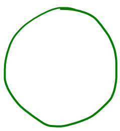
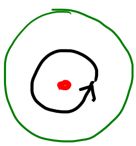
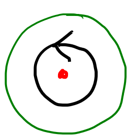
$$A^g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} dx$$

Remark

$$A = \begin{pmatrix} 0 & 1 \\ T(x) & 0 \end{pmatrix} dx$$

the Drinfeld - Sokolov slice

# 2D - 1D duality

| 2D | 1D | type |
| --- | --- | --- |
| $J\Big($  $\Big)$ L | $\mathcal{O}_{-\frac{c}{2}L^2}$ | $hyp_0$ |
| $J\Big($  $\Big)$ | $\mathcal{O}_{-\frac{c}{2}}$ | Teichmüller |
| $J\Big($  $\Big)$ conic singularity $\theta = 2\pi\lambda$ | $\mathcal{O}_{\frac{c}{2}\lambda^2}$ | elliptic |
| $J\Big($  $\Big)$ $\theta = 2\pi n$ Question: equivalence relation | $\mathcal{O}_{\frac{c}{2}n^2}$ | $special_n$ |
| $J(\ ?\ )$ | $\mathcal{O}_{T(x)}$ | $hyp_n$ $par^{\pm}_n$ |

# Stanford - Witten integrals

- $\mathcal{O}_{T=\mathrm{const},\, \neq \frac{cn^2}{2}} \cong \mathrm{Diff}^+(S^1)/S^1$

- $S^1 \subset \mathrm{Diff}^+(S^1)$ rigid rotations

$$f(x) = x + \theta$$

Facts :
- $\mathcal{O}_T^{S^1} = \{T\}$
- $T \leq \frac{c}{2} \Rightarrow \mu(\mathcal{O}_T) = (-\infty, \mu(T)]$

Karshon - Tolman spaces ?

## Formal Duistermaat - Heckman integrals:

$$\int_{\mathcal{O}_T} e^{t\mu(m)} \, d\nu = \frac{e^{t\mu(T)}}{\prod_{k=1}^{\infty} \left(\frac{t}{2\pi} k\right)}$$

$\zeta$ - function regularization

$$= \left(\frac{t}{2\pi}\right)^{1/2} e^{2\pi T t}$$

Teichmüller orbit:

$$\int_{\mathcal{O}_{\frac{c}{2}}} e^{t\mu(m)}\, d\nu = \left(\frac{t}{2\pi}\right)^{3/2} e^{\pi c t}$$

**Theorem** [ACY] Global $S^1$-equivariant Darboux charts on $\mathcal{O}_T$ for $T<0$:

- $\lambda = (-2T/c)^{1/2}$
- $u(x) = \alpha f(x) + \log f'(x)$

$\Rightarrow$ $\omega = \frac{c}{2}\int_{S^1} \delta u \wedge \delta u'\, dx = ic\sum_{n=1}^{\infty} n\, d\bar{u}_n \wedge du_n$

$\Rightarrow$ $\mu = 2\pi T - 2\pi c\sum_{n=1}^{\infty} n^2 |u_n|^2$

$\Rightarrow$ Gaussian integral

Questions:
- elliptic $0<T<\frac{c}{2}$ ?
- Teichmüller ?

Thank you !