Solving Cartan's Realization Problem

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Overview

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Main message: Lie groupoids and Lie algebroids (with extra structure) provide the right language to solve equivalence problems.

Based on joint work with Ivan Struchiner (USP):

– The Global Solutions to a Cartan's Realization Problem, `arXiv:1907.13614`. To appear in *Memoirs of the AMS*

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Starting from the classical correspondence:

$$\text{Geometric structures} \quad \longleftrightarrow \quad G\text{-structures(with connection)}$$

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The main steps of the program:

Classification problem for a class of geometric structures	$\longleftrightarrow$	G-structure algebroid (with connection)
Solutions to classification problem	$\longleftrightarrow$	Integrate G-structure algebroid to G-structure groupoid (with connection)

Plan

Lecture 1:

- Recollection of G-structures
- Finite type vs infinite type through examples
- Cartan's Realization Problem and algebroids

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Lecture 3:

- G-integrability
- Solving Cartan's Realization Problem
- Moduli space of solutions
- The example of extremal Kähler metrics on surfaces

G-principal bundles

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Example

For an effective orbifold M with $\dim M = n$, its **frame bundle**:

$$\pi : \mathrm{F}(M) \to M$$

is a principal $\mathrm{GL}(n, \mathbb{R})$-bundle.

Connections on G-principal bundles

A principal connection on $\pi : P \to M$ is a subbundle $H \subset TP$ satisfying:

(i) horizontal: $TP = \ker \mathrm{d}\pi \oplus H$;

(ii) G-invariance: $H_{pg} = g_* H_p$, for all $g \in G$, $p \in P$.

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Equivalently:

A principal connection on $\pi : P \to M$ is a form $\omega \in \Omega^1(P, \mathfrak{g})$ satisfying:

(i) vertical: $\omega(\alpha_P) = \alpha$, for all $\alpha \in \mathfrak{g}$;

(ii) G-invariance: $g^*\omega = \mathrm{Ad}_{g^{-1}}\, \omega$, for all $g \in G$.

ω is called the **connection 1-form**.

$$H = \operatorname{Ker} \omega$$

G-structures

If $G \subset \mathrm{GL}(n, \mathbb{R})$ is a closed subgroup:

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G-structures allow to encode many geometric structures, e.g.

- Coframes $\Longleftrightarrow$ $\{e\}$-structures;
- Riemannian structures $\Longleftrightarrow$ O_n-structures;
- Almost complex structures $\Longleftrightarrow$ $\mathrm{GL}_n(\mathbb{C})$-structures;
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Note: We will assume G compact and connected. Results extend to more general cases with appropriate properness assumptions.

Tautological form

The tautological form of a G-structure $\pi : \mathrm{F}_G(M) \to M$ is

$$\theta \in \Omega^1(\mathrm{F}_G(M), \mathbb{R}^n), \quad \theta(\xi) := p^{-1}(\mathrm{d}_p\pi(\xi))$$

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$\theta \in \Omega^1(\mathrm{F}_G(M), \mathbb{R}^n)$ satisfies:

(i) pointwise surjective: $\theta_p : T_p\mathrm{F}_G(M) \twoheadrightarrow \mathbb{R}^n$

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Proposition

A G-principal $\pi : P \to M$ is a G-structure if and only if it carries a 1-form $\widetilde{\theta} \in \Omega^1(P, \mathbb{R}^n)$ satisfying (i)–(iii). Each such form gives a unique isomorphism $P \simeq \mathrm{F}_G(M)$ identifying $\widetilde{\theta} \simeq \theta$.

Equivalence of G-structures

A (local) diffeomorphism $\phi : M_1 \to M_2$ lifts to a (local) isomorphism of the frame bundles:

$$\phi_* : \mathrm{F}(M_1) \to \mathrm{F}(M_2).$$

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A principal bundle map $\Phi : \mathrm{F}_G(M_1) \to \mathrm{F}_G(M_2)$ is an equivalence if and only if $\Phi^*\theta_2 = \theta_1$.

Structure equations of a G-structure with connection

Theorem

Let $\pi : \mathrm{F}_G(M) \to M$ *be G-structure with tautological form* $\theta \in \Omega^1(\mathrm{F}_G(M), \mathbb{R}^n)$ *and connection 1-form* $\omega \in \Omega^1(\mathrm{F}_G(M), \mathfrak{g})$*. Then the following* **structure equations** *hold:*

$$\begin{cases} \mathrm{d}\theta = c(\theta \wedge \theta) - \omega \wedge \theta \\ \mathrm{d}\omega = R(\theta \wedge \theta) - \omega \wedge \omega \end{cases}$$

where:

- $c : \mathrm{F}_G(M) \to \mathrm{Hom}(\wedge^2\mathbb{R}^n; \mathbb{R}^n)$ *is the* **torsion***;*
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Remark: The pair

$$(\theta, \omega) : T\mathrm{F}_G(M) \to \mathbb{R}^n \oplus \mathfrak{g}$$

gives a coframe at each p so $\mathrm{F}_G(M)$ is parallelizable.

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Differential analysis

– (M^2, g, Ω, J) – extremal Kähler surface

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The classification problem amounts to:

> Find all $U(1)$-structures $P \to M$ with tautological form θ, connection form ω and function $(K, T, U) : P \to \mathbb{R} \oplus \mathbb{C} \oplus \mathbb{R}$, such that the pde's above hold.

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$$\begin{cases} \mathrm{d}K = \cos\phi\,\theta^1 + \sin\phi\,\theta^2, & \text{where} \quad \phi : P \to \mathbb{R}, \\ \mathrm{d}\phi = \omega + J_1(-\sin\phi\,\theta^1 + \cos\phi\,\theta^2), & J_1 : P \to \mathbb{R} \\ \mathrm{d}J_1 = -(K + J_1^2)(\cos\phi\,\theta^1 + \sin\phi\,\theta^2) + J_2(-\sin\phi\,\theta^1 + \cos\phi\,\theta^2), & J_2 : P \to \mathbb{R} \\ \quad \dots & \quad \dots \\ \mathrm{d}J_k = F_k(K, J_1, \dots, J_k)(\cos\phi\,\theta^1 + \sin\phi\,\theta^2)+ & \\ \qquad\qquad + J_{k+1}(-\sin\phi\,\theta^1 + \cos\phi\,\theta^2), & J_{k+1} : P \to \mathbb{R} \end{cases}$$

The method does not apply (yet) to such **infinite type** problems.

3) Cartan's Realization Problem

One is given **Cartan Data**:

(i) a connected, closed, Lie subgroup $G \subset \mathrm{GL}(n, \mathbb{R})$;

(ii) a proper G-manifold X with infinitesimal action $\psi : X \times \mathfrak{g} \to TX$;

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$$c : X \to \mathrm{Hom}(\wedge^2\mathbb{R}^n, \mathbb{R}^n),\ R : X \to \mathrm{Hom}(\wedge^2\mathbb{R}^n, \mathfrak{g}),\ F : X \times \mathbb{R}^n \to TX$$

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Cartan's Realization Problem - Aims

- Characterize all solutions up to equivalence
- Determine group of symmetries/Lie algebra of symmetries of solutions
- Find if moduli space of solutions has some differential or stacky structure
- Determine if ”complete” solutions (e.g., metric complete solutions) exist

Associated algebroid

Cartan Data (G, X, c, R, F) determines:

– vector bundle $A = X \times (\mathbb{R}^n \oplus \mathfrak{g}) \to X$

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The triple $(A, [\cdot, \cdot], \rho)$ is an example of a Lie algebroid

Example: extremal Kähler surfaces

- $X = \mathbb{R} \times \mathbb{C} \times \mathbb{R}$, with coordinates (K, T, U)
- $A = X \times (\mathbb{C} \oplus i\mathbb{R}) \to X$
- Bracket of constant sections:

$$[(z,\alpha),(w,\beta)]|_{(K,T,U)} := (\alpha w - \beta z, -\frac{K}{2}(z\bar{w} - \bar{z}w))$$

- Anchor:

$$\rho(z,\alpha)|_{(K,T,U)} := \left(-T\bar{z} - \bar{T}z, Uz - \alpha T, -\frac{K}{2}T\bar{z} - \frac{K}{2}\bar{T}z\right)$$

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Remark. In this formulation, there are no more unknown objects!!

Next time: What does it mean to solve the problem, in this Lie algebroid language?

Solving Cartan’s Realization Problem

Lecture 2

Overview

Starting from the classical correspondence:

$$\text{Geometric structures} \quad \longleftrightarrow \quad G\text{-structures (with connection)}$$

The main steps of the program:

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Last time

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Extremal Kähler surfaces. To find such metrics amounts to find all $U(1)$-structures $P \to M$ with tautological form θ, connection form ω and function $(K, T, U) : P \to \mathbb{R} \oplus \mathbb{C} \oplus \mathbb{R}$, such that

$$\begin{cases} \mathrm{d}\theta = -\omega \wedge \theta \\ \mathrm{d}\omega = \frac{K}{2}\,\theta \wedge \bar{\theta} \\ \mathrm{d}K = -(\bar{T}\theta + T\bar{\theta}) \\ \mathrm{d}T = U\theta - T\omega \\ \mathrm{d}U = -\frac{K}{2}(\bar{T}\theta + T\bar{\theta}) \end{cases}$$

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$$A = (\mathbb{R} \times \mathbb{C} \times \mathbb{R}) \times (\mathbb{C} \oplus i\mathbb{R}) \longrightarrow X = \mathbb{R} \times \mathbb{C} \times \mathbb{R}$$

(with global coordinates (K, T, U))

Lie bracket: $[(z, \alpha), (w, \beta)]|_{(K,T,U)} := (\alpha w - \beta z, -\frac{K}{2}(z\bar{w} - \bar{z}w))$

Anchor: $\rho(z, \alpha)|_{(K,T,U)} := \left(-T\bar{z} - \bar{T}z, Uz - \alpha T, -\frac{K}{2}T\bar{z} - \frac{K}{2}\bar{T}z\right)$

It comes with a right $U(1)$-action:

$$(K, T, U, z, \alpha)g = (K, g^{-1}T, U, g^{-1}z, \alpha).$$

Another example

Metrics of constant sectional curvature. To find such metrics amounts to find all $SO(n)$-structures $P \to M$ with tautological form θ, connection form ω and function $K : P \to \mathbb{R}$, such that

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Associated algebroid:

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(with global coordinate K)

Lie bracket: $[(u, \alpha), (v, \beta)]|_K := (\alpha v - \beta u, [\alpha, \beta] - R(u, v))$

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Plan

Lecture 1:

- Recollection of *G*-structures
- Finite type vs infinite type through examples
- Cartan's Realization Problem and algebroids

Lecture 2:

- Algebroids and groupoids
- *G*-structure groupoids
- *G*-structure algebroids
- Construction of solutions

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Lecture 3:

- G-integrability
- Solving Cartan's Realization Problem
- Moduli space of solutions
- The example of extremal Kähler metrics on surfaces

1) Crash course on Lie algebroids and groupoids

A Lie algebroid is a vector bundle $A \to X$ with:

1. A Lie bracket $[\cdot,\cdot]_A ; \Gamma(A) \times \Gamma(A) \to \Gamma(A)$;
2. A anchor map $\rho_A : A \to TX$;

satisfying:

$$[s_1, f\, s_2]_A = f[s_1, s_2]_A + \rho(s_1)(f)\, s_2.$$

Main idea: Think of $(A, [\cdot,\cdot]_A, \rho_A)$ as a *generalized tangent bundle*.

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Alternative definition:

A Lie algebroid is a vector bundle $A \to X$ with a linear operator:

$$\mathrm{d}_A : \Omega^{\bullet}(A) \to \Omega^{\bullet+1}(A),$$

satisfying:

1. $\mathrm{d}_A^2 = 0$;
2. $\mathrm{d}_A(\alpha \wedge \beta) = \mathrm{d}_A \alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge \mathrm{d}_A \beta$.

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A Lie algebroid morphism is a vector bundle map

$$\begin{array}{ccc} A_1 & \xrightarrow{\Phi} & A_2 \\ \downarrow & & \downarrow \\ X_1 & \xrightarrow[\phi]{} & X_2 \end{array}$$

that intertwines the differentials: $\Phi^*\mathrm{d}_{A_2} = \mathrm{d}_{A_1}\Phi^*$.

Geometry on Lie algebroids

Basic properties of $(A, \rho, [\cdot, \cdot])$:

- **characteristic foliation** of X: integrates the (singular) distribution $\mathrm{Im}\,\rho \subset TX$;
- **isotropy Lie algebras**: for each $x \in X$, $\mathfrak{g}_x := \mathrm{Ker}\,\rho_x$ is a finite dim Lie algebra.

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One works with A as if it was the tangent bundle. For example:

- *A*-**symplectic form**: $\omega \in \Omega^2(A)$ such that $\mathrm{d}_A\omega = 0$ and $A \to A^*$, $\alpha \mapsto i_\alpha\omega$, is isomorphism;
- *A*-**complex structure**: $J : A \to A$ such that $J^2 = -I$ and

$$N_J(\alpha, \beta) := [J\alpha, J\beta] - J([J\alpha, \beta] + [\alpha, J\beta]) - [\alpha, \beta] = 0.$$

- *A*-**connection**: $\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(A)$ a $\mathbb{R}$-bilinear map such that:

$$\nabla_{f\alpha} s = f\nabla_\alpha s, \quad \nabla_\alpha fs = f\nabla_\alpha s + \rho(\alpha)(f)s.$$

Some classes of examples

- Tangent bundles TX;
- Lie algebras $\mathfrak{g}$;
- Bundle of Lie algebras;
- Lie algebra actions $\psi : \mathfrak{g} \to \mathfrak{X}(X)$;
- Prequantization (X, ω);
- Poisson structures (X, π)
- (...)

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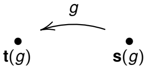

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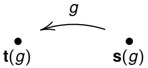

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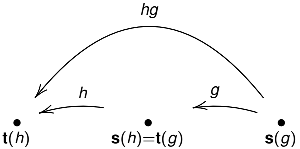
- **product**:



$$\Gamma^{(2)} = \{(h,g) \in \Gamma \times \Gamma : \mathbf{s}(h) = \mathbf{t}(g)\}$$

$$m : \Gamma^{(2)} \to \Gamma$$

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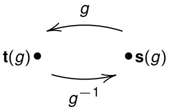
  $u : X \hookrightarrow \Gamma$

- **inverse**: $\iota : \Gamma \longrightarrow \Gamma$



Groupoids

A **groupoid** is a small category where every morphism is an isomorphism.

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This means we have a map $\mathcal{F} : \Gamma_1 \to \Gamma_2$ between the sets of arrows, and a map $f : X_1 \to X_2$ between the sets of objects, such that:

- if $g : x \longrightarrow y$ is in Γ_1, then $\mathcal{F}(g) : f(x) \longrightarrow f(y)$ in Γ_2.
- if $g, h \in \Gamma_2$ are composable, then $\mathcal{F}(gh) = \mathcal{F}(g)\mathcal{F}(h)$.
- if $x \in X_1$, then $\mathcal{F}(1_x) = 1_{f(x)}$.
- if $g : x \longrightarrow y$, then $\mathcal{F}(g^{-1}) = \mathcal{F}(g)^{-1}$.

Groupoids: basic concepts

- **right multiplication by** $g : y \longleftarrow x$ is a bijection between **s**-fibers:

$$R_g : \mathbf{s}^{-1}(y) \longrightarrow \mathbf{s}^{-1}(x), \quad h \mapsto hg.$$

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$$\mathcal{O}_x := \mathbf{t}(\mathbf{s}^{-1}(x)) = \{y \in M : \exists\, g : x \longrightarrow y\}$$

Lie groupoids

Definition

A Lie groupoid is a groupoid $\Gamma \rightrightarrows X$ whose spaces of arrows and objects are both manifolds, the structure maps $\mathbf{s}, \mathbf{t}, u, m, i$ are all smooth maps and such that $\mathbf{s}$ and $\mathbf{t}$ are submersions.

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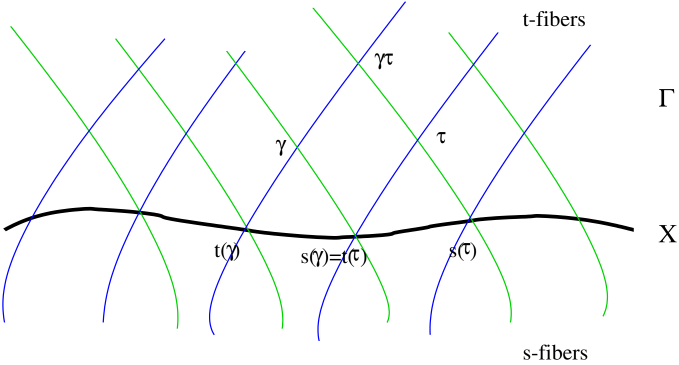
Basic Properties For a Lie groupoid $\Gamma \rightrightarrows X$ and $x \in X$, one has that:

1. the isotropy groups Γ_x are Lie groups;
2. the orbits $\mathcal{O}_x$ are (regular immersed) submanifolds in X;
3. the unit map $u : X \to \Gamma$ is an embedding;
4. $\mathbf{t} : \mathbf{s}^{-1}(x) \to \mathcal{O}_x$ is a principal Γ_x-bundle.

Some classes of examples

- Pair groupoid $X \times X \rightrightarrows X$;
- Fundamental groupoid $\Pi(X) \rightrightarrows X$;
- Lie group $G \rightrightarrows \{*\}$;
- Bundle of Lie groups;
- Lie group actions $G \times X \to X$;
- Gauge groupoid of principal bundle $G \curvearrowright P \to X$;
- Symplectic groupoids $(\Sigma, \Omega) \rightrightarrows X$.
- (...)

From Lie groupoids to Lie algebroids



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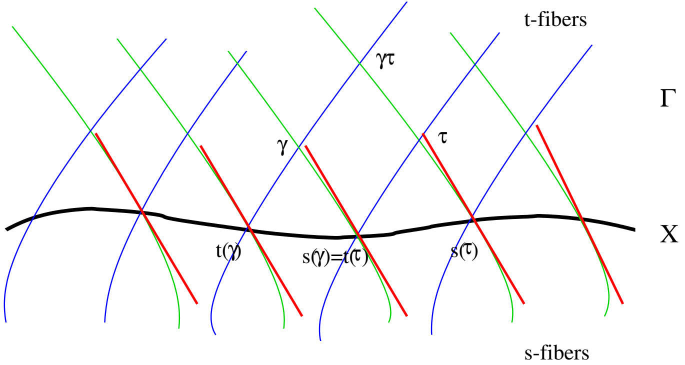

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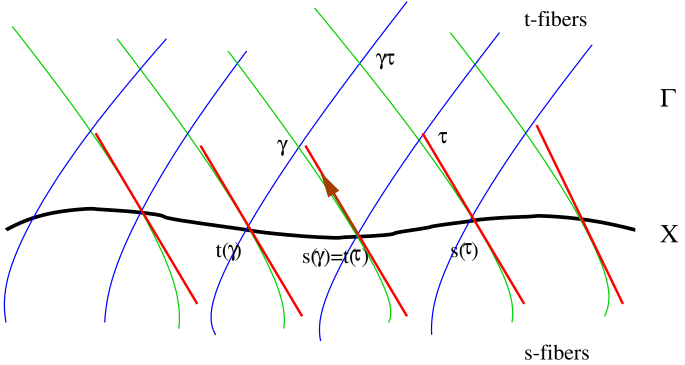

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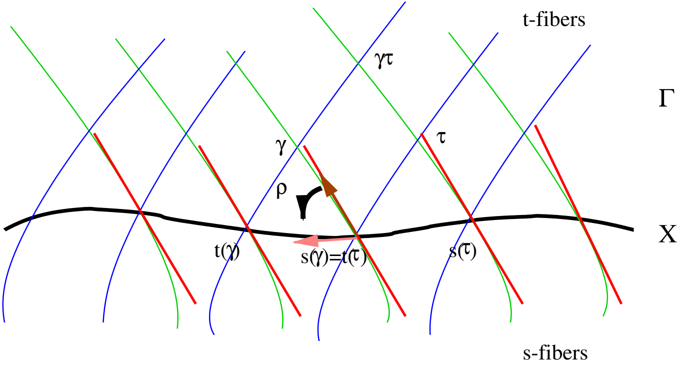

$A = T^s_X \Gamma$ $\quad\quad$ $\rho = dt\,\big|_A$

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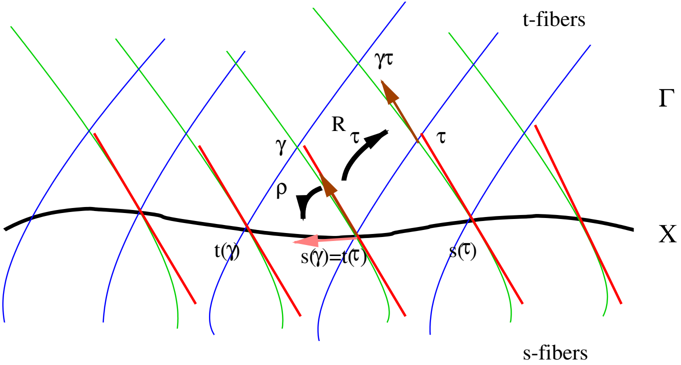

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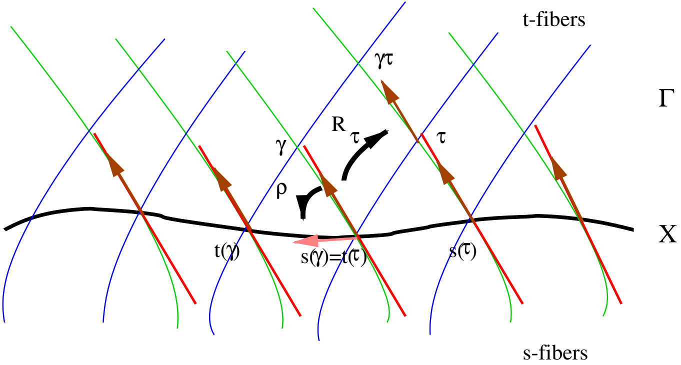

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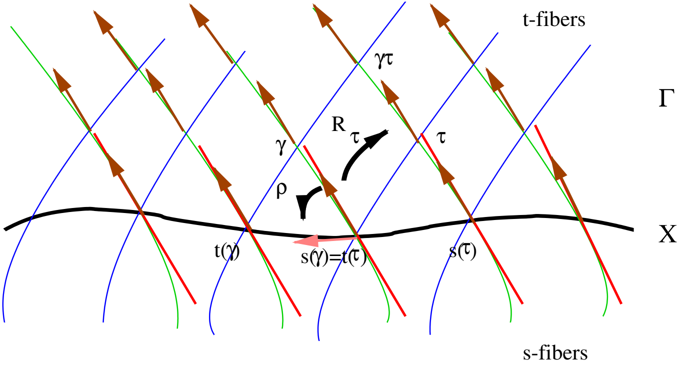

$A = T^s_X \Gamma$ $\rho = dt\,\Big|_A$ $[\alpha,\beta] = [X^\alpha, X^\beta]\,\Big|_X$

PS: Can also use **t**-fibers and left-invariant vector fields! That is our convention here.

2) G-structure groupoids

Definition

• A **G-principal groupoid** is a Lie groupoid $\Gamma \rightrightarrows X$ with a principal action of G satisfying:

$$(\gamma_1 \cdot \gamma_2)\, g = \gamma_1 \cdot (\gamma_2\, g), \quad \forall(\gamma_1, \gamma_2) \in \Gamma^{(2)},\ g \in G.$$

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Alternative point of view: **action morphism**

$$\left\{ \begin{matrix} G\text{-principal groupoid} \\ \Gamma \rightrightarrows X \end{matrix} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{matrix} \text{groupoid morphism } \iota : X \rtimes G \to \Gamma \\ \text{locally injective and effective} \end{matrix} \right\}$$

Connections and G-structures on groupoids

A **connection 1-form** on a G-principal groupoid $\Gamma \rightrightarrows X$ is a $\mathfrak{g}$-valued, left-invariant 1-form, $\Omega \in \Omega^1_L(\Gamma; \mathfrak{g})$ satisfying:

(i) vertical: $\Omega(\alpha_\Gamma) = \alpha$, for all $\alpha \in \mathfrak{g}$

(ii) G-equivariance: $g^*\Omega = \mathrm{Ad}_{g^{-1}}\,\Omega$.

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Let $G \subset \mathrm{GL}(n, \mathbb{R})$ be a closed subgroup:

A ***G*-structure groupoid** is a G-principal groupoid $\Gamma \rightrightarrows X$ with a pointwise surjective left-invariant form $\Theta \in \Omega^1_L(\Gamma; \mathbb{R}^n)$ such:

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$\Longrightarrow$ each fiber $\mathbf{t}^{-1}(x)$ is a G-structure with tautological form $\theta_x = \Theta|_{\mathbf{t}^{-1}(x)}$.

G-structure groupoids with connection

Proposition

Let $\Gamma \rightrightarrows X$ be a G-structure groupoid with connection. The tautological form $\Theta \in \Omega^1_L(\Gamma; \mathbb{R}^n)$ and the connection form $\Omega \in \Omega^1_L(\Gamma; \mathfrak{g})$ satisfy:

$$\mathrm{d}\Theta = -\Omega \wedge \Theta + \mathrm{Tors}(\Omega)$$
$$\mathrm{d}\Omega = -\Omega \wedge \Omega + \mathrm{Curv}(\Omega)$$

In this proposition:

- d denotes the **t**-foliated de Rham differential;
- $\mathrm{Tors}(\Omega) \in \Omega^2_L(\Gamma; \mathbb{R}^n)$ is given by $\mathrm{Tors}(\Omega)(v, w) = \mathrm{d}\Theta(h(v), h(w))$;
- $\mathrm{Curv}(\Omega) \in \Omega^2_L(\Gamma; \mathfrak{g})$ is given by $\mathrm{Curv}(\Omega)(v, w) = \mathrm{d}\Omega(h(v), h(w))$.

3) *G*-structure algebroids

• A ***G*-principal algebroid** is a Lie algebroid $A \to X$ with a G-action by automorphisms and an injective morphism $i : X \rtimes \mathfrak{g} \to A$ such that:

$$\hat{\psi}(\alpha) = [i(\alpha), \cdot].$$

• A **morphism of *G*-principal algebroids** is a morphism $\Phi : A_1 \to A_2$ which is G-equivariant and intertwines the action morphisms:

$$\Phi \circ i_1 = i_2 \circ (\phi \times I).$$

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Proposition

• *If $\Gamma \rightrightarrows X$ is a G-principal groupoid then its Lie algebroid $A \to X$ is a G-principal algebroid.*

• *If $\Phi : \Gamma_1 \to \Gamma_2$ is a morphism of G-principal groupoids then $(\Phi)_* : A_1 \to A_2$ is a morphism of G-principal algebroids.*

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A **connection 1-form** on a G-principal algebroid $A \to X$ is a $\mathfrak{g}$-valued A-form $\omega \in \Omega^1(A; \mathfrak{g})$ satisfying:

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Let $G \subset \mathrm{GL}(n, \mathbb{R})$ be closed:

A ***G*-structure algebroid** is a G-principal algebroid $A \to X$ equipped with a fiberwise surjective A-form $\theta \in \Omega^1(A; \mathbb{R}^n)$ satisfying:

(i) horizontal: $\theta_x(\xi) = 0 \quad \text{iff} \quad \xi = i(x, \alpha),$ for some $\alpha \in \mathfrak{g}$.

(ii) G-equivariance: $g^*\theta = g^{-1} \cdot \theta, \quad \forall g \in G$.

θ is called the **tautological form** of the G-structure algebroid.

G-structure algebroids with connection

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Let $A \to X$ be a G-structure algebroid with connection. The tautological form $\theta \in \Omega^1(A; \mathbb{R}^n)$ and the connection form $\omega \in \Omega^1(A; \mathfrak{g})$ satisfy:

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Proposition

Fix any G-principal groupoid $\Gamma \rightrightarrows X$ with Lie algebroid $A \to X$. Then there are 1:1 correspondences:

$$\left\{\begin{matrix}\text{connection 1-forms on } \Gamma \\ \Omega \in \Omega^1_L(\Gamma; \mathfrak{g})\end{matrix}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{matrix}\text{connection 1-forms on } A \\ \omega \in \Omega^1(A; \mathfrak{g})\end{matrix}\right\}$$

$$\left\{\begin{matrix}\text{tautological forms on } \Gamma \\ \Theta \in \Omega^1_L(\Gamma; \mathbb{R}^n)\end{matrix}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{matrix}\text{tautological forms on } A \\ \theta \in \Omega^1(A; \mathbb{R}^n)\end{matrix}\right\}$$

4) Construction of solutions

Theorem

Any G-structure algebroid with connection $A \to X$ is naturally isomorphic to one in **canonical form**.

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Any G-structure algebroid with connection $A \to X$ is naturally isomorphic to one in **canonical form**. Under the isomorphism

$$(\theta, \omega) : A \xrightarrow{\cong} X \times (\mathbb{R}^n \oplus \mathfrak{g}), \quad \xi_x \mapsto (x, \theta(\xi), \omega(\xi)).$$

one has that:

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one has that:

- the action morphism becomes $i : X \rtimes \mathfrak{g} \to A$, $(x, \alpha)) \mapsto (x, 0, \alpha)$;
- the tautological form becomes $\theta : X \times (\mathbb{R}^n \oplus \mathfrak{g}) \to \mathbb{R}^n$;
- the connection form becomes $\omega : X \times (\mathbb{R}^n \oplus \mathfrak{g}) \to \mathfrak{g}$;
- the G-action on A becomes $(x, u, \alpha)\, g = (x\, g, g^{-1}\, u, \mathrm{Ad}_{g^{-1}} \cdot \alpha)$;

Moreover, the anchor and bracket on constant sections become:

$$\rho(u, \alpha) = F(u) + \psi(\alpha),$$
$$[(u, \alpha), (v, \beta)] = (\alpha \cdot v - \beta \cdot u - c(u, v), [\alpha, \beta]_{\mathfrak{g}} - R(u, v)),$$

where $c : X \to \mathrm{Hom}(\wedge^2 \mathbb{R}^n, \mathbb{R}^n)$, $R : X \to \mathrm{Hom}(\wedge^2 \mathbb{R}^n, \mathfrak{g})$ and $F : X \times \mathbb{R}^n \to TX$ are G-equivariant maps.

Construction of solutions

Conclusion:

$$\left\{\begin{matrix}\text{Cartan Data}\\(G, X, c, R, F)\end{matrix}\right\} \quad \overset{1-1}{\longleftrightarrow} \quad \left\{\begin{matrix}G\text{-structure algebroids}\\\text{with connection } A \to X\end{matrix}\right\}$$

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Theorem

Given Cartan Data with associated G-structure algebroid with connection $(A, \theta, \omega) \to X$, let $(\Gamma, \Theta, \Omega) \rightrightarrows X$ be a G-structure groupoid integrating it. Then for each $x \in X$

$$(\mathbf{t}^{-1}(x), \Theta|_{\mathbf{t}^{-1}(x)}, \Omega|_{\mathbf{t}^{-1}(x)})$$

is a G-structure with connection over $M = \mathbf{t}^{-1}(x)/G$ which solves Cartan's realization problem with $h := \mathbf{s} : \mathbf{t}^{-1}(x) \to X$.

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$\Longrightarrow$ integrations gives rise to family of solutions

Easy example: Metrics of constant sectional curvature

Associated SO(n)-structure algebroid with connection:

$$A = \mathbb{R}^n \times (\mathbb{R}^n \oplus \mathfrak{so}(n, \mathbb{R})) \longrightarrow X = \mathbb{R}$$

(with global coordinate K)

Lie bracket: $[(u, \alpha), (v, \beta)]|_K := (\alpha v - \beta u, [\alpha, \beta] - K (\langle \cdot, v\rangle u - \langle \cdot, u\rangle v))$

Anchor: $\rho(u, \alpha)|_K := 0$

SO(n)-action: $(K, u, \alpha)g = (K, g^{-1}u, g^{-1}\alpha g)$

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Associated SO(n)-structure groupoid with connection:

Bundle of Lie groups $p = \mathbf{s} = \mathbf{t} : \Gamma \to \mathbb{R}$ with fibers

$$\mathbf{t}^{-1}(K) \simeq \begin{cases} \mathrm{SO}(n+1), & \text{if } K > 0 \\ \mathrm{SO}(n) \ltimes \mathbb{R}^n, & \text{if } K = 0 \\ \mathrm{SO}^+(n, 1), & \text{if } K < 0 \end{cases}$$

These SO(n)-structures are the oriented orthogonal frame bundles of the 1-connected space forms:

$$\mathbf{t}^{-1}(x)/\,\mathrm{SO}(n) \simeq \begin{cases} \mathbb{S}^n, & \text{if } K > 0 \\ \mathbb{R}^n, & \text{if } K = 0 \\ \mathbb{H}^n, & \text{if } K < 0 \end{cases}$$

Construction of solutions: dictionary

Several important questions left:

- Do we get all solutions in this way?
- Do integrations/solutions all exist?
- What can we say about symmetries of solutions and their moduli spaces?
- Can this be used in “real” problems?

... to be discussed in the next lecture.

Solving Cartan’s Realization Problem

Lecture 3

Overview

Starting from the classical correspondence:

$$\text{Geometric structures} \quad \longleftrightarrow \quad G\text{-structures (with connection)}$$

The main steps of the program:

Classification problem for a finite type class of geometric structures	$\longleftrightarrow$	Cartan's realization problem (Cartan Data)
Cartan Data	$\longleftrightarrow$	G-structure algebroid (with connection)
Solutions to classification problem	$\longleftrightarrow$	Integrate G-structure algebroid to G-structure groupoid (with connection)

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Associated SO(n)-structure groupoid with connection:

Bundle of Lie groups $p = \mathbf{s} = \mathbf{t} : \Gamma \to \mathbb{R}$ with fibers

$$\mathbf{t}^{-1}(K) \simeq \begin{cases} \mathrm{SO}(n+1), & \text{if } K > 0 \\ \mathrm{SO}(n) \ltimes \mathbb{R}^n, & \text{if } K = 0 \\ \mathrm{SO}^+(n,1), & \text{if } K < 0 \end{cases}$$

These SO(n)-structures are the oriented orthogonal frame bundles of the 1-connected space forms:

$$\mathbf{t}^{-1}(x)/\,\mathrm{SO}(n) \simeq \begin{cases} \mathbb{S}^n, & \text{if } K > 0 \\ \mathbb{R}^n, & \text{if } K = 0 \\ \mathbb{H}^n, & \text{if } K < 0 \end{cases}$$

Plan

Lecture 1:

- Recollection of G-structures
- Finite type vs infinite type through examples
- Cartan's Realization Problem and algebroids

Lecture 2:

- Algebroids and groupoids
- G-structure groupoids
- G-structure algebroids
- Construction of solutions

Lecture 3:

- G-integrability
- Solving Cartan's Realization Problem
- The example of extremal Kähler metrics on surfaces
- ~~Moduli space of solutions~~

1) G-Integrability

Theorem (Lie I)
Let Γ be a Lie groupoid with Lie algebroid A. There exists a unique (up to isomorphism) source 1-connected Lie groupoid $\widetilde{\Gamma}$ with Lie algebroid A.

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Let Γ_1 and Γ_2 be Lie groupoids with Lie algebroids A_1 and A_2, where Γ_1 is source 1-connected. Given a Lie algebroid homomorphism $\phi : A_1 \to A_2$, there exists a unique Lie groupoid homomorphism $\Phi : \Gamma_1 \to \Gamma_2$ with $(\Phi)_* = \phi$.

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. . . Lie III does not hold!

Obstructions to integrability

Theorem [Crainic & RLF, 2003]
For a Lie algebroid A, there exist *monodromy groups* $N_x \subset A_x$ such that A is integrable iff the groups N_x are uniformly discrete for $x \in X$.

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Example. For prequantization algebroid A defined by $\omega \in \Omega^2_{\mathrm{cl}}(M)$:

$$N_x = \left\{ \int_\sigma \omega : [\sigma] \in \pi_2(M) \right\} \subset \mathbb{R} = \mathfrak{g}_x.$$

So A is integrable if and only if ω has discrete spherical periods.

Lie Functor for G-principal groupoids/algebroids

Theorem (Lie I)
Let Γ be a G-principal groupoid with Lie algebroid A. There exists a unique (up to isomorphism) G-principal groupoid $\widetilde{\Gamma}_G$ with Lie algebroid A and $\mathbf{s}^{-1}(x)/G$ all 1-connected.

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Theorem (Lie II)

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Note: Lie III fails even when A is integrable. In general,

$$A \text{ is integrable} \quad \not\Rightarrow \quad A \text{ is } G\text{-integrable}$$

G-Integrability

Problem. When is a G-principal algebroid $A \to X$ **G-integrable**?

We are looking for:

- a Lie groupoid $\Gamma \rightrightarrows X$ which integrates A;
- a morphism $\iota : X \rtimes G \to \Gamma$ which integrates $i : X \rtimes \mathfrak{g} \to A$.

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Remark. We only care about G-principal groupoids: if A has a tautological form or a connection form they "integrate for free".

Extended G-Monodromy

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Definition.
The **extended G-monodromy at** $x \in X$ is the image $\widetilde{\mathcal{N}}_x^G$ of the map

$$\partial_x^G : \pi_1(G) \to \widetilde{\Gamma}_x, \quad g \mapsto \tilde{\iota}(x, g).$$

These groups assemble to a normal sub-bundle of groups contained in the center of the isotropy groups:

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In this case the **canonical G-integration of** A is:

$$\widetilde{\Gamma}_G = \widetilde{\Gamma}/\widetilde{\mathcal{N}}_x^G.$$

Computing G-Monodromy

The G**-monodromy at** $x \in X$ is the subgroup $\mathcal{N}_x^G \subset Z(\ker \rho_x)$ such that

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compatible with the action morphism $i : X \rtimes \mathfrak{g} \to A$ and with center-valued curvature 2-form:

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Proposition. If the action is locally free at x and the leaf $L \subset X$ admits a G-splitting $\sigma : TL \to A|_L$ then

$$\mathcal{N}_x^G = \left\{ \int_c \Omega_\sigma \mid c : D^2 \to L,\ c|_{\partial D^2} \subset x \cdot G \right\}.$$

2) Solving Cartan's Realization Problem

Theorem (local solutions).

Let (G, X, c, R, F) be Cartan Data defining a G-structure Lie algebroid with connection $A \to X$. For each $x \in X$ there exists a G-invariant, open neighborhood $x \in U \subset L$ such that $A|_U$ is G-integrable.

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In particular, there exists a solution $(\mathrm{F}_G(M), (\theta, \omega), h)$ with $x \in \mathrm{Im}\, h$ and:

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Remark. According to Bryant, local existence was known to E. Cartan. I am not so sure...

Complete solutions

Restrict to the metric type (but there is a general theory!):

A G-structure algebroid with connection (=Cartan data (G, X, c, R, F)) is said to be of **metric type** if $G \subset O(n, \mathbb{R})$ and $c = 0$.

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$$K_A((u, \alpha), (v, \beta)) := \langle u, v \rangle_{\mathbb{R}^n} + \langle \alpha, \beta \rangle_{\mathfrak{g}} \quad (u, \alpha), (v, \beta) \in A,$$

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$\Longrightarrow$ leaves of A have a natural Riemannian structure.

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(i) If $M = P/G$ is metric complete and 1-connected, then P is isomorphic to a fiber $\mathbf{t}^{-1}(x)$ of the canonical G-integration of $A|_L$ for some leaf $L \subset X$. Moreover, this leaf is metric complete.

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(ii) Conversely, if a leaf L is metric complete and $A|_L$ is G-integrable, then any $\mathbf{t}$-fiber of the canonical G-integration of $A|_L$ yields a metric complete solution.

3) Example: Extremal Kähler Metrics

- $X = \mathbb{R} \times \mathbb{C} \times \mathbb{R}$ - Coordinates: (K, T, U);
- $U(1)$-Action: $(K, T, U) \cdot g = (K, g^{-1}T, U)$;
- $A = X \times (\mathbb{C} \oplus i\mathbb{R})$;
- Bracket of constant sections:

$$[(z, \alpha), (w, \beta)]|_{(K,T,U)} := (\alpha w - \beta z, -\frac{K}{2}(z\bar{w} - \bar{z}w));$$

- Anchor:

$$\rho(z, \alpha)|_{(K,T,U)} := \left(-T\bar{z} - \bar{T}z, Uz - \alpha T, -\frac{K}{2}T\bar{z} - \frac{K}{2}\bar{T}z\right).$$

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This Lie algebroid is not $U(1)$-integrable!

Need to investigate $U(1)$-integrability of $A|_L$, for each leaf L.

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In real coordinates: $\alpha = i\lambda$, $z = a + ib$, $T = X + iY$:

$$\rho(z, \alpha)|_{(K,T,U)} = a\left(-2X\frac{\partial}{\partial K} + U\frac{\partial}{\partial X} - KX\frac{\partial}{\partial U}\right) + \\ + b\left(-2Y\frac{\partial}{\partial K} + U\frac{\partial}{\partial Y} - KY\frac{\partial}{\partial U}\right) + \lambda\left(Y\frac{\partial}{\partial X} - X\frac{\partial}{\partial Y}\right)$$

For constant sections $e_1 = (1, 0)$, $e_2 = (i, 0)$, $e_3 = (0, i)$:

$$[e_1, e_2] = Ke_3, \quad [e_1, e_3] = -e_2, \quad [e_2, e_3] = e_1.$$

Action morphism $\iota : X \rtimes i\mathfrak{u}(1) \to A$:

$$\iota(x, i\lambda) = \lambda e_3.$$

Leaves and Isotropy of A

Functions constant on the leaves of A:

$$I_1 = \frac{K^2}{4} - U, \qquad I_2 = X^2 + Y^2 + KU - \frac{1}{6}K^3,$$

These two functions are independent everywhere except at $X = Y = U = 0$ when the anchor vanishes.

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Leaves and Isotropy Lie algebras:

- the points $(K, 0, 0, 0)$ with isotropy Lie algebra $\mathfrak{so}(3, \mathbb{R})$ (if $K > 0$), $\mathfrak{sl}(2, \mathbb{R})$ (if $K < 0$) and $\mathfrak{so}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ (if $K = 0$);
- the 2-dimensional submanifolds of $\mathbb{R}^4$ given by the connected components of the common level sets of I_1 and I_2, with isotropy Lie algebra $\mathbb{R}$.

Fixed Points

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It remains to analyze the 2-dimensional leaves...

2-d Leaves of A

- I_1 and I_2 only depend on the radius $|T|^2 = X^2 + Y^2$;
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$$\begin{cases} I_1 = c_1 \\ I_2 = c_2 \end{cases} \quad \Leftrightarrow \quad \begin{cases} U = \frac{K^2}{4} - c_1 \\ |T|^2 = -\frac{1}{12}K^3 + c_1 K + c_2 \end{cases}$$

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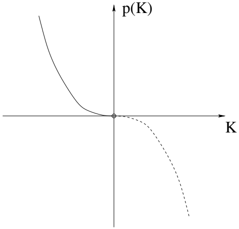
- Use K as a parameter;
- Depending on the values of c_1 and c_2, the shape of the curve will determined if leaves have topology and hence also monodromy and/or G-monodromy;
- Note that the cubic
$$p(K) = -\frac{1}{12}K^3 + c_1 K + c_2$$
has discriminant:
$$\Delta = \frac{1}{48}(16c_1^3 - 9c_2^2).$$
- A 0-dimensional leaf $(K, 0, 0, 0)$ belongs to a common level set $I_1 = c_1$, $I_2 = c_2$, if and only if

$$\begin{cases} K^2 = 4c_1 \\ 0 = -\frac{1}{12}K^3 + c_1 K + c_2 \end{cases} \quad \Rightarrow \quad 16c_1^3 - 9c_2^2 = 0 \quad \Leftrightarrow \quad \Delta = 0.$$

$\Delta = 0$, $c_1 = c_2 = 0$



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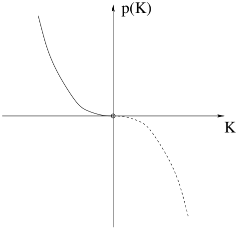

$p(K)$ has triple root: Level set consists of one single leaf obtained by rotating the curve

$$\begin{cases} U = K^2 \\ |T|^2 = -\frac{1}{12}K^3 \end{cases} \qquad K \in]-\infty, 0[.$$

The value $K = 0$ is excluded since the origin $(0, 0, 0, 0)$ is a 0-dim leaf.

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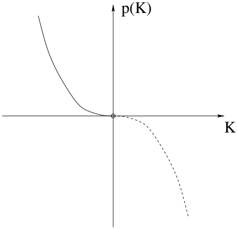

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$$\pi_1(L) = \mathbb{Z}, \quad \pi_2(L) = 1.$$

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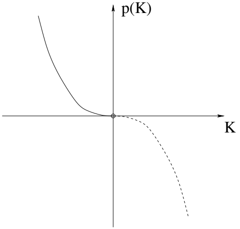

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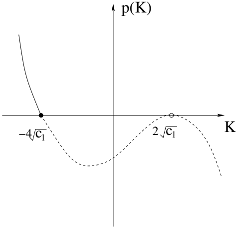
The extended monodromy is trivial, $\widetilde{\Gamma}_x^0 \simeq \mathbb{R}$ and $\pi_0(\widetilde{\Gamma}_x) = \mathbb{Z}$.

The restricted $U(1)$-monodromy is also trivial, so $A|_L$ is $U(1)$-integrable.

$\Delta = 0,\ c_2 < 0$



$p(K)$ has 1 single real root $-4\sqrt{c_1}$ and 1 double real root $2\sqrt{c_1}$.
Level set consists of isolated point $(2\sqrt{c_1}, 0, 0, 0)$ and 2-d leaf obtained by rotation of:

$$\left\{ \begin{array}{l} U = \frac{1}{4}K^2 - c_1 \\ |T|^2 = -\frac{1}{12}(K - 2\sqrt{c_1})^2(K + 4\sqrt{c_1}) \end{array} \right. \qquad K \in]-\infty, -4\sqrt{c_1}].$$

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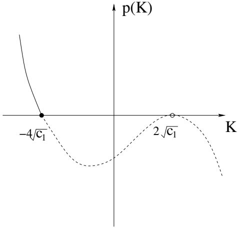

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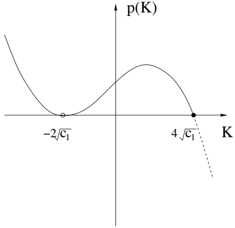
Leaf is topologically a plane:

$$\pi_1(L) = \pi_2(L) = 1.$$

$A|_L$ is $U(1)$-integrable.

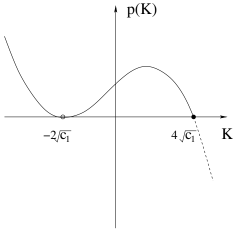
$\Delta = 0,\ c_2 > 0$



$p(K)$ has 1 single real root $4\sqrt{c_1}$ and 1 double real root $-2\sqrt{c_1}$.
Level set consists of a fixed point $(-2\sqrt{c_1}, 0, 0, 0)$ and two 2-dimensional leaves obtained by rotating the curve

$$\begin{cases} U = \frac{1}{4}K^2 - c_1 \\ |T|^2 = -\frac{1}{12}(K + 2\sqrt{c_1})^2(K - 4\sqrt{c_1}) \end{cases} \qquad K \in]-\infty, -2\sqrt{c_1}[\cup] - 2\sqrt{c_1}, 4\sqrt{c_1}].$$

$\Delta = 0$, $c_2 > 0$



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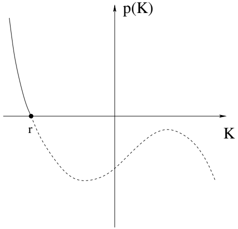
One leaf is a cylinder and the other is a plane:

$$\pi_1(L) = 1 \text{ or } \mathbb{Z}, \quad \pi_2(L) = 1.$$

$A|_L$ is $U(1)$-integrable.

$\Delta < 0$

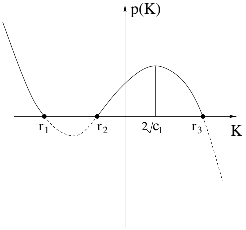


$p(K)$ has 1 real root r and two complex conjugate roots. The level set consists of a 2-dimensional leaf obtained by rotating the curve

$$\begin{cases} U = \frac{1}{4}K^2 - c_1 \\ |T|^2 = -\frac{1}{12}(K - r)(K^2 + rK + r^2 - 12c_1) \end{cases} \qquad K \in]-\infty, r].$$

Leaf is a plane, so $A|L$ is $U(1)$-integrable.

$\Delta > 0$



$p(K)$ has 3 real roots $r_1 < r_2 < r_3$. The level set consists of two 2-dimensional leaves obtained by rotating the curve

$$\begin{cases} U = \frac{1}{4}K^2 - c_1 \\ |T|^2 = -\frac{1}{12}(K - r_1)(K - r_2)(K - r_3) \end{cases} \qquad K \in]-\infty, r_1] \cup [r_2, r_3].$$

One leaf is a plane L_1 and the other leaf $L_2 \simeq \mathbb{S}^2$.

L_1 is $U(1)$-integrable.

L_2 could fail to be $U(1)$-integrable.

G-integrability over L_2

Parameterization of L_2:

$$\gamma(K,\theta) = (K, p(K)^{\frac{1}{2}} e^{i\theta}, K^2/4 - c_1), \quad (K,\theta) \in [r_2, r_3] \times [0, 2\pi],$$

G-Splitting:

$$\sigma\left(\frac{\partial\gamma}{\partial K}\right) = \left(-\tfrac{1}{2} p(K)^{-\frac{1}{2}} e^{i\theta}, 0\right)$$
$$\sigma\left(\frac{\partial\gamma}{\partial\theta}\right) = \frac{1}{p(K)+(K^2/4-c_1)^2}\left(p(K)^{\frac{1}{2}}(K^2/4-c_1)ie^{i\theta}, -p(K)i\right)$$

Curvature:

$$\Omega_\sigma\left(\frac{\partial\gamma}{\partial K}, \frac{\partial\gamma}{\partial\theta}\right) = \left[\sigma\left(\frac{\partial\gamma}{\partial K}\right), \sigma\left(\frac{\partial\gamma}{\partial\theta}\right)\right] = \frac{\partial}{\partial K}\left(\frac{K^2/4-c_1}{p(K)+(K^2/4-c_1)^2}\right) s_0$$

Monodromy:

$$\mathcal{N} = 8\pi\mathbb{Z}\left(\frac{1}{r_3^2-4c_1} + \frac{1}{4c_1-r_2^2}\right) s_0.$$

G-Monodromy:

$$\mathcal{N}^{U(1)} = \mathcal{N} \cup 2\pi\mathbb{Z}\left(\frac{1}{r_3^2/4-c_1}\right) s_0 = \left\{8\pi\left(\frac{n_1}{r_3^2-4c_1} + \frac{n_2}{4c_1-r_2^2}\right) s_0 \,:\, n_1, n_2 \in \mathbb{Z}\right\}.$$

$A|_L$ is $U(1)$-integrable if and only if $\frac{4c_1-r_2^2}{r_3^2-4c_1} \in \mathbb{Q}$.

Table: 1-connected extremal Kähler surfaces

Conditions	$U(1)$-frame bundle: $\mathbf{s}^{-1}(x)$	Solutions: $\mathbf{s}^{-1}(x)/U(1)$
$K = 0$	$SO(2) \ltimes \mathbb{R}^2$	$\mathbb{R}^2$
$K = c > 0$	$\mathbb{S}^3$	$\mathbb{S}^2$
$K = c < 0$	$SO(2, 1)$	$\mathbb{H}^2$
$\Delta = 0$, $c_1 = c_2 = 0$	$(\mathbb{R}^2 \times \mathbb{R})/\mathbb{Z}$	$\mathbb{R}^2$
$\Delta = 0$, $c_2 < 0$	$\mathbb{R}^2 \times \mathbb{S}^1$	$\mathbb{R}^2$
$\Delta = 0$, $c_2 > 0$	$(\mathbb{R}^2 \times \mathbb{R})/\mathbb{Z}$ $(\mathbb{R}^2 \times \mathbb{S}^1)$	$\mathbb{R}^2$
$\Delta < 0$	$\mathbb{R}^2 \times \mathbb{S}^1$	$\mathbb{R}^2$
$\Delta > 0$ (if $\frac{4c_1 - r_2^2}{r_3^2 - 4c_1} = \frac{p}{q}$)	$\mathbb{R}^2 \times \mathbb{S}^1$ $\mathbb{S}^3$	$\mathbb{R}^2$ $\mathbb{CP}^1_{p,q}$

Complete extremal Kähler surfaces

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- No need! One can compute the induced metric on each leaf L explicitly and determine when it is complete.

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The 1-connected complete extremal Kähler metrics on a surface are the constant scalar curvature metrics $\mathbb{R}^2$, $\mathbb{S}^2$, $\mathbb{H}^2$, and two special families of metrics: one on a disk $\mathbb{D}^2$ and the other on the weight projective space $\mathbb{CP}^1_{p,q}$.

Note: $\mathbb{D}^2$ corresponds to the branch $]-2\sqrt{c_1}, 4\sqrt{c_1}]$ in the case $\Delta = 0$, $c_2 > 0$.

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BIG OPEN QUESTION: What about infinite type?