THE PLAN FOR THESE TALKS

1. Explain how one can hope to define a knot invariant by counting minimal surfaces in $\mathbb{H}^4$ which fill a given knot $K \subseteq S^3$

   This is an example of "Fredholm differential topology" (Smale, Donaldson, ...)

2. Explain why this is actually a "Gromov - Witten invariant" for a special SINGULAR symplectic manifold.

3. Describe what new avenues this opens up:

   - Define invariants of knots $K \subseteq Y^3$ by counting minimal surfaces in certain M^4 with $\partial M = Y$??

- Define invariants of knots $K \subseteq Y^3$ by counting J-hol. curves in certain symplectic Z^6 with $\partial Z = Y \times S^2$??

- Are these invariants related to classical knot invariants ??

  Eg: I conjecture that the Alexander polynomial (or, more generally, HOMFLYPT) actually counts minimal surfaces in $\mathbb{H}^4$.

4. Introduce the technical tools needed in this whole story:

   - Fredholm theory of DEGENERATE elliptic operators.

   - Special behaviour of minimal surfaces in $\mathbb{H}^4$ near infinity.

Knots, minimal surfaces and J-holomorphic curves

I. The main result in brief:

Let $K \subseteq S^3$ be a knot.

ie image of a smooth embedding $S^1 \to S^3$

Regard S^3 as boundary at infinity of hyperbolic 4-space $\mathbb{H}^4$.

Eg. Poincaré ball model,

$$g_{hyp} = \frac{4\, dx^2}{(1-|x|^2)^2}$$

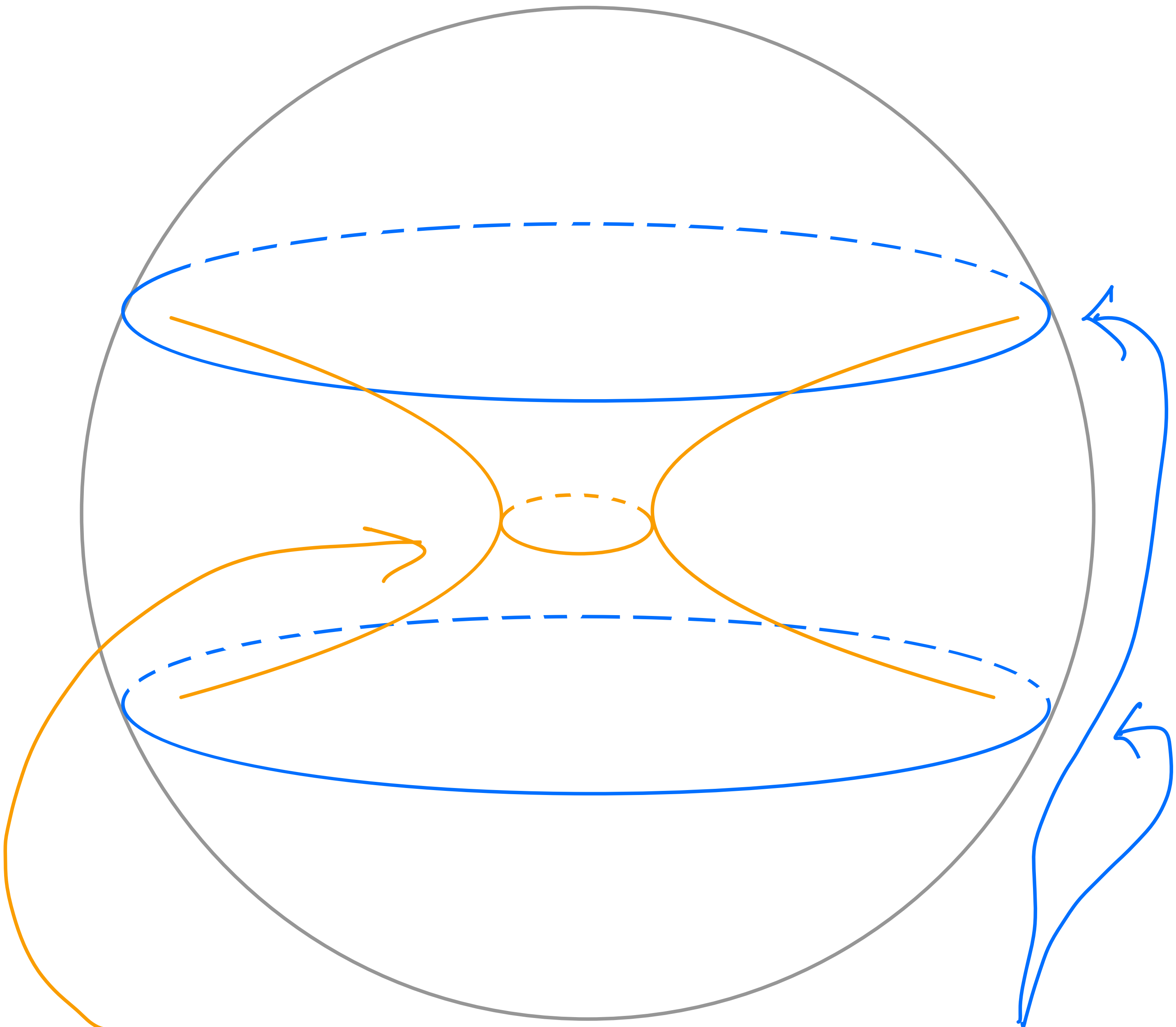
on $B^4 = \{ x \mid |x| < 1 \}$

$\partial B^4 = S^3$.

3D Example: "hyperbolic catenoid"

Due to Mori

$S^2 = \partial_\infty \mathbb{H}^3$



Theorem

The number of complete minimal discs in $\mathbb{H}^4$ which have boundary at infinity equal to K is a KNOT INVARIANT of K

1. The number $n(K)$ of minimal discs filling K is FINITE

2. If K_0 and K_1 can be joined by a path of knots K_t, then

$$\boxed{n(K_0) = n(K_1)}$$

(There are caveats but I will come to them ...)

Example

Consider $H = \{(x_1, x_2, 0, 0) \in B^4\}$

H is totally geodesic copy of $\mathbb{H}^2$

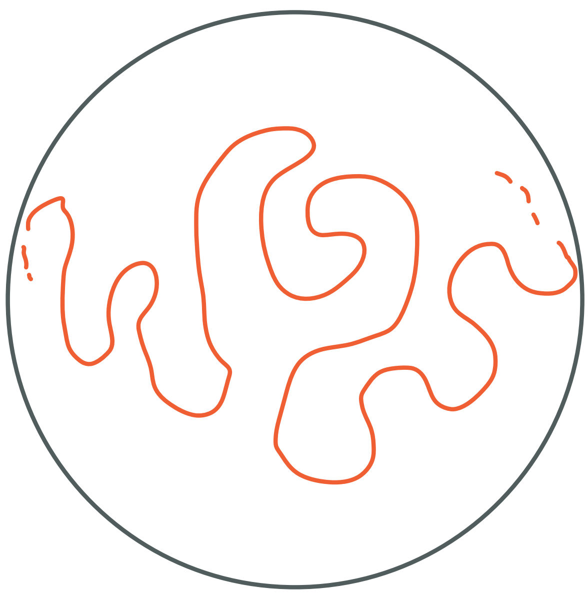
$\partial H = \mathcal{U} \subseteq S^3$, a copy of the UNKNOT.

So we have one minimal filling of $\mathcal{U}$.

Can use maximum principle to show there are no others, (I'll explain how later.)

So $\boxed{n(\mathcal{U}) = \underline{1}}$.

Now let $\hat{\mathcal{U}}$ be ANY unknot, no matter how wiggly:



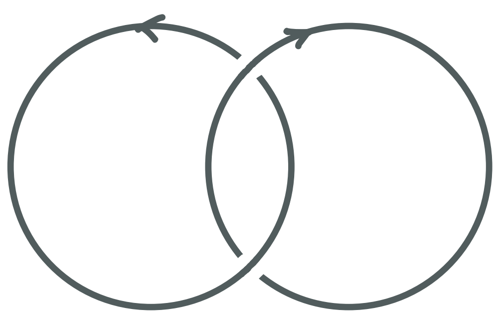
Since $n(\hat{U}) = n(U) = 1$,
there is a minimal disc filling $\hat{U}$!

(This was already known by other methods)

And for more complicated surfaces?

Let $L \subseteq S^3$ be an ORIENTED LINK with k components.

ie image of k smoothly embedded disjoint copies of S^1, plus direction



HOPF LINK

WHITEHEAD LINK

Conjecture

It is possible to count connected oriented minimal surfaces of genus g in $\mathbb{H}^4$ which fill L (with correct orientation) and obtain a topological link invariant $n_g(L)$.

I will explain what is known in this direction and what remains to be done.

II. The Alexander polynomial

Suppose we can define $n_g(L)$

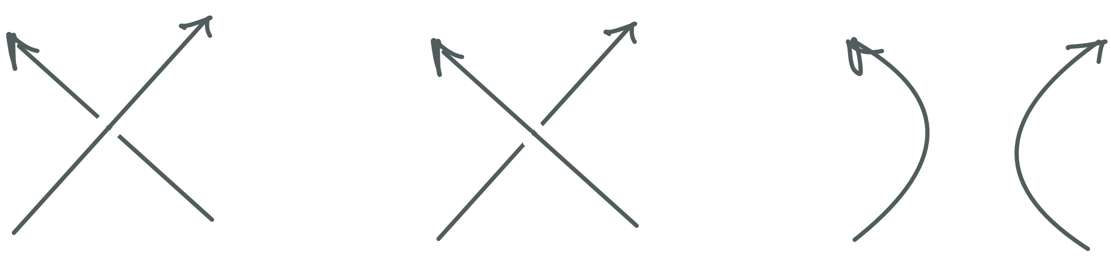
How could we actually compute it?

I strongly believe it is related to the Alexander polynomial.

Here is Conway's approach to the Alexander polynomial:

- To each link L, we have a polynomial $A_L(z)$ in a single variable z.

- Take a diagram of a link and focus on a crossing. We can "swap" the crossing or "resolve" the crossing and in this way we get two other links:



L_+ $\quad$ L_- $\quad$ L_0

The Alexander polynomial $A_L(z)$ is the unique polynomial which satisfies:

1. Conway's Skein relation:

$$A_{L_+}(z) - A_{L_-}(z) = z A_{L_0}(z)$$

2. Normalisation:

$$A_u(z) = 1. \qquad (u = \text{unknot})$$

Calculating A is quite easy!

Since $L_+ = L_-$ we see that

$$A_{u \amalg u} = 0$$

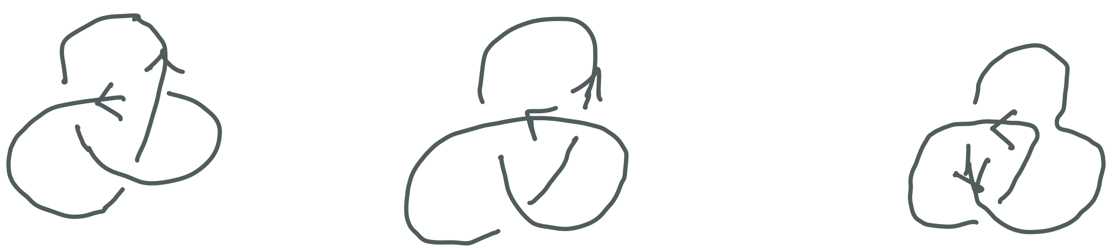
More generally A vanishes for ANY SPLIT LINK.

($L = L_1 \sqcup L_2$ is SPLIT if you can find disjoint open balls containing L_1 and L_2 respectively.)

$L_+ = H$ $\qquad$ $L_- = U \sqcup U$ $\qquad$ $L_0 = U$

So Alexander poly of Hopf link is

$$A_H(z) = z.$$



$L_+ = T$ $\qquad$ $L_- = U$ $\qquad$ $L_0 = H$

So Alexander polynomial of trefoil is

$$A_T(z) = 1 + z^2$$

Conjecture

The Alexander polynomial of an oriented link L with k components is given by counting connected oriented minimal fillings in $\mathbb{H}^4$ of $L \subseteq S^3$ by the formula:

$$\boxed{A(z) = \sum_g n_g(L)\, z^{2g-k+1}}$$

Remarks:

1. $A_H(z) = z$, $\quad H =$ Hopf link

   So the conjecture predicts that any Hopf link is filled by a minimal annulus.

   For certain symmetric Hopf links this has been verified by M. T. Nguyen.

2. $A_T(z) = 1 + z^2$ $\quad T =$ trefoil

So the conjecture predicts that any trefoil is filled by a minimal disc and a minimal surface of genus one.

3. In general, $A_L(z)$ is relatively easy to compute via the skein relation

But minimal surfaces are very hard to find. You need to solve a non-linear PDE!

So proving this conjecture would give a fantastic existence theorem for minimal surfaces!

Two tests

1. In the expression $\sum_g n_g(L) z^{2g-1+k}$

when k is even, only odd powers of z appear and when k is odd only even powers appear.

The same happens for $A_L(z)$. One can prove this from the skein relation. (Exercise!)

2. For a split link $L = L_1 \amalg L_2$, $A_L(z) = 0$.

M. T. Nguyen has proved that if L_1 and L_2 are very far apart, near opposite poles of S^3 then there is no connected minimal surface filling $L_1 \amalg L_2$.

Assuming $n_g(L)$ can be defined, this then implies that $n_g(L_1 \amalg L_2) = 0$ as we would hope.

Time permitting, I will explain why I believe this conjecture and how one might try and prove it.

There is also a conjecture relating minimal surfaces to the HOMFLYPT polynomial, which I might have time to explain ...

III Strategy for defining $n_g(L)$

Recall the definition of the degree of a map.

Let $\beta: X \to Y$ be a smooth proper map between manifolds of the same dimension, with Y connected.

The degree of β is given by "counting solutions x to $\beta(x) = y$ for generic $y \in Y$"

In more detail:

$Y \ni y$ is a regular value of β if for all $x \in \beta^{-1}(y)$, $d\beta_x$ is surjective

When y is regular, $\beta^{-1}(y) \subseteq X$ is a submanifold of dimension $\dim X - \dim Y$.

Sard's Theorem: Almost all $y \in Y$ are regular values of β.

Since $\dim X = \dim Y$, $\beta^{-1}(y)$ is a 0-dim submanifold and so a set of points

Since β is proper this set is finite

Since X, Y are oriented, each point $x \in \beta^{-1}(Y)$ comes with a sign:

$$d\beta_x : T_x X \to T_y Y \quad \text{is an isomorphism}$$

If $d\beta_x$ preserves orientations we say x is POSITIVE.

If $d\beta_x$ reverses orientations we say x is NEGATIVE.

$$\boxed{\deg \beta := \begin{array}{c}\text{signed count of}\\ \text{points in } \beta^{-1}(y)\\ \text{for a regular value } y\end{array}}$$

Need to check that this doesn't depend on the choice of regular value $y \in Y$ that we use.

Let $y_0, y_1 \in Y$ be two regular values of β.

A path $y_t \in Y$ for $t \in [0,1]$ is <u>transverse to β</u> if for any $x \in \beta^{-1}(y_t)$

$$d\beta_x + \langle y_t' \rangle = T_{y_t} Y$$

When $\{y_t : t \in [0,1]\}$ is transverse to β

$$\bigcup_{t \in [0,1]} \beta^{-1}(y_t) \quad \text{is a submanifold of } X.$$

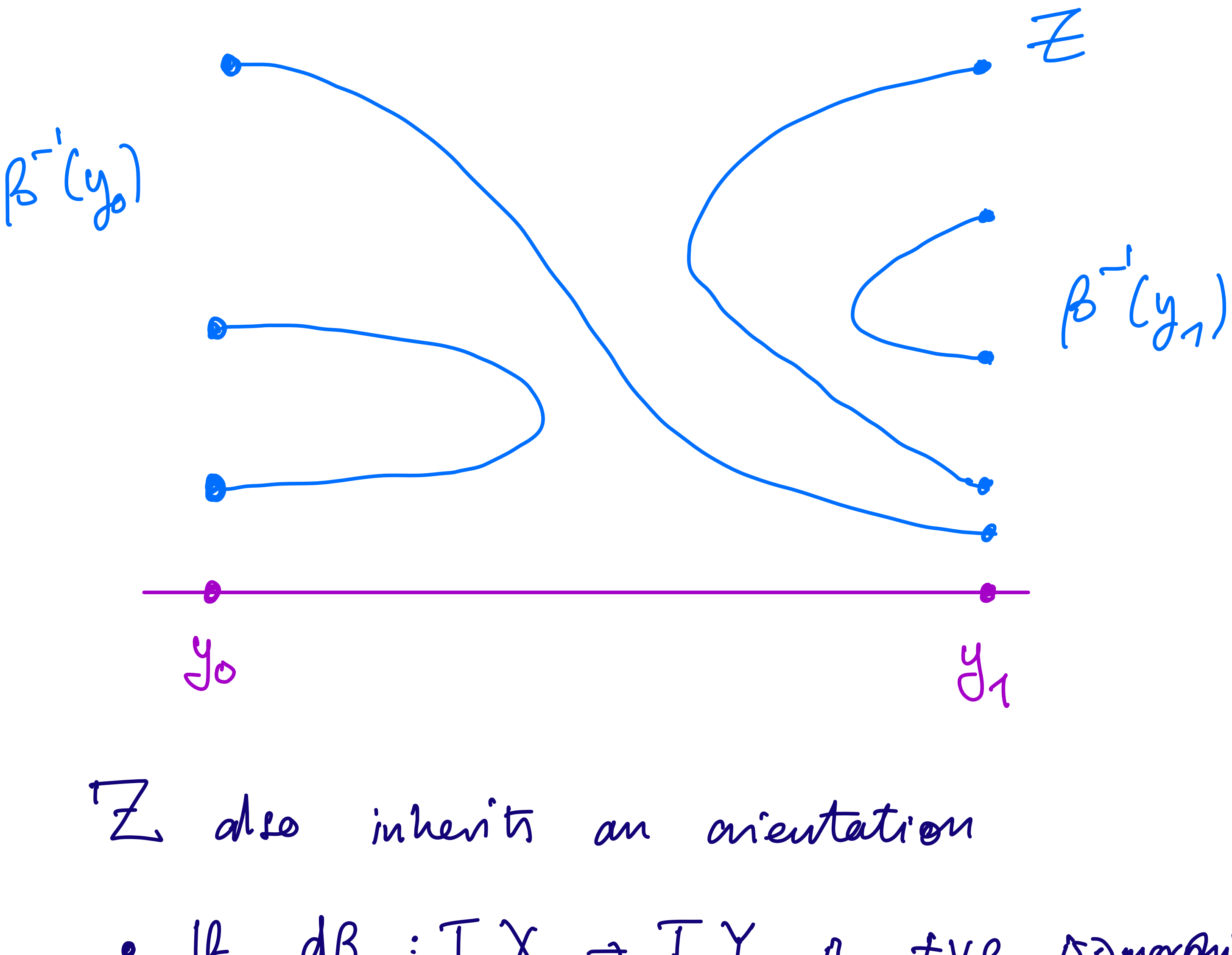
with boundary $\beta^{-1}(y_0) \cup \beta^{-1}(y_1)$

Important fact: If y_0 and y_1 are regular values of β (and Y is connected) there is a path y_t joining them that is transverse to β.

Suppose y_t is transverse to β, so that

$Z = \bigcup_{t\in[0,1]} \beta^{-1}(y_t)$ is 1D submanifold with boundary.

Since β is proper, Z is compact, so a union of closed intervals.



Z also inherits an orientation

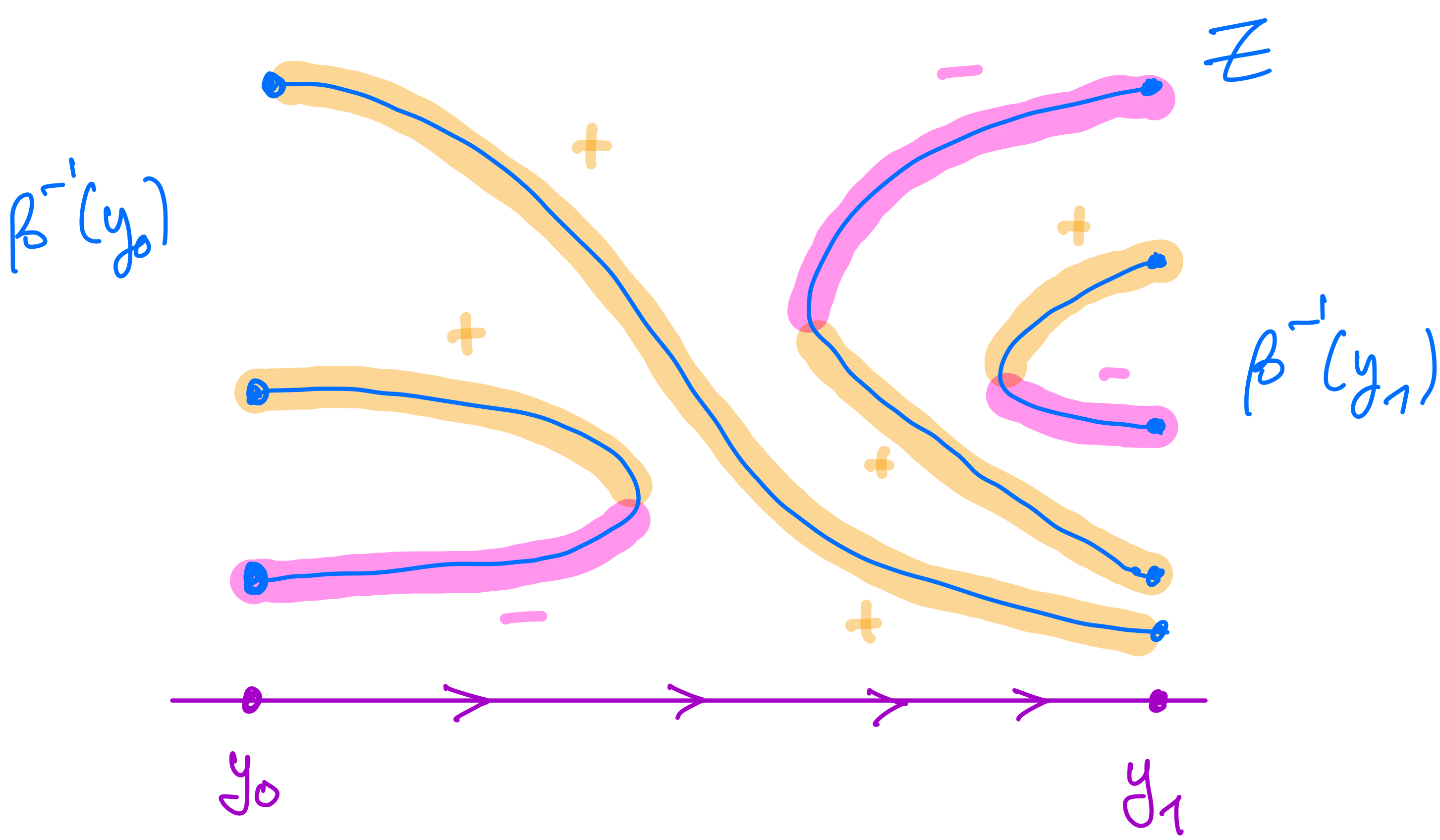
- If $d\beta_x : T_xX \to T_yY$ is +ve isomorphism then we pull back orientation from path y_t to T_xZ.

- If $d\beta_x : T_x X \to T_y Y$ is $-$ve isomorphism then we pull back MINUS the orientation from the path y_t to $T_x Z$.

- Can check these match up across the critical points, $x \in Z$ where $d\beta_x$ has 1D kernel.

So Z is oriented cobordism from $\beta^{-1}(y_0)$ to $\beta^{-1}(y_1)$

Hence the signed count of points in each agrees!



In this example:

$$\beta^{-1}(y_0) = + + - , \qquad \# \beta^{-1}(y_0) = 1$$

$$\beta^{-1}(y_1) = - + - + + , \quad \# \beta^{-1}(y_1) = 1 .$$

Both counts agree and give

$$\deg(\beta) = 1 .$$

Smale's Fredholm degree:

Let $\beta : \mathcal{X} \to \mathcal{Y}$ be a smooth map between infinite dimensional Banach manifolds, with $\mathcal{Y}$ connected.

- Suppose that β is Fredholm. I.e. that for every $x \in \mathcal{X}$

$$d\beta_x : T_x \mathcal{X} \to T_{\beta(x)} \mathcal{Y}$$

is a Fredholm map

This means that:

1. $\ker d\beta_x$ and $\operatorname{coker} d\beta_x$ are both finite dimensional

2. $\operatorname{Im} d\beta_x \subseteq T_{\beta(x)}\mathcal{Y}$ is closed.

Suppose moreover that index of $d\beta_x$ is 0

$$\operatorname{ind} d\beta_x = \dim \ker d\beta_x - \dim \operatorname{coker} d\beta_x$$

Finally suppose also that β is PROPER

Then the whole story above goes through, apart from the discussion of signs

We define $\deg \beta \in \mathbb{Z}_2$ as # of solutions to $\beta(x) = y$ for y a regular value of β, mod 2.

To get $\mathbb{Z}$-valued degree we need analogue of orientations.

Given $x \in \mathcal{X}$ define

$$(\operatorname{Ind} \beta)_x = \left(\Lambda^{top} \ker d\beta_x\right) \otimes \left(\Lambda^{top} \operatorname{coker} d\beta_x\right)^*$$

These copies of $\mathbb{R}$ fit together to give a line bundle $\operatorname{Ind}\beta \to \mathcal{X}$

We assume we have a trivialisation of this bundle

If $y = \beta(x)$ is regular value, so coker $= 0$, then $\ker d\beta_x = T_x(\beta^{-1}(y))$

So trivialisation of $(\operatorname{Ind}\beta)_x \cong \mathbb{R}$ orients $\beta^{-1}(y)$.

With this extra data we can now make sense of $\deg \beta \in \mathbb{Z}$ as a signed count, just as before.

Using Fredholm degree to define $n_g(L)$

Let $\mathcal{X}_{g,k}$ be the set of complete minimal surfaces in $\mathbb{H}^4$ which are diffeomorphic to the interior of a compact surface of genus g and with with k bdry components.

Let $\mathcal{Y}_k$ denote the set of k-component links in S^3

Each connected component of $\mathcal{Y}_k$ corresponds to a topological class of links

Sending a minimal surface to its boundary defines a map

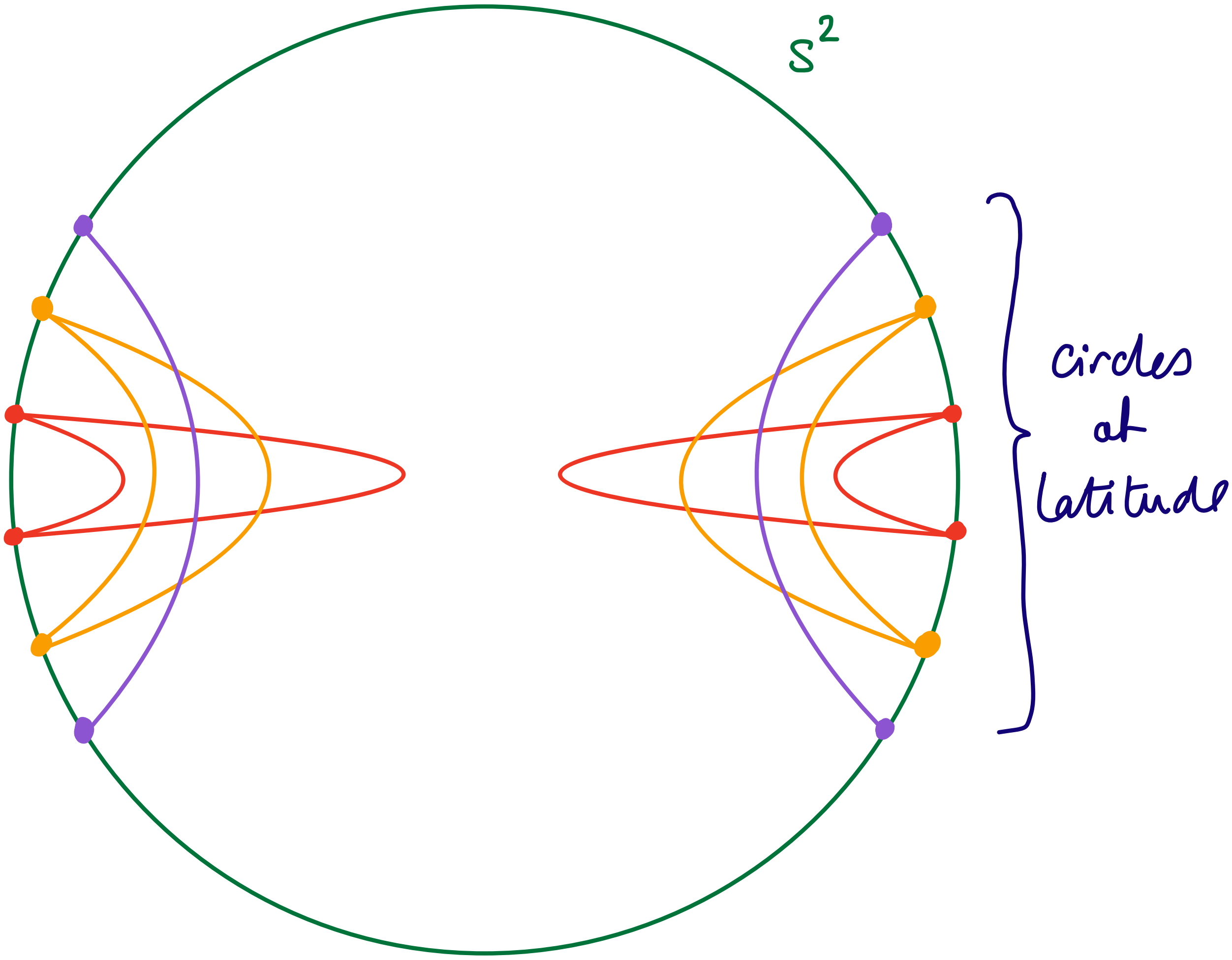
$$\beta : \mathcal{X}_{g,k} \longrightarrow \mathcal{Y}_k .$$

Suppose we could define the degree of β as $\# \beta^{-1}(L)$ for a regular value $L \in \mathcal{Y}_k$ of β.

Then we would have our link invariant

$n_g(L) :=$ $\deg(\beta)$ over connected component containing L

Example: Mori's catenoids in 3D.



- Purple circles are critical value at β.
- Below we have two solutions, opposite signs
- Above we have NO solutions.

To make this happen we need to do the following things:

0. Prove $\mathcal{Y}_k$ is a Banach manifold.

This is standard (providing we use eg Hölder regularity, not C^∞ links).

1. Prove $\mathcal{X}_{g,k}$ is a Banach manifold

(Again we need to use appropriate function spaces here.)

2. Prove β is Fredholm

3. Prove index of β is zero.

4. Trivialise $\operatorname{ind}(d\beta)$

5. Show β is proper.

Points 1, 2, 3, 4 are now theorems,

$$\boxed{\text{BUT } \beta \text{ IS NOT PROPER!}}$$

(Inspired by Alexakis – Mazzeo who counted minimal surfaces in $\mathbb{H}^3$ this way. There β really is proper!)

Recall from yesterday:

$$\mathcal{X}_{g,k} = \left\{ \begin{array}{l} \text{oriented minimal surfaces} \\ \text{in } \mathbb{H}^4, \text{ complete, genus } g \\ \text{meet } \partial_\infty \mathbb{H}^4 \text{ in embedded} \\ k\text{- component link.} \end{array} \right\}$$

$\mathcal{Y}_k$ = k- component links.

$\beta : \mathcal{X}_{g,k} \longrightarrow \mathcal{Y}_k$ send surface to boundary

Claim β is Fredholm map, index zero, between Banach manifolds.

$$\boxed{\text{BUT } \beta \text{ IS NOT PROPER!}}$$

Example of what can go wrong (M.T. Nguyen)

Use the ball model of $\mathbb{H}^4$

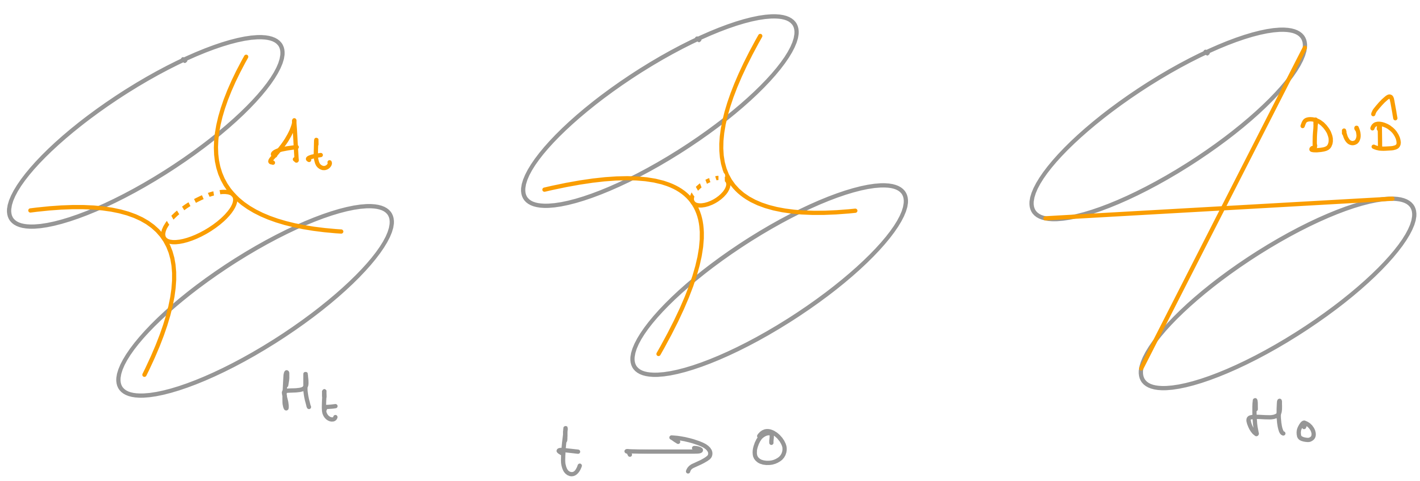
Consider $D = \{ (x_1, x_2, 0, 0) \in \dot{B}^4 \}$
and $\hat{D} = \{ (0, 0, x_3, x_4) \in B^4 \}$
a pair of minimal discs.

$\partial(D \cup \hat{D}) = H_0$, "Standard" Hopf link in S^3.

Theorem (Nguyen)

1. $D \cup \hat{D}$ is the ONLY minimal filling of H_0

2. There is a path H_t of Hopf links $t \in [0,1]$ such that for each $t > 0$ H_t is filled by a minimal annulus A_t.



As $t \to 0$ the waist of A_t pinches and the annulus degenerates into the pair of discs $D \cup \hat{D}$.

(Compare holomorphic curves $zw = t$ in $\mathbb{C}^2$.)

Conjecture

For each g, k there is a codimension 2 subset $\mathcal{B}_{g,k} \subseteq \mathcal{Y}_k$ of "bad" links such that

1. $\beta : \mathcal{X}_{g,k} \to \mathcal{Y}_k$ is proper over $\mathcal{Y}_k \setminus \mathcal{B}_{g,k}$.

2. $\mathcal{Y}_k \setminus \mathcal{B}_{g,k}$ has the same connected components as $\mathcal{Y}_k$. ($\mathcal{B}_{g,k}$ has codim 2.)

When $k = 1$ and $g = 0$, so we are counting minimal discs filling knots this is a theorem.

To understand why the conjecture might be true we need a completely new perspective on minimal surfaces.

IV The Eells – Salamon correspondence

Short interlude :

An almost complex structure on a manifold X is an endomorphism $J : TX \to TX$ with $J^2 = -1$.

J makes TX into a complex vector bundle.

One important source of examples are complex manifolds. Charts take values in $\mathbb{C}^n$ Transition functions are holomorphic.

Recall: $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ is holomorphic if $d\varphi \circ i = i \circ d\varphi$. So $J := \times i$ makes sense on TX.

Not all almost complex structures come this way ! Alg. top gives simple condition for existence of J.

If (X, J_X) and (Y, J_Y) are almost complex manifolds, a map $f: X \to Y$ is called (J_X, J_Y)-holomorphic if:

$$df \circ J_X = J_Y \circ df$$

Now, for minimal surfaces in $\mathbb{R}^3$:

- $\Sigma \subseteq \mathbb{R}^3$ an oriented surface
- Rotation by $90°$ on each $T\Sigma$ defines almost complex structure on Σ.

  Actually a 1D complex manifold, ie a Riemann surface.

- $n: \Sigma \to S^2$ the Gauss map.

Weierstrass's Theorem

Σ is minimal if and only if n is anti-holomorphic, ie $dn \circ i = -i \circ dn$

4D analogue is due to Jim Eells and Simon Salamon. Uses twistor spaces (themselves invented by Roger Penrose).

Let (M^4, g) be oriented Riemannian 4-mfd.

Given $x \in M$, write

$$Z_x = \left\{ \begin{array}{l} J : T_x M \to T_x M \text{ linear st} \\ \bullet\ J^2 = -1 \\ \bullet\ J \text{ is orthogonal} : g(Ju, Jv) = g(u,v) \\ \bullet\ J\text{-orientation is positive} \end{array} \right\}$$

$$\cong \frac{SO(4)}{U(2)} \cong S^2$$

Get S^2-bundle $Z \xrightarrow{\pi} M$, called the twistor space of M

Z has a "tautological almost complex structure," actually 2 of them:

Write $V = \ker d\pi \subseteq TZ$

V is vertical tangent bundle
ie those tangent vectors on Z which are tangent to fibres of π.

Metric g defines Levi-Civita connection and so a connection on all associated bundles, including Z.

So can define "horizontal vectors" $H \leq TZ$, a complement to V.

At $z \in Z$, $d\pi : H_z \longrightarrow T_{\pi(z)} M$
is an isomorphism.

$$T_z Z = V_z \oplus H_z \cong V_z \oplus T_{\pi(z)} M$$

$$J_{\pm} = \pm J_V \oplus j_z$$

J_+ is the "Atiyah-Hitchin-Singer" structure, sometimes integrable.
(Z, J_+) is a whole story in itself, but not for us today...

J_- is the "Eells – Salamon" structure
Never integrable, but very important for minimal surfaces ....

Let $f : \Sigma \to M$ be an immersion from an oriented surface

$f^* g$ makes Σ a Riemann surface.

Let $x = f(p)$, $df(T_p\Sigma) \leq T_x M$ is 2-dim subspace.

<u>Lemma</u> There is a unique $z \in Z_x$ st $df(T_p\Sigma)$ is J_z – complex line with correct orientation:

J_z preserves $df(T_p\Sigma)$ and so also the orthogonal complement

Now J_z is completely determined by requirement that it respects orientations of $T_x M$ and $T_p\Sigma$.

Given $f: \Sigma \to M$ the twistor lift of f is the map $u: \Sigma \to Z$ defined by

$$u(p) = z \quad \text{st} \quad J_z \text{ preserves } df(T_p\Sigma)$$

Theorem (Eells - Salamon)

1. Let $f: \Sigma \longrightarrow M$ be conformal map from a Riemann surface.

The twistor lift $u: \Sigma \to Z$ is J_- - holomorphic if and only if $f(\Sigma)$ is minimal.

2. If $u: \Sigma \to Z$ is J_- - hol. map from a Riemann surface, and $f = \pi \circ u: \Sigma \to M$ is not constant, then f is conformal, $f(\Sigma)$ is minimal and u is the twistor lift of f.

So we have 1-1 correspondence:

$$\left\{ \begin{array}{c} \text{conformally} \\ \text{parametrised} \\ \text{minimal} \\ \text{surfaces} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} J_- \text{ hol. maps} \\ \text{into } Z \end{array} \right\}$$

Very important point!

$u : \Sigma \to Z$ holomorphic, can have critical points, where $du = 0$

Can also have self intersections.

$f := \pi \circ u : \Sigma \to M^4$ can have even more critical points, where $u(\Sigma)$ is tangent to a fibre of π

Can also have self-intersections, where $u(\Sigma)$ meets a fibre of π in more than one point.

When we study J-hol. curves in Z we must accept singularities and self intersections in the minimal surfaces.

Twistor space of $\mathbb{H}^4$.

Look at definition of $Z \to \mathbb{H}^4$.

It only sees the metric g_{hyp} on $\mathbb{H}^4$ up to scale.

$$g_{hyp} = \frac{4\, g_{Euclidean}}{(1-|x|^2)^2}$$

So $g_{Euclidean}$ and g_{hyp} define the same twistor space.

Write $\overline{Z} \to \overline{B}^4$ for twistor space of closed 4-ball

$\overline{Z}$ is compact manifold with boundary

$$\partial Z \cong S^3 \times S^2$$

Interior Z is twistor space of $\mathbb{H}^4$.

Now $T_z Z = V_z \oplus H_z$ is defined by Levi-Civita connection of $\mathbb{H}^4$.

This is DIFFERENT for g_{hyp} and $g_{Euclidean}$ and so horizontal bundle DOESN'T extend up to ∂Z

And nor does Eells-Salamon J_- !

As we will soon see it has a particular type of singularity at ∂Z.

<u>Aside</u> Atiyah-Hitchin-Singer structure J_+ is actually conformally invariant and so $(\bar{Z}, J_+)$ is the same for g_{hyp} and $g_{Euclidean}$. This is not obvious from the way I've described things ...

From now on (Z, J) is twistor space of $\mathbb{H}^4$ with $J = J_-$ (Eells-Salamon)

V. The Moduli Space of J-hol curves

Now let $\bar{\Sigma}$ be compact surface of genus g, with k bdry components, and interior Σ.

We want to study pairs (u, j) where

$$
* \begin{cases}
\bullet \; j \text{ is a complex structure on } \bar{\Sigma} \\
\bullet \; u^{-1}(\partial Z) = \partial \Sigma \\
\bullet \; \pi \circ u : \partial \Sigma \longrightarrow S^3 = \partial \mathbb{H}^4 \text{ is an embedding} \\
\bullet \; u : (\Sigma, j) \to (Z, J) \text{ is holomorphic} \\
\qquad \text{ie } du \circ j = J \circ du
\end{cases}
$$

We also divide out by diffeomorphisms of $\bar{\Sigma}$ (pulling back j and reparametrising the map u).

$$\mathcal{X}_{g,k} = \{(u, j) \mid * \text{ holds}\} / \mathrm{Diff}(\bar{\Sigma})$$

is the moduli space of J-hol curves

Or, equivalently, genus g minimal surfaces.

Want to apply theory of J-hol. curves (Fredholm, compactness) to try and define the degree of the boundary map

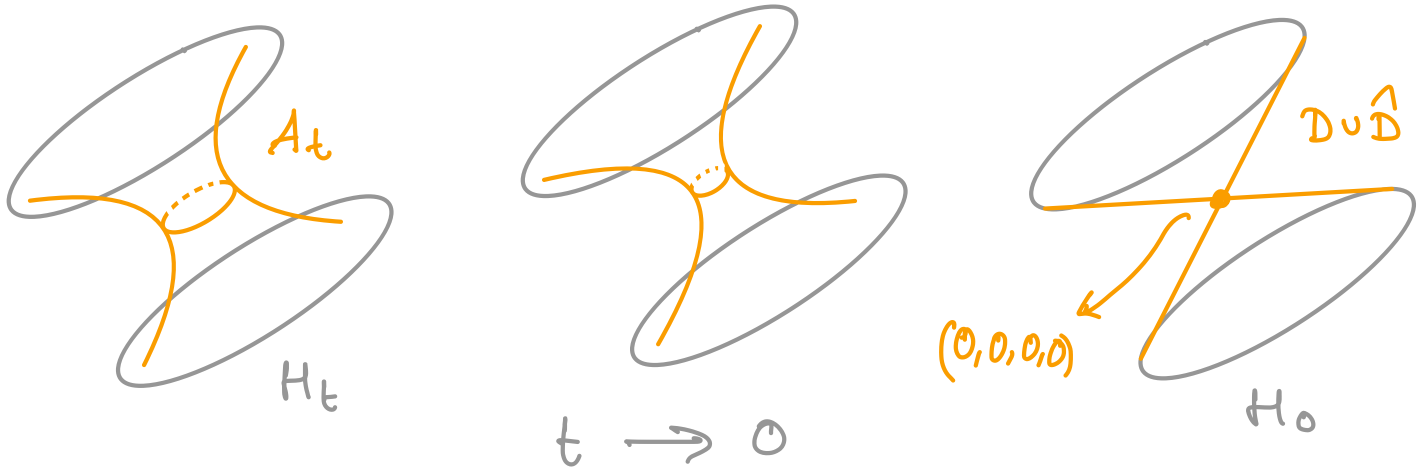
$$\beta : \mathcal{X}_{g,k} \to \mathcal{Y}_k .$$

$$\beta[u,j] = \pi(u(\partial\Sigma))$$

Intersections and properness of β

First, let's see why this shows a possible solution to non-properness of β.

Recall Nguyen's example for Hopf links:



If we look at the twistor lifts we see that the limit is a pair of J-hol. discs in Z which meet at the point z which corresponds to j_z on $T_{(0,0,0,0)}\mathbb{H}^4$ which makes BOTH

$T_{(0,0,0,0)}D$ AND $T_{(0,0,0,0)}\hat{D}$ complex lines.

Now in general, asking for a pair of discs to meet in a 6-manifold is to impose a 2-dimensional constraint

So from view-point of Z we should expect that the set of Hopf links in $\mathcal{Y}_2$ which are filled by a pair of hol. discs in Z which have non-trivial intersection should be codim 2.

This is a theorem that I'll explain in a bit, but here's evidence.

One way to make a Hopf link is to take a pair of 2D linear subspaces $V, W \leq \mathbb{R}^4$ which meet transversely at the origin.

Then $(V \cup W) \cap S^3 =: H(V,W)$ is a Hopf link.

$H(V,W)$ is filled by a unique pair of minimal discs:

$$D(V) = V \cap B^4$$
$$D(W) = W \cap B^4$$

Obviously they meet at 0.

These lift to a pair of J-hol. discs $\hat{D}(V)$ and $\hat{D}(W)$ in Z

But these discs meet in Z if and only if V and W are complex lines FOR THE SAME j.

V alone already determines j, so we're asking for the real 2D subspace W to actually be j-complex

This is asking for $W \in Gr(2,4)$ (dim 4) to lie in a copy of $\mathbb{CP}^1$ (dim 2)

So $\hat{D}(V) \cap \hat{D}(W) \neq \emptyset$ defines a codimension 2 subset in the space of all pairs (V,W)

The corresponding Hopf links $H(V,W)$ are among the "bad" links in $\mathcal{Y}_2$ that we should exclude.

VI. The J-hol. equation is degenerate

We want to prove that $\mathcal{X}_{g,k}$ is a Banach manifold.

Proof is technical, but idea is quite easy,

Look at ambient space of maps $\mathcal{A}_{g,k}$ and almost complex structures $\mathcal{J}$ on $\overline{\Sigma}$

Ask maps $u : \overline{\Sigma} \to \overline{Z}$ to have

- $u(\overline{\Sigma}) \cap \partial Z = \partial \Sigma$ transversely
- $u|_{\partial \Sigma}$ an embedding

$\mathrm{Diff}(\overline{\Sigma})$ acts on $\mathcal{A}_{g,k} \times \mathcal{J}$ and quotient is smooth Banach manifold $\mathcal{Z}_{g,k}$

$\mathcal{X}_{g,k} \subseteq \mathcal{Z}_{g,k}$ is zero locus of an equation, that $du + J \circ du \circ j = 0$

$\mathcal{X}_{g,k}$ is smooth submanifold provided linearised equations are surjective.

I won't give details, but I will try and motivate this, and explain why the map,
$\beta : \mathcal{X}_{g,k} \longrightarrow \mathcal{Y}_k$ is Fredholm

The Eells-Salamon J at infinity

Use half space coordinates (x, y_1, y_2, y_3) on $\mathbb{H}^4$

$$g_{hyp} = \frac{dx^2 + dy^2}{x^2}$$

There are coordinates on twistor space too

Define j_1, j_2, j_3 by $j_i(\partial_x) = \partial_{y_i}$

So
$$
\begin{aligned}
j_1(\partial_x) &= \partial_{y_1} \\
j_1(\partial_{y_1}) &= -\partial_x \\
j_1(\partial_{y_2}) &= \partial_{y_3} \\
j_1(\partial_{y_3}) &= -\partial_{y_2}
\end{aligned}
$$

Similarly for j_2, j_3

Given $z = (z_1, z_2, z_3)$ with $z_1^2 + z_2^2 + z_3^2 = 1$.
write $j_z = z_1 j_1 + z_2 j_2 + z_3 j_3$

$$j_z^2 = -|z|^2 = -1.$$

Now in these coordinates we can write down the Eells–Salamon J:

$$J \begin{pmatrix} \partial x \\ \partial y_i \\ \partial z_i \end{pmatrix} = \begin{pmatrix} 0 & -z^T & 0 \\ z & R(z) & 0 \\ 0 & 2x^{-1}P(z) & -R(z) \end{pmatrix} \begin{pmatrix} \partial x \\ \partial y_i \\ \partial z_i \end{pmatrix}$$

where $z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$

$$R(z) = \begin{pmatrix} 0 & -z_3 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{pmatrix}$$ cross product w/. z

$$P(z) = 1 - zz^T$$ projection onto plane $\perp$ to z.

KEY POINT is the $2x^{-1}P(z)$ which blows up as $x \to 0$!

Matrix for J is 7×7, because we've used 7D coordinates (x, y_i, z_i). But actually we're interested in $|z| = 1$.

The tangent space to Z corresponds to vectors $a\partial_x + b^i \partial_{y^i} + c^i \partial_{z^i}$ for which $z_i c^i = 0$

You can check the above matrix acts on these vectors and there, $J^2 = -1$.

Look at J-hol. map $u : S^1 \times [0,\delta) \to \overline{Z}$

Coords (s,t) on $S^1 \times [0,\delta)$, $j(\partial_s) = \partial_t$

Assume that $u^{-1}(\partial Z) = S^1 \times \{0\}$

Write $x, y_i, z_i : \Sigma \to \mathbb{R}$ for $x \circ u$ etc.

Assume $x(s,t)$, $y_i(s,t)$, $z_i(s,t)$ have Taylor series expansions in t and try to solve
$du + J \circ du \circ j = 0$ term-by-term:

One checks that...

$$x(s,t) = |\dot\gamma(s)|t + \mu(s)t^3 + O(t^4)$$
$$y(s,t) = \gamma(s) + \eta(s)t^2 + \nu(s)t^3 + O(t^4)$$
$$z(s,t) = -\frac{\ddot\gamma(s)}{|\dot\gamma(s)|} + \zeta(s)t + \xi(s)t^2 + O(t^3)$$

and η and ξ also determined by γ (but formulae aren't so pretty).

μ, ν, ζ are undetermined but once you pick them, everything else in a formal power series is determined.

Compare this with ordinary holomorphic functions $\varphi : S^1 \times [0,\delta) \to \mathbb{C}$

$$\varphi = \varphi_0(s) + \varphi_1(s)t + \varphi_2(s)t^2 + \cdots$$

$(\partial_s + i\partial_t)\varphi = 0$ means

$$\dot\varphi_n + (n+1)i\varphi_{n+1} = 0$$

So φ_0 determines all the other coefficients.

Formally at least we can pick φ_0 and get a holomorphic function on $S^1 \times [0,\delta)$.

For J-hol curves in $\overline{Z}$, formally at least we can pick $\gamma : S^1 \to \mathbb{R}^3$ and the other coefficients μ, ν, ξ, and get a J-holomorphic map
$$u : S^1 \times [0,\delta) \longrightarrow \overline{Z}.$$

If we want this to be "boundary data" of a hol. fnc. $\varphi : \overline{\Sigma} \to \mathbb{C}$ or a J-hol map $u : \overline{\Sigma} \to \overline{Z}$ with closed domain, we can only specify "half" the boundary data.

For $\varphi : \overline{\Sigma} \to \mathbb{C}$ the real part of φ_0 is the "right" amount of freedom.

For $u : \overline{\Sigma} \to \overline{Z}$, γ is the "right" amount of freedom.

When we also divide out by diffeomorphisms of $\overline{\Sigma}$ we see that prescribing image of γ ie $\pi(u(\partial\Sigma)) \subseteq S^3$ is the "right" amount of freedom.

This leads to the hope that solving $u : \overline{\Sigma} \to \overline{Z}$ with $\pi(u(\partial\Sigma)) = L$ FIXED should be a Fredholm problem.

Fredholm means linearised equations have finite dimensional cokernel, WHEN WE PERTURB u BUT DON'T MOVE L!

In $\mathcal{X}_{g,k}$ we are also allowed to move L

This is an infinite dimensional degree of freedom which fills out the finite dimensional cokernel.

So we expect $\mathcal{X}_{g,k}$ to be transversely cut out, hence smooth

And we expect $\beta : \mathcal{X}_{g,k} \to \mathcal{Y}_h$ to be Fredholm.

Can prove this rigorously. Need to use 0-calculus of Mazzeo-Melrose, because equation ISN'T ELLIPTIC. It degenerates as $t \to 0$, symbol tends to zero there.

As for the fact that index$(\beta) = 0$ ...

There is no index theorem known for the 0-calculus, so one has to prove ind$(\beta) = 0$ "by hand." That is a story for another day...

Aside

Recall J is singular at $x=0$:

$$J\begin{pmatrix}\partial_x\\ \partial_{y_i}\\ \partial_{z_i}\end{pmatrix} = \begin{pmatrix} 0 & -z^T & 0\\ z & R(z) & 0\\ 0 & 2x^{-1}P(z) & -R(z)\end{pmatrix}\begin{pmatrix}\partial_x\\ \partial_{y_i}\\ \partial_{z_i}\end{pmatrix}$$

However, if we use the following tangent vectors:

$$x\,\partial_x,\quad x\,\partial_{y_i},\quad \partial_{z_i}$$

then the matrix for J becomes ...

$$J\begin{pmatrix}x\partial_x\\ x\partial_{y_i}\\ \partial_{z_i}\end{pmatrix} = \begin{pmatrix} 0 & -z^T & 0\\ z & R(z) & 0\\ 0 & -2P(z) & -R(z)\end{pmatrix}\begin{pmatrix}x\partial_x\\ x\partial_{y_i}\\ x\partial_{z_i}\end{pmatrix}$$

THIS EXTENDS SMOOTHLY UP TO $x=0$!

Moral : life will be easier if we use $x\partial_x$, $x\partial_{y_i}$ and ∂_{z_i}

These vector fields generate a bundle called the edge tangent bundle ${}^eTZ$

${}^eTZ \subseteq \mathbb{R}^7 \times \overline{Z}$ is the sub-bundle of (a, b^i, c^i) st $c^i z_i = 0$

There is a map ${}^eTZ \to T\overline{Z}$ given by

$$(a, b, c) \longmapsto a\, x\partial_x + b^i x\partial_{y_i} + c^i \partial_{z_i}$$

called the anchor map

Away from ∂Z this is an isomorphism

Along ∂Z the image is exactly $V \leq TZ$ the vertical tangent vectors.

$\left\{ \text{sections of } {}^eTZ \right\} = \left\{ \begin{array}{l} \text{vector fields on } \bar{Z} \\ \text{which at } \partial Z \text{ are} \\ \text{tangent to fibres of } \pi \end{array} \right\}$

$J : {}^eTZ \to {}^eTZ$ is an <u>edge almost complex structure</u>

${}^eTZ \longrightarrow T\bar{Z}$ is an example of a <u>Lie algebroid</u>. Interesting to ask:

$$\boxed{\begin{array}{l} \text{Which Lie algebroids } E \to TX \\ \text{with "}E\text{"-almost complex structure} \\ \text{give rise to interesting Fredholm} \\ \text{theory for } J\text{-holomorphic curves?} \end{array}}$$

Recall:

$Z \to \mathbb{H}^4$ is twistor space of $\mathbb{H}^4$

J Eells–Salamon almost complex structure

$\mathcal{X}_{g,k}$ = moduli space of genus g, J-hol curves $u : (\Sigma, j) \to (Z, J)$ filling k-component links in $S^3 = \partial_\infty \mathbb{H}^4$

$\mathcal{Y}_k$ = k-component links

$\beta : \mathcal{X}_{g,k} \to \mathcal{Y}_k$ boundary map

$$\beta[u, j] = \pi(u(\partial\Sigma)) \subseteq S^3$$

β is Fredholm map of index 0 between Banach manifolds.

$\boxed{\text{Want to understand properness of } \beta}$

Let $u_n : (\overline{\Sigma}, j_n) \to \overline{Z}$ be a sequence of J-holomorphic curves for which $L_n := \pi(u_n(\partial\Sigma)) \subseteq S^3$ converges to a link L_∞

Want to understand when (u_n, j_n) has convergent subsequence.

Write $f_n = \pi \circ u_n : \overline{\Sigma} \longrightarrow \overline{\mathbb{H}}^4$

Conformal parametrisation of minimal surface

COMPACTNESS THEOREM A

Suppose that $j_n \longrightarrow j_\infty$, a complex structure on $\overline{\Sigma}$. Then, up to a subseq.,
$f_n \longrightarrow f_\infty : (\overline{\Sigma}, j_\infty) \to \overline{\mathbb{H}}^4$ conformal param. of a minimal surface filling L_∞.

If, moreover, f_∞ has no critical points then $u_n \longrightarrow u_\infty$, the twistor lift of f_∞

Why worry about critical points of f_∞?

Let $u : (\Sigma, j) \to Z$ be J-hol., twistor lift of $f = \pi \circ u$.

Fix metric h on Σ compatible with j

$u(p) = J$ on $T_{f(p)} \mathbb{H}^4$ which equals j on $df(T_p\Sigma)$

identify skew endomorphism J with element of $\Lambda^2 T\mathbb{H}^4$ via metric g_{hyp}.

h-orthonormal frame e_1, e_2 at $p \in \Sigma$.

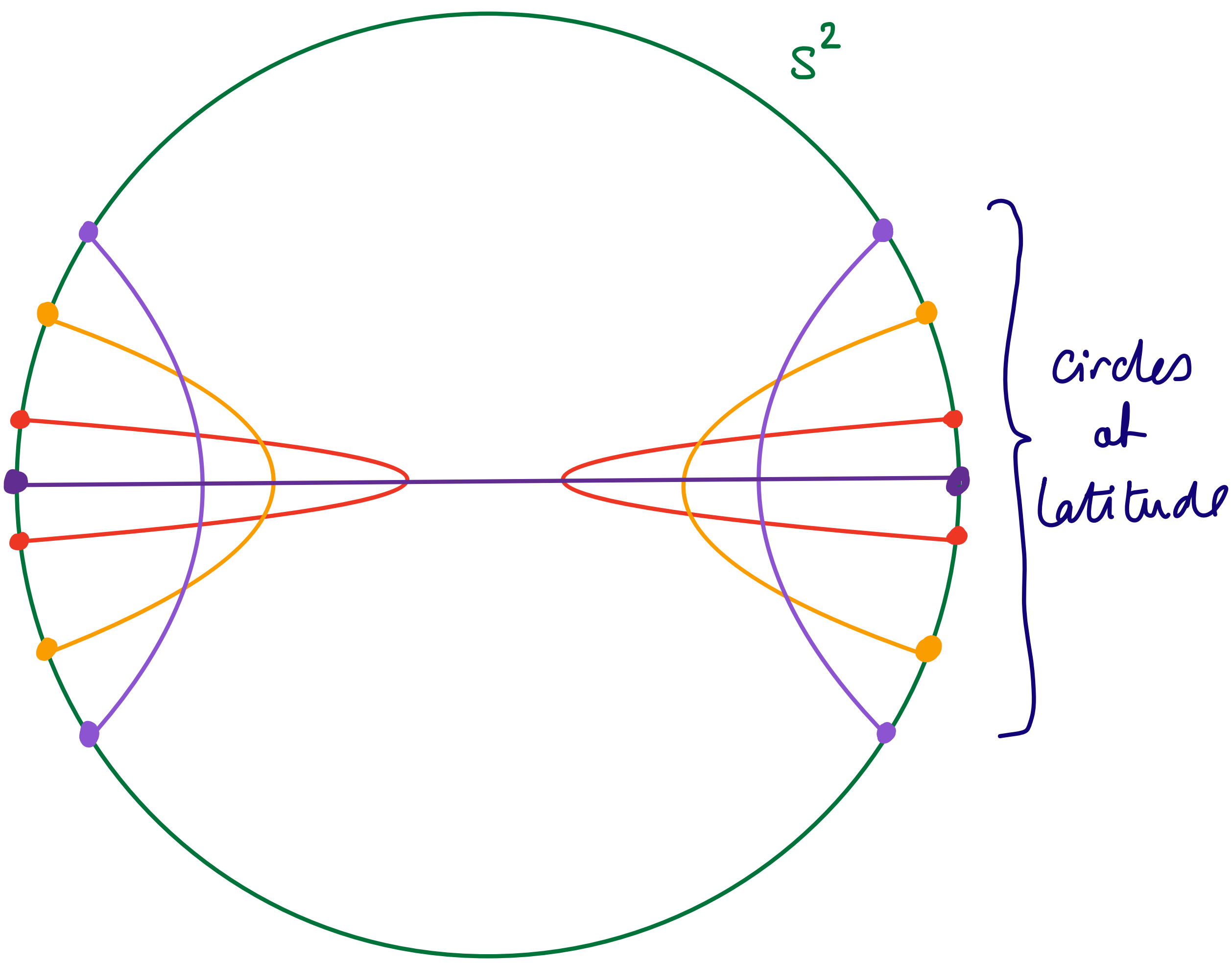
$$u(p) = \frac{df(e_1) \wedge df(e_2) + *(df(e_1) \wedge df(e_2))}{|df(e_1) \wedge df(e_2)|_{g_{hyp}}}$$

Denominator vanishes at critical point of f!

For a single f its no problem: critical points are isolated and u extends smoothly over them, defined on all of Σ

But for a sequence of f_n which develop a critical point in the limit, u_n might develop a bubble.

This really can happen! We've already seen a 3D example:



In the limit as boundary converges to 2 × equator, the surfaces converge to 2 × totally geodesic disc.

But Gauss map progressively covers more and more of S^2

In the limit, we get Gauss map of totally geodesic disc (constant) AND a "bubble" of the whole of S^2 at the origin.

(This is not quite an example fitting hypotheses of theorem above: j_n on annuli don't converge, boundary links don't converge.)

Let $Z \subseteq \mathcal{X}_{g,k}$ be those $[u,j]$ for which $f = \pi \circ u$ has a critical point.

Theorem Z has codimension 2.

Justification: Consider J-hol curves with a marked point $p \in \Sigma$

Moduli space $\mathcal{X}^1_{g,k} = \dfrac{\{(u,j,p) \mid p \in \Sigma\}}{\mathrm{Diff}(\Sigma)}$

It's a Banach manifold, fibres over $\mathcal{X}_{g,k}$ by forgetting the point p:

$$\Sigma \hookrightarrow \mathcal{X}^1_{g,k} \longrightarrow \mathcal{X}_{g,k}$$

$$df_p \in \mathrm{Hom}_{\mathbb{C}}\left(T^*_p\Sigma, H_{u(p)}\right)$$

rank 2 complex vector space.

These vector spaces fit together to give rank 2 complex vector bundle

$$\mathcal{V} \longrightarrow \mathcal{X}^1_{g,k}$$

and df_p gives a section s of $\mathcal{V}$

$$s[u,j,p] = 0 \quad \text{iff} \quad df_p = 0$$

Fact: s vanishes transversely and so $s^{-1}(0) \subseteq \mathcal{X}^1_{g,k}$ is (real) codimension 4.

Fibres of $\mathcal{X}^1_{g,k} \to \mathcal{X}_{g,k}$ are 2-dimensional

$$s^{-1}(0) \longrightarrow \mathcal{Z}$$

So $\mathcal{Z}$ is codimension 2.

Consequence: we can ignore links which are filled by f with critical points.

Main Theorem:

We can count minimal discs! I.e. Degree of $\beta: \mathcal{X}_{0,1} \to \mathcal{Y}_1$ is well-defined.

Proof Up to diffeomorphisms, there is a UNIQUE j on $\overline{D}$, so we just fix it.

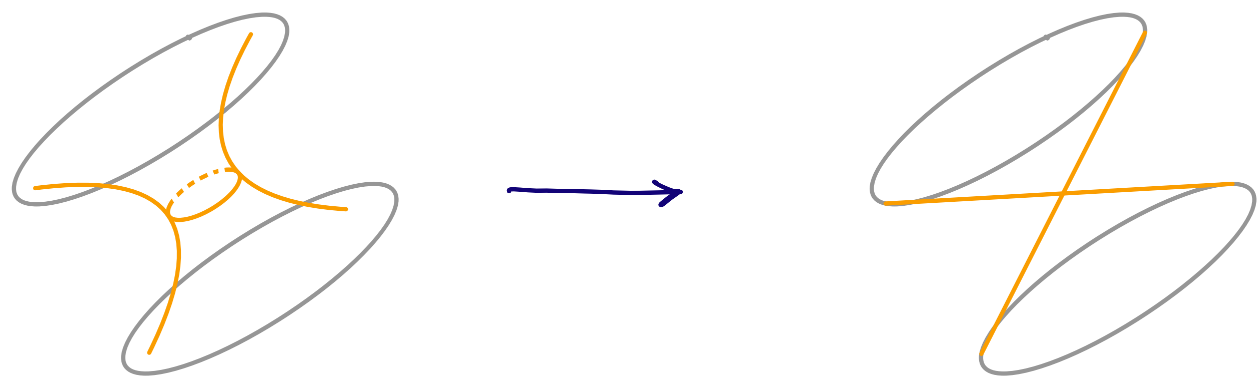
Let $\mathcal{B} \subseteq \mathcal{Y}_1$ be those knots such that if $\beta(u) \in \mathcal{B}$ then f has branch points.

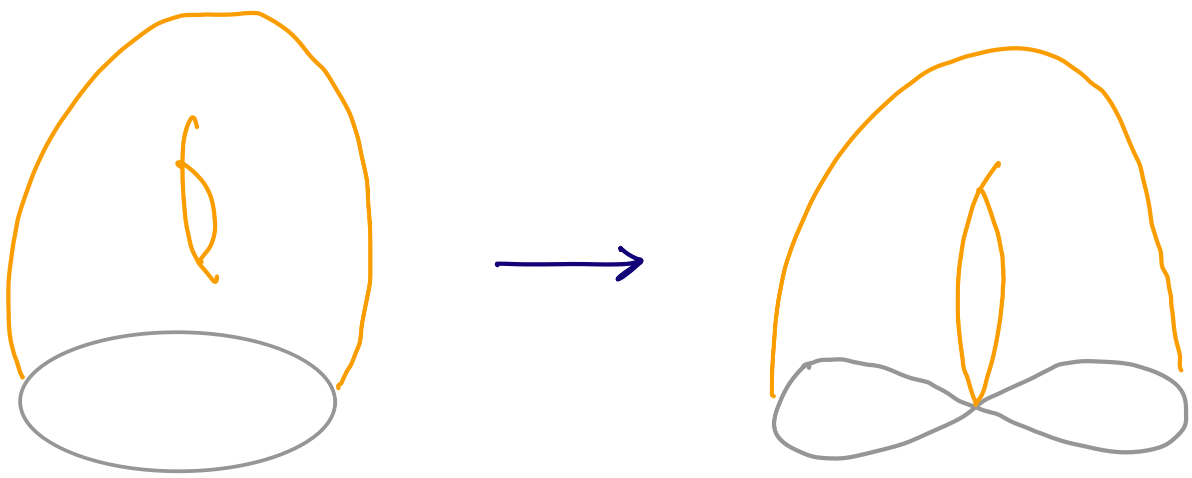
$\mathcal{B}$ has codimension 2, and by Compactness Theorem A above, β is proper over $\mathcal{Y}_1 \setminus \mathcal{B}$. ☺

When J_n don't converge.

Deligne – Mumford: after acting by diffeomorphisms we can make a subsequence of J_n converge to a NODAL limit.

ie a Riemann surface with some points "glued together" to make nodes





"boundary node"

COMPACTNESS THEOREM B

If $J_n \longrightarrow J_\infty$, a complex structure on a NODAL Riemann surface $\bar{S}$ with boundary, then

1). $\bar{S}$ has NO boundary nodes.

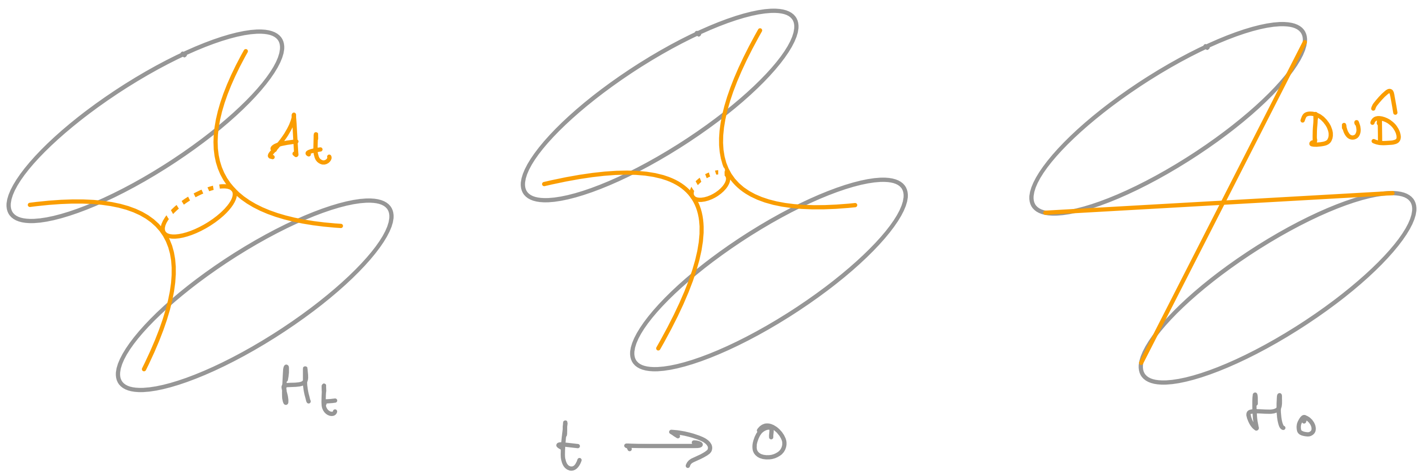
2) A subsequence of the f_n converges to a conformal parametrisation of a nodal minimal surface

$$f_\infty : \bar{S} \to \bar{\mathbb{H}}^4$$

i.e. f_∞ is ordinary conformal map on each component of $\bar{S}$ and it agrees on points which are glued to make nodes.

Moreover, $f_\infty(\partial \bar{S}) = L_\infty$.

We've already seen an explicit example of this (due to M. T. Nguyen):



At first sight it looks like we're nearly done with the definition of $n_g(L)$

Suppose that the u_n converge to the twistor lift $u_\infty : \bar{S} \to \bar{Z}$ of f_∞. Then we have intersecting J-hol. curves in 6D Z, which is a codim 2 phenomenon. We just ignore these "bad" links.

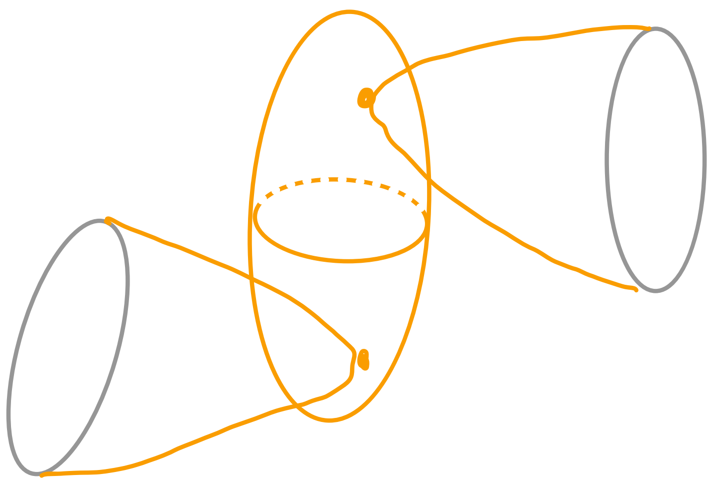
This is what happens for Nguyen's example.

But something else can go wrong.

Even if f_∞ has no critical points, u_n can still bubble exactly where the node forms.

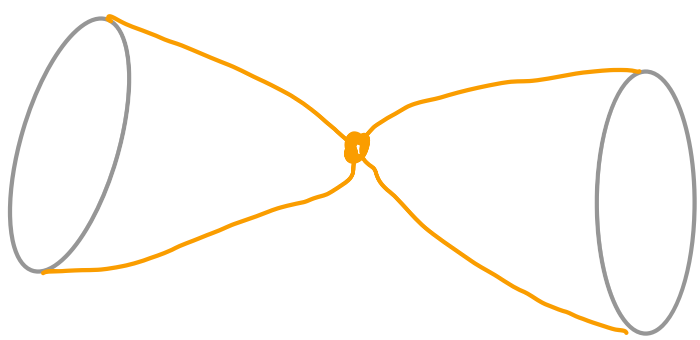
This is what happens for Mori's catenoids.

Up in twistor space the picture is



$\subseteq \overline{Z}$

$\downarrow \pi$



$\subseteq \overline{\mathbb{H}}^4$

This is not avoidable!

But it still "SHOULDN'T" happen.

Twistor fibre is J-holomorphic curve. It has index 0, so should be isolated

If that were true then the above picture would be codimension 4 in the space of links.

But the twistor fibre moves in 4D family (the points of H^4!)

Linearised J-hol. equation for twistor fibre has 4D cokernel.

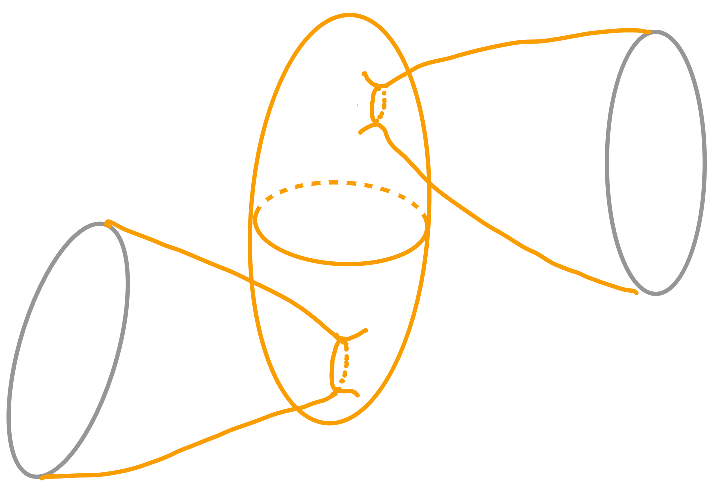
This is what I hope should save the definition of $n_g(L)$.

Conjecture

You can't avoid singular configurations like in the above picture, but they should only arise as limits of smooth things in codimension 4.

Justification (but NOT a proof!)

Take a singular configuration of J-hol. curves as above and smooth it out, gluing in annuli to replace nodes.



Result is APPROXIMATELY J-holomorphic.
at least when annuli are very small

Try to perturb to genuine solution.

If linearised equations are surjective
then there is a solution nearby

But they won't be, they'll have
4D cokernel coming from fact that
twistor fibre has cokernel.

So can only use implicit function
theorem if error lies in codim 4
space.

To arrange this, you'll have to
move boundary knot or link,
giving 4D constraint on Y_k.

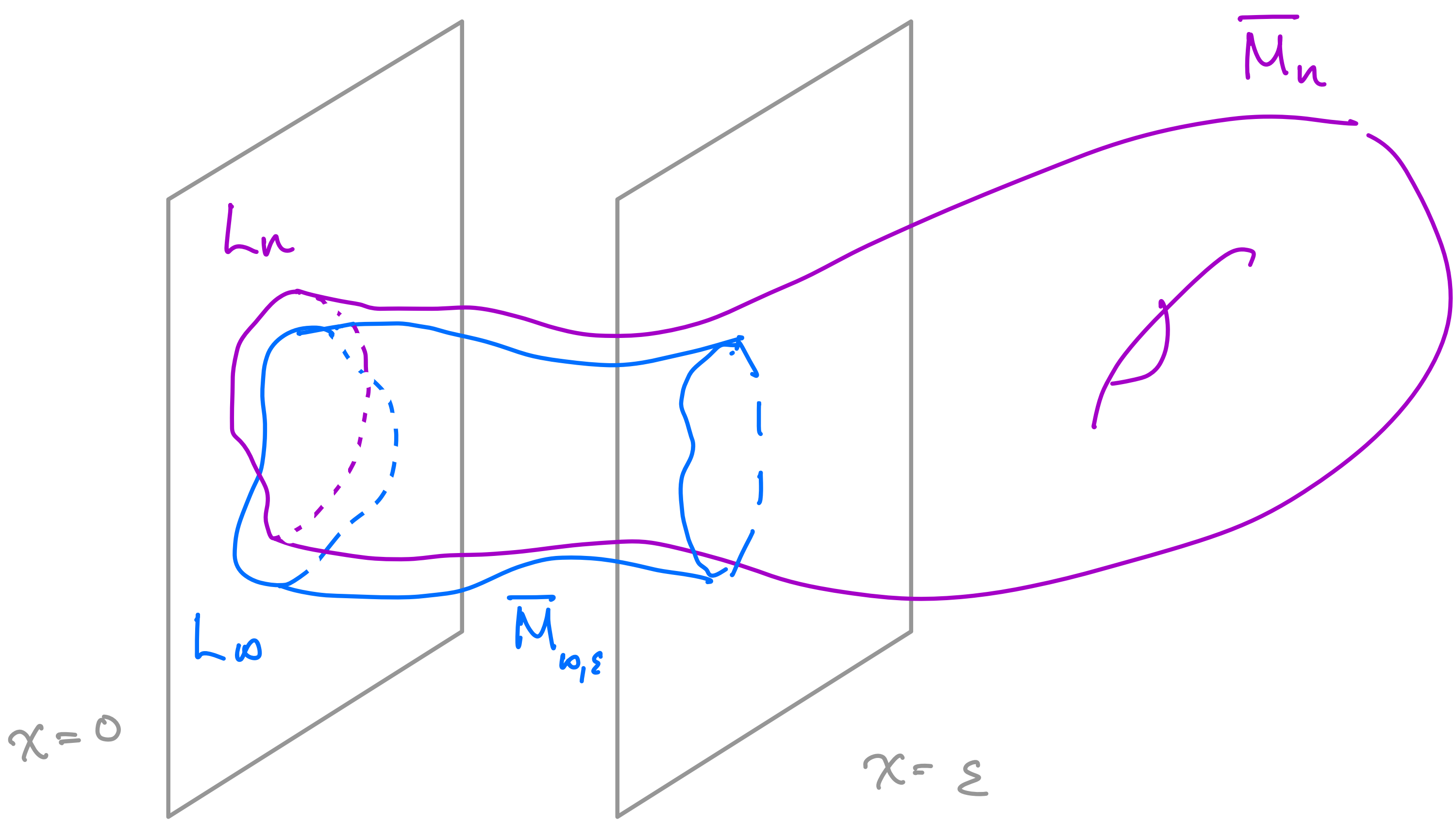
Main new ideas in proofs of compactness theorems: Convergence near infinity.

Put $\overline{M}_n = f_n(\overline{\Sigma})$, minimal surface.

Theorem. There exists $\varepsilon > 0$ such that, after passing to a subsequence,

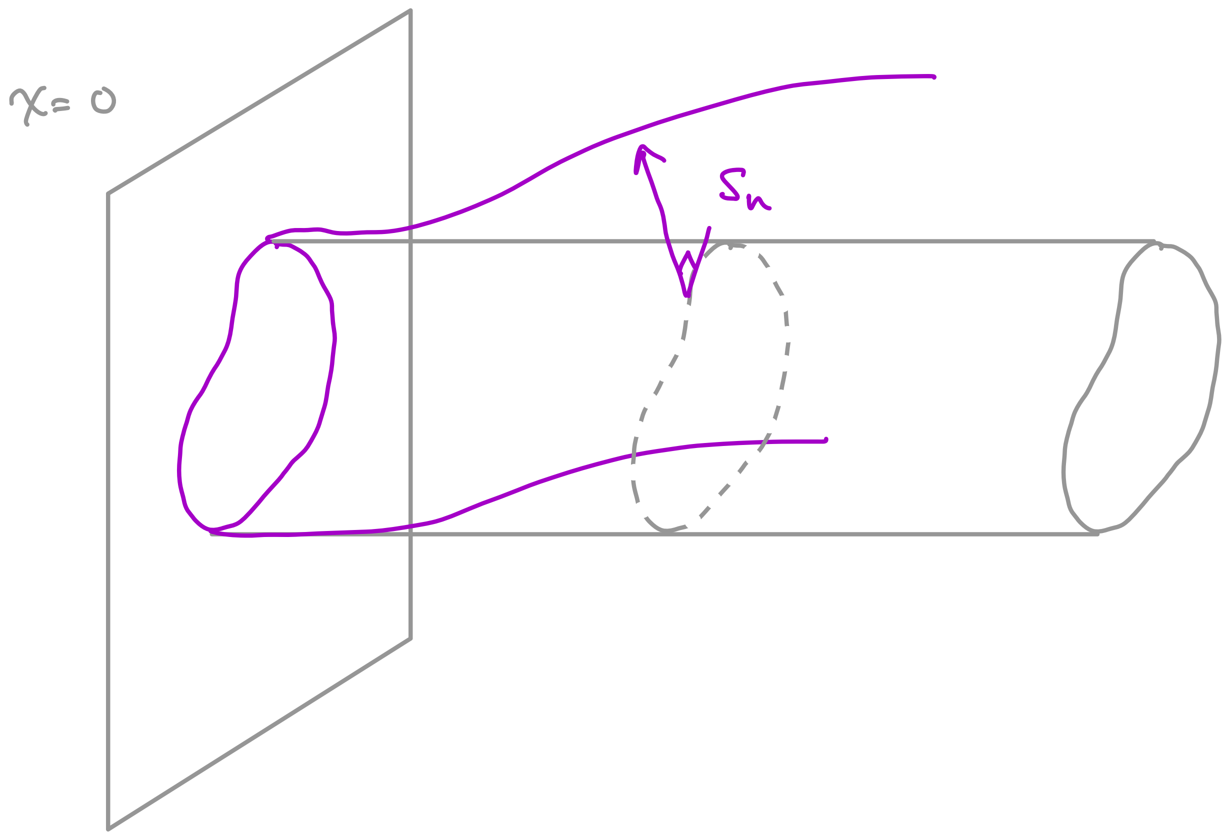
$$\overline{M}_{n,\varepsilon} = \overline{M}_n \cap \{x \leq \varepsilon\}$$

converges to a minimal surface $\overline{M}_{\infty,\varepsilon}$ filling L, and diffeomorphic to $L \times [0, \varepsilon]$



Ideas in the proof

Near $x=0$, each $\overline{M}_n$ is a graph over a cylinder :



$s_n : L_n \times [0, \varepsilon_n] \longrightarrow \mathbb{R}^3$,

$s_n(p,\xi)$ normal to $T_p L_n$.

Proposition There is $\varepsilon > 0$ INDEPENDENT OF n so that we can write $\overline{M}_{n,\varepsilon}$ as a graph over $L_n \times [0,\varepsilon]$, of map s_n

Moreover, $|\nabla s_n| \leq 1$ on all of $L_n \times [0, \varepsilon]$.

Sketch of proof

We can do this on $[0, \varepsilon_n]$, for $\overline{M}_n$.

Suppose for a contradiction that $\varepsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$.

Suppose ε_n is the first time $|\nabla s_n| = 1$

(Other things could go wrong eg $\overline{M}_n$ has double point or branch point at $x = \varepsilon_n$, but let's worry just about $|\nabla s_n| = 1$ for now. Same techniques work in other cases)

Idea is to RESCALE BY ε_n^{-1}

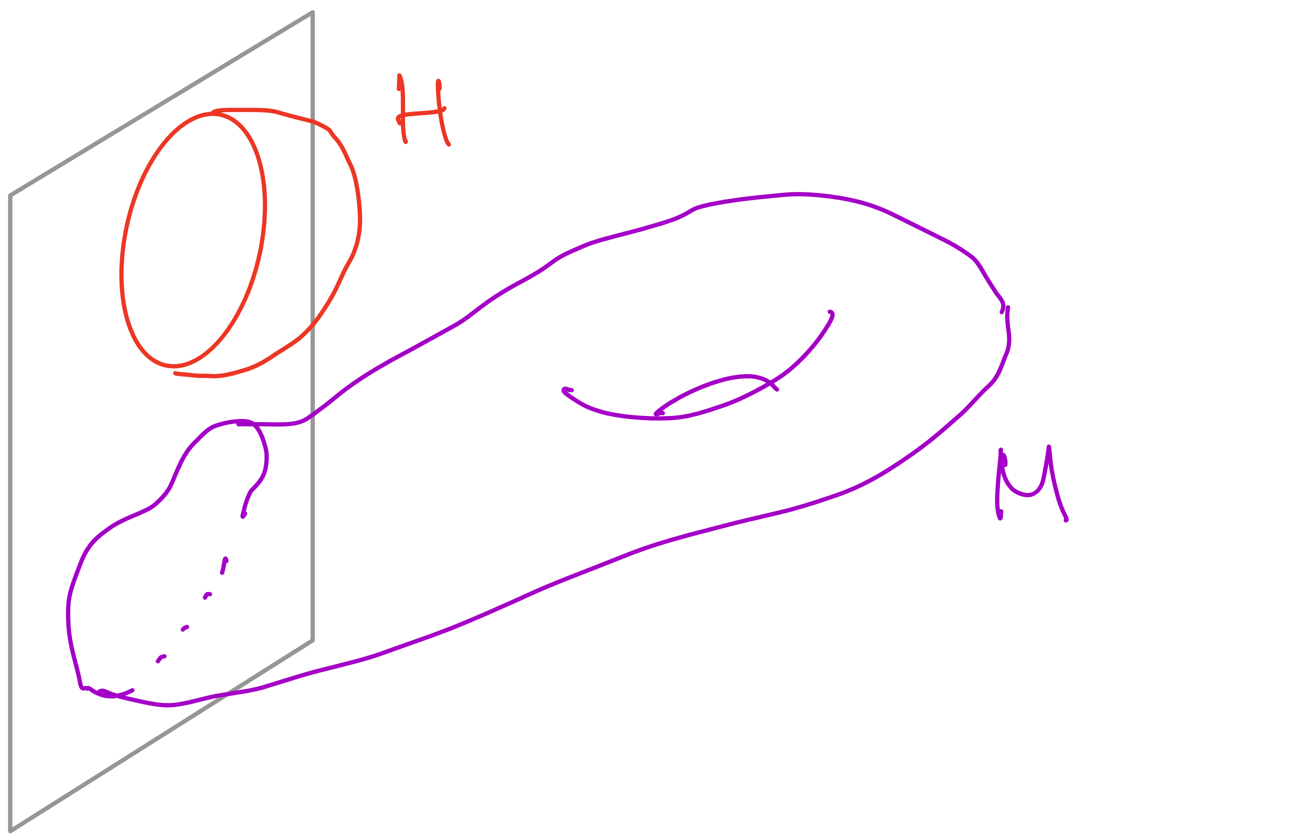
To take limit we use barriers:

Let $S \subseteq \mathbb{R}^3$ be a round 2-sphere

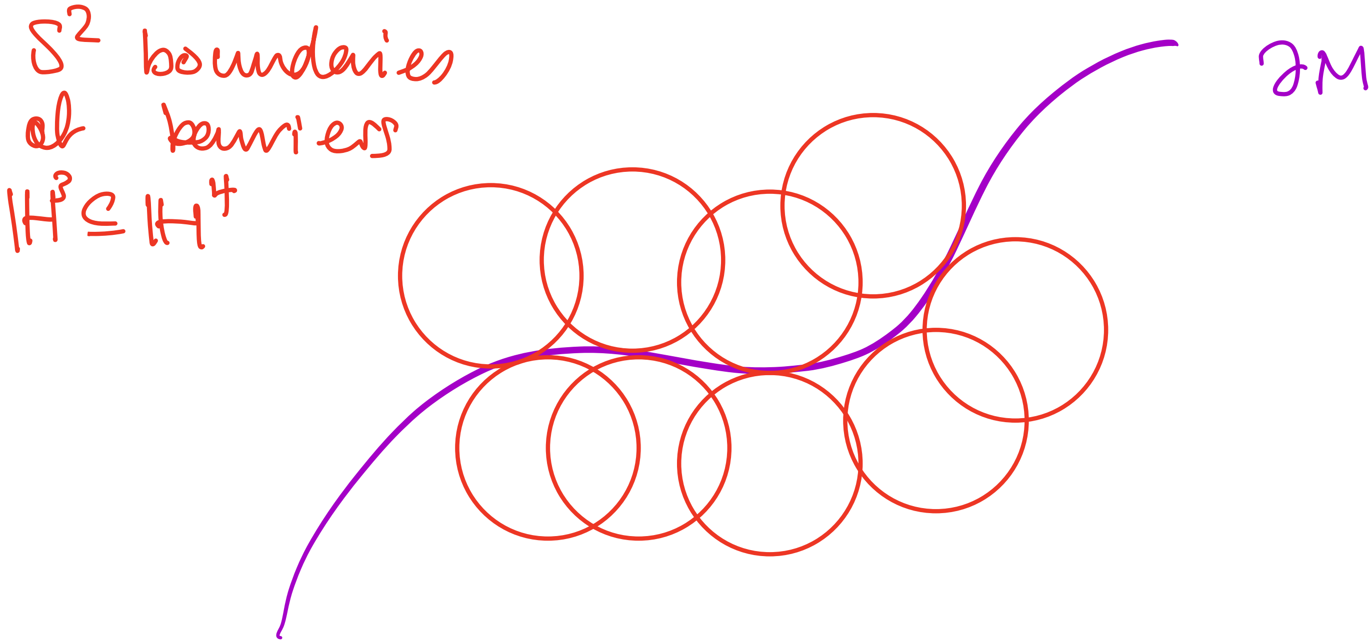
$S = \partial H$ for some totally geodesic copy $H \subseteq \mathbb{H}^4$ of $\mathbb{H}^3$.

Suppose $M \subseteq \mathbb{H}^4$ is a minimal surface with boundary outside of S

Then M must be outside of H by maximum principle: If $d : M \to \mathbb{R}$ is distance to H then $\Delta d \leq 0$.



So M is constrained to lie in a region determined by C^2 geometry of ∂M:



Now, we have points $p_n \in L_n$ st
$$|\nabla s_n(p_n, \varepsilon_n)| = 1$$

Assume $p_n \to p$.

"Rescale" centred at p_n by factor ε_n^{-1} in half space coordinates:
$$(x, y) \mapsto (\varepsilon_n^{-1} x, \ \varepsilon_n^{-1}(y - p_n))$$

THIS IS AN ISOMETRY OF $\mathbb{H}^4$

$\overline{M}_n$ gets mapped to a new minimal surface $\overline{X}_n$

$\partial \overline{X}_n$ passes through $(0,0)$

Assume $L_n \to L_\infty$ in C^2, then $\partial \overline{X}_n \to \Lambda$ a line in $\mathbb{R}^3$

Λ is $T_p L_\infty$ translated to go through 0

We can hit bigger and bigger barriers so $\overline{X}_n$ converges to the copy $H \subseteq \mathbb{H}^4$ of $\mathbb{H}^2$ with boundary Λ

Barriers give C^0 convergence $\overline{X}_n \to H$

Deep results for minimal surfaces (due to White and Allard) imply

$\bar{X}_n \to H$ in C^∞ on compact sets in $\mathbb{H}^4$

Meanwhile $\bar{X}_n$ is still a graph, now over $[0,1]$, of section $\tilde{s}_n$

And $\nabla \tilde{s}_n = \nabla s_n$ since we rescaled both domain and range equally.

So $$|\nabla \tilde{s}_n (p_n, 1)| = 1$$

Since $\bar{X}_n \to H$, $\tilde{s}_n \to 0$ in C^∞ on compact sets. But $|\nabla \tilde{s}_n (p_n, 1)| = 1$ for all n and thats our contradiction!

We now have uniform C^1 control of s_n near infinity so, by Arzela–Ascoli we have a subsequence that converges in C^0. But we need derivatives to converge too.

2. The s_n solve a PDE of the form
$F(s_n, \nabla s_n, \nabla^2 s_n) = 0$, because
the graphs are minimal.

Use analysis of PDE to show that
the s_n converge in $C^{k,\alpha}$

Actually, we assume $L_n \to L_\infty$ in $C^{2,\alpha}$
then we need to show $s_n \to s_\infty$
in $C^{2,\alpha}$

2 so we can use barriers, & so we
can use elliptic estimates.

Control of higher derivatives

Graph of s_n is minimal surface, so

$$F(s_n, \nabla s_n, \Delta s_n) = 0$$

for some particular F

Would like to rearrange to get

$$\Delta s_n = G(s_n, \nabla s_n)$$

Then bounds on s_n, ∇s_n imply bounds on Δs_n and so on 2nd derivatives of s_n.

If F were "quasilinear elliptic" this would work.

$$\boxed{\text{BUT F DEGENERATES AS } x \to 0}$$

This is because g_{hyp} blows up as $x \to 0$.

We can only play this game for $x \geq x_0 > 0$

Any bounds we get this way will blow up as $x_0 \to 0$.

Willmore comes to the rescue!

Minimal surface eqn is Euler–Lagrange eqn for the area functional.

For the Willmore eqn. we use a different functional: for $\Sigma^2 \subseteq (M^n, g)$ with mean curvature μ, we consider

$$W(\Sigma) := \int_\Sigma \left(|\mu|^2 + \mathrm{Sec}_g(T\Sigma) \right) d\mathrm{vol}$$

(where dvol is induced volume form on Σ.)

Euler–Lagrange equation for W is called the Willmore equation, elliptic, non-linear.

Two key facts:

1. If Σ is minimal it is automatically a solution of Willmore equation

2. W is conformally invariant and so is the Willmore equation.

Our surfaces $\overline{M}_n \subseteq \overline{B^4}$ are g_{hyp} - minimal

So they are g_{hyp} - Willmore

So they are $g_{Euclidean}$ - Willmore

But the Euclidean metric extends smoothly across ∂B^4.

The Euclidean Willmore equation ISN'T DEGENERATE at $x = 0$!

From here one can use more "traditional" methods of geometric analysis to show S_n converge in $C^{2,\alpha}$...