Supersymmetries of geometric structures I

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Super Vector Spaces

A super-vector space is a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$.
Dim: $(m|n) \equiv m + \epsilon n$ ($\epsilon^2 = 1$), c-dim: $m + n$, s-dim: $m - n$.
Changing parity yields a superspace ΠV of dimension $(n|m)$.

Define the tensor product $V \otimes W$ by

$$(V \otimes W)_{\bar{0}} = (V_{\bar{0}} \otimes W_{\bar{0}}) \oplus (V_{\bar{1}} \otimes W_{\bar{1}}),\ (V \otimes W)_{\bar{1}} = (V_{\bar{0}} \otimes W_{\bar{1}}) \oplus (V_{\bar{0}} \otimes W_{\bar{1}})$$

and similarly define $\mathrm{Hom}(V, W) = V^* \otimes W \simeq W \otimes V^*$.

A superalgebra structure on $A = A_{\bar{0}} \oplus A_{\bar{1}}$ is defined via $\mathbb{Z}_2$-homogeneous $\mu \in \mathrm{Hom}(A \otimes A, A)_{\bar{0}}$. It is commutative if

$$ab = (-1)^{|a||b|} ba \quad \text{(sign rule)}.$$

An example is the Grassmann algebra in n variables $\Lambda(n) = \Lambda^{\text{even}} \oplus \Lambda^{\text{odd}}$ of dimension $(2^{n-1}|2^{n-1})$.
Another example is the tensorial algebra $T(V)$. In particular,

$$\dim(S^2 V) = \left(\tbinom{m+1}{2} + \tbinom{n}{2} \,|\, mn\right),\ \dim(\Lambda^2 V) = \left(\tbinom{m}{2} + \tbinom{n+1}{2} \,|\, mn\right).$$

Examples of 2-tensors

Even nondegenerate symmetric str on $\mathbb{R}^{m|2n}(x,\xi)$ has the form:

$$g = \sum_{i=1}^{n} \epsilon_i dx_i^2 + \sum_{j=1}^{m} d\xi_{2j-1} d\xi_{2j} \qquad (\epsilon_i = \pm 1).$$

Change of parity gives nondegenerate skew-symmetric structure $\omega(v,w) = g(\Pi v, \Pi w)$ on $\mathbb{R}^{2n|m}(x,\xi)$:

$$\omega = \sum_{i=1}^{m} dx_{2i-1} \wedge dx_{2i} + \sum_{j=1}^{n} \epsilon_j d\xi_j \wedge d\xi_j.$$

Odd nondegenerate symmetric str on $\mathbb{R}^{n|n}(x,\xi)$ has the form:

$$q = \sum_{i=1}^{n} dx_i \otimes d\xi_i.$$

There is a bijection: odd ndg symm $\leftrightsquigarrow$ skew-symm str.

Odd complex structure on $\mathbb{R}^{n|n}(x,\xi)$ has normal form

$$J = \sum_{i=1}^{n} \partial_{\xi_i} \otimes dx_i - \partial_{x_i} \otimes d\xi_i.$$

Lie superalgebras (LSA)

A Lie superalgebra structure on $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ is defined by $\mathbb{Z}_2$-homogeneous bracket $[,] \in \mathrm{Hom}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})_{\bar 0}$ that is skew-symm and satisfies the Jacobi in super-sense (sign rule). Examples:

- $\mathfrak{gl}(m|n) = \mathrm{End}(\mathbb{R}^{m|n})$ with supercommutator
$$[A, B] = AB - (-1)^{|A|\,|B|} BA$$
- $\mathfrak{sl}(m|n) = \left\{ A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \mathrm{str}(A) = \mathrm{tr}(\alpha) - \mathrm{tr}(\delta) = 0 \right\}$.
- $\mathfrak{osp}(m|2n)$ preserves even ndg symmetric structure on $\mathbb{R}^{m|2n}$ ($\simeq$)
- ◦ $\mathfrak{spo}(2n|m)$ preserves even ndg skew-symm structure on $\mathbb{R}^{2n|m}$,
- $\mathfrak{pe}(n)$ preserves odd ndg symmetric structure on $\mathbb{R}^{n|n}$ ($\simeq$)
- ◦ $\mathfrak{pe}^{\mathsf{sk}}(n)$ preserves odd ndg skew-symmetric structure on $\mathbb{R}^{n|n}$,
- $\mathfrak{q}(n)$ preserves odd complex structure on $\mathbb{R}^{n|n} \rightsquigarrow \mathfrak{sq}(n), \mathfrak{psq}(n)$
- $G(3) = (\mathfrak{g}(2) \oplus \mathfrak{sl}(2)|\mathbb{R}^7 \otimes \mathbb{R}^2)$ exceptional $\mathfrak{ag}(2)$,
- $F(4) = (\mathfrak{spin}(7) \oplus \mathfrak{sl}(2)|\mathbb{R}^8 \otimes \mathbb{R}^2)$ exceptional $\mathfrak{ab}(3)$,
- $D(2|1;\alpha) = (\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)|\mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2)$.

Digression: Lie algebra from representation

For a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of a Lie algebra, one can (non-uniquely) recover the structure of $\mathfrak{g}$ by representation $\mathfrak{h} \to \mathrm{End}(\mathfrak{m} = \mathfrak{g}/\mathfrak{h})$ and some (cohomological) data. This is esp simple in the reductive case, when $\exists$ $\mathfrak{h}$-invariant complement $\mathfrak{m} \subset \mathfrak{g}$.

Consider, for example, the case $\mathfrak{h} = \mathfrak{su}(3)$, $\mathfrak{m} = \mathbb{C}^3$. Then the brackets on $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ are given by the subalgebra structure of $\mathfrak{h}$, its representation and an $\mathfrak{h}$-equivariant map $\beta : \Lambda^2\mathfrak{m} \to \mathfrak{g}$. We split

$$(\Lambda^2\mathfrak{m})_{\mathbb{C}} = \Lambda^{2,0}(\mathfrak{m}) \oplus \Lambda^{1,1}(\mathfrak{m}) \oplus \Lambda^{0,2}(\mathfrak{m}).$$

Thus, $\Lambda^{2,0}(\mathfrak{m}) \oplus \Lambda^{0,2}(\mathfrak{m}) \simeq \mathfrak{m}_{\mathbb{C}}$, $\Lambda^{1,1}(\mathfrak{m}) = \mathbb{C} \oplus \Lambda^{1,1}_0(\mathfrak{m}) \simeq \mathbb{C} \oplus \mathfrak{h}_{\mathbb{C}}$, and we decompose into $\mathfrak{h}$-irreps $\Lambda^2\mathfrak{m} = \mathfrak{m} \oplus \mathbb{R} \oplus \mathfrak{h}$. Now by Schur's lemma β is given by the matrix

$$\beta = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \end{pmatrix}.$$

The Jacobi identity should be checked only for all 3 arguments from $\mathfrak{m}$, and this gives $4b + a^2 = 0$. Thus either the algebra is flat $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{m}$ ($a = b = 0$) or rescaling leads to $\mathfrak{g} = G(2)$.

LSA example: $G(3)$ and $F(4)$ (over $\mathbb{C}$)

For $\mathfrak{g} = G(3)$: $\mathfrak{g}_{\bar{0}} = G(2) \oplus A(1)$, $\mathfrak{g}_{\bar{1}} = \mathbb{C}^7 \boxtimes \mathbb{C}^2$. We split

$$S^2\mathfrak{g}_{\bar{1}} = (\Lambda^2\mathbb{C}^7 \boxtimes \Lambda^2\mathbb{C}^2) \oplus (S^2\mathbb{C}^7 \boxtimes S^2\mathbb{C}^2) = G(2) \boxtimes \mathbb{C} \oplus [10\,0] \oplus [20\,2] \oplus \mathbb{C} \boxtimes A(1)$$

into $\mathfrak{g}_{\bar{0}}$-irreps, whence by Schur the $\mathfrak{g}_{\bar{0}}$-equivariant map $\beta : S^2\mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$ has matrix $\beta = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}$ and Jacobi $[x,[x,x]] = 0 \ \forall x \in \mathfrak{g}_{\bar{1}}$ uniquely fixes non-flat $[a:b]$ yielding $G(3)$.

For $\mathfrak{g} = F(4)$: $\mathfrak{g}_{\bar{0}} = B(2) \oplus A(1)$, $\mathfrak{g}_{\bar{1}} = \mathbb{C}^8 \boxtimes \mathbb{C}^2$. We split

$$S^2\mathfrak{g}_{\bar{1}} = (\Lambda^2\mathbb{C}^8 \boxtimes \Lambda^2\mathbb{C}^2) \oplus (S^2\mathbb{C}^8 \boxtimes S^2\mathbb{C}^2) = B(3) \boxtimes \mathbb{C} \oplus [010\,0] \oplus [002\,2] \oplus \mathbb{C} \boxtimes A(1)$$

into $\mathfrak{g}_{\bar{0}}$-irreps, whence by Schur the $\mathfrak{g}_{\bar{0}}$-equivariant map $\beta : S^2\mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$ has matrix $\beta = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}$ and Jacobi $[x,[x,x]] = 0 \ \forall x \in \mathfrak{g}_{\bar{1}}$ uniquely fixes non-flat $[a:b]$ yielding $F(4)$.

Another exception: $D(2|1;\alpha)$

For $\mathfrak{g} = D(2|1;\alpha)$: $\mathfrak{g}_{\bar{0}} = A(1) \oplus A(1) \oplus A(1)$, $\mathfrak{g}_{\bar{1}} = \mathbb{C}^2 \boxtimes \mathbb{C}^2 \boxtimes \mathbb{C}^2$.
We split into $\mathfrak{g}_{\bar{0}}$-irreps (where $S^2 = S^2\mathbb{C}^2$, $\Lambda^2 = \Lambda^2\mathbb{C}^2$)

$$S^2\mathfrak{g}_{\bar{1}} = (S^2 \boxtimes S^2 \boxtimes S^2) \oplus (S^2 \boxtimes \Lambda^2 \boxtimes \Lambda^2) \oplus (\Lambda^2 \boxtimes S^2 \boxtimes \Lambda^2) \oplus (\Lambda^2 \boxtimes \Lambda^2 \boxtimes S^2)$$
$$= A(1) \boxtimes A(1) \boxtimes A(1) \oplus A(1) \boxtimes \mathbb{C} \boxtimes \mathbb{C} \oplus \mathbb{C} \boxtimes A(1) \boxtimes \mathbb{C} \oplus \mathbb{C} \boxtimes \mathbb{C} \boxtimes A(1)$$

so by Schur the $\mathfrak{g}_{\bar{0}}$-equivariant map $\beta : S^2\mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$ has matrix
$\beta = \begin{pmatrix} 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}$ and Jacobi yields $\lambda_1 + \lambda_2 + \lambda_3 = 0$.
Assuming not all λ_i zero, denote $\alpha = [\lambda_1 : \lambda_2] \in \mathbb{P}^1$. Then the natural S_3 action on the λ-plane is given by

$$\alpha \mapsto 1/\alpha, \quad \alpha \mapsto -(1+\alpha).$$

Here $\alpha \notin \{0, -1, \infty\}$. The orbit $\alpha \in \{1, -\frac{1}{2}, -2\}$ corresponds to non-deformed $D(2|1) = \mathfrak{osp}(4|2)$. The singular orbit $\alpha = e^{\pm 2\pi i/3}$ corresponds to vertices of (any) fundamental domain in $\mathbb{P}^1$.

Classical Lie superalgebras

Definition

A simple Lie superalgebra $\mathfrak{g}$ is called classical if the representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is completely reducible.

Theorem

A simple Lie superalgebra $\mathfrak{g}$ is classical if and only if $\mathfrak{g}_{\bar{0}}$ is reductive.

Remark

The module $S^2\mathfrak{g}_{\bar{1}}$ contains every irrep $\Gamma_\lambda \subset \mathfrak{g}_{\bar{0}}$ with multiplicity 1.

$\mathbb{E}$x: check this with the orthosymplectic algebra

$$\mathfrak{osp}(2m+1|2n)_{\bar{0}} = B_m \oplus C_n,$$
$$\mathfrak{osp}(2m|2n)_{\bar{0}} = D_m \oplus C_n$$

Types

I: $\mathfrak{g}_{\bar 0} \to \mathrm{End}(\mathfrak{g}_{\bar 1})$ reducible, then $\mathfrak{g}_{\bar 1} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ is the direct sum of two irreps of $\mathfrak{g}_0 = \mathfrak{g}_{\bar 0}$, and $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a $\mathbb{Z}$ grading.

II: $\mathfrak{g}_{\bar 0} \to \mathrm{End}(\mathfrak{g}_{\bar 1})$ irreducible, hence $\mathfrak{g}_{\bar 0}$ is semi-simple. (Otherwise $S^2\mathfrak{g}_{\bar 1} \to \mathfrak{g}_{\bar 0}$ is not $\mathfrak{z}(\mathfrak{g}_{\bar 0})$ equivariant.) $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

Killing form $K_{\mathfrak{g}}(x, y) = \mathrm{str}(\mathrm{ad}_x \, \mathrm{ad}_y)$ is even, supersymmetric and $\mathfrak{g}$-invariant. It may however be zero. Any invariant bilinerar supersymmetric even form K on a simple LSA is either nondegenerate or zero, hence any two such forms are proportional.

LSA is called basic if it possesses a nondegenerate form K.
Examples when such is lacking:

$$P(n) = \left\{ \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} \in \mathfrak{sl}(n+1, n+1) \,:\, \mathrm{tr}(a) = 0,\ b \text{ symm},\ c \text{ skeq} \right\},$$

$$Q(n) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in \mathfrak{sl}(n+1, n+1) \,:\, \mathrm{tr}(b) = 0 \right\} / \langle \mathbf{1} \rangle.$$

Classification of classical LSA

	Type I	Type II
BASIC (ndg Killing)	$A(m,n),\ m>n\geq 0$ $C(n+1),\ n>0$	$B(m,n),\ m\geq 0,\ n>0$ $D(m,n),\ \begin{cases} m>1, n>0 \\ m\neq n+1 \end{cases}$ $F(4),\ G(3)$
BASIC (zero Killing)	$A(n,n),\ n>0$	$D(n+1,n),\ n>0$ $D(2\|1,\alpha)$
STRANGE (no ndg K)	$P(n),\ n>1$ (periplectic)	$Q(n),\ n>1$ (queer)

Here

$$A(m,n)=\mathfrak{sl}(m+1,n+1),\ \ m\neq n$$

$$A(n,n)=\mathfrak{sl}(n+1,n+1)/\langle\mathbf{1}\rangle$$

$$B(m,n)=\mathfrak{osp}(2m+1,2n)$$

$$C(n)=\mathfrak{osp}(2,2n-2)$$

$$D(m,n)=\mathfrak{osp}(2m,2n).$$

Root space decomposition

A Cartan subalgebra (CSA) of LSA $\mathfrak{g}$ is a maximal nilpotent self-normalizing subalgebra. For a classical LSA its CSA $\mathfrak{h} \subset \mathfrak{g}_{\bar{0}}$ and it is diagonalizable, whence root space decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha \oplus \mathfrak{h}, \text{ where } \mathfrak{g}_\alpha = \{v \in \mathfrak{g} : [h, v] = \alpha(h)\ \forall h \in \mathfrak{h}\}.$$

Root system $\Delta = \{\alpha \in \mathfrak{h}^* : \mathfrak{g}_\alpha \neq 0\} = \Delta_{\bar{0}} \cup \Delta_{\bar{1}}$, sets of even and odd roots may intersect and have multiplicities as for $Q(n)$ (but the latter case is special as CSA contains even and odd parts).

For all classical LSA we have $\mathfrak{g}_0 = \mathfrak{h}$ and $\dim \mathfrak{g}_\alpha = 1\ \forall\ \alpha \neq 0$ if $\mathfrak{g} \neq A(1,1), P(2), P(3), Q(n)$. For basic LSA

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0 \text{ if } \alpha, \beta, \alpha + \beta \in \Delta, \quad [e_\alpha, e_{-\alpha}] = \langle e_\alpha, e_{-\alpha} \rangle h_\alpha;$$
$$\langle \mathfrak{g}_\alpha, \mathfrak{g}_\beta \rangle = 0 \text{ if } \alpha \neq -\beta, \quad \text{pairing } \langle , \rangle|_{\mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha}} \text{ is ndg .}$$

In addition, if $\alpha \in \Delta$ and $k\alpha \in \Delta$ for $k \in \mathbb{Z}$, $k \neq \pm 1$, then $k = \pm 2$, $\alpha \in \Delta_{\bar{1}}$, $\langle \alpha, \alpha \rangle \neq 0$.

Cartan matrix

Contragradient LSA $\mathfrak{g}(A,\tau) \equiv$ (possessing Cartan matrix) are constructed as follows: $A = (a_{ij})_{r\times r}$, $\tau \subset \{1,\dots,r\}$. Local LSA $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ has basis f_i, h_i, e_i, $i \in I$ with parity $|e_i| = |f_i| = \bar{0}$ iff $i \in \tau$, $|h_i| = \bar{0}$ $\forall i$. Relations:

$$[e_i, f_j] = \delta_{ij} h_j,\ [h_i, h_j] = 0,\ [h_i, e_j] = a_{ij} e_j,\ [h_i, f_j] = -a_{ij} f_j.$$

There exists "minimal" $\mathbb{Z}$-graded LSA $\mathfrak{g} = \oplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ with the above local part. (Serre relations + supplementary conditions!)

The map $(h_i, f_i) \mapsto (ch_i, cf_i)$, $c \neq 0$, rescales the i-th row of A. If $a_{ii} \neq 0$ normalize it to $a_{ii} = 2$ for $|i| = \bar{0}$ and to $a_{ii} = 1$ if $|i| = \bar{1}$, if $a_{ii} = 0$ normalize the row to contain integers without common divisor.

Let $\mathfrak{h} = \langle h_1, \dots, h_r \rangle$. Define simple roots α_j by $\alpha_j(h_i) = a_{ij}$ (opposite to Bourbaki; classically $\frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}$). Each basic LSA has a distinguished simple root system with only one odd root.

Dynkin diagrams

Let $\mathfrak{g}$ be a basic LSA with CSA $\mathfrak{h}$ of rank $r = \dim \mathfrak{h}$. Let $\Delta^0 = (\alpha_1, \dots, \alpha_r)$ be a simple root system of $\mathfrak{g}$, $A = (a_{ij})_{r \times r}$ the associated Cartan matrix. A Dynkin diagram is given as follows.

- $\bigcirc$ white node for even simple root,
- $\otimes$ grey node for odd isotropic simple root,
- $\bullet$ black node for odd non-isotropic simple root.

The i-th and j-th nodes are joined by $\eta_{ij} = \max(|a_{ij}|, |a_{ji}|)$ lines with an arrow from i-th to j-th node when $|a_{ij}| > |a_{ji}|$. (For $D(2|1;\alpha)$ the recipe is different.)

The distinguished Dynkin diagram is the Dynkin diagram associated to a distinguished simple root system.
It is constructed as follows. Consider the distinguished $\mathbb{Z}$-gradation $\mathfrak{g} = \oplus \mathfrak{g}_i$. Then the even nodes are given by the Dynkin diagram of $\mathfrak{g}_{\bar{0}}$ (may be not connected) and the odd node corresponds to the lowest weight of the $\mathfrak{g}_{\bar{0}}$-representation $\mathfrak{g}_{\bar{1}}$.

Weyl groupoid

For even root α one has $\langle\alpha,\alpha\rangle \neq 0$ and the reflection at α

$$S_\alpha(\beta) = \beta - \frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha, \qquad \beta \in \mathfrak{h}^*$$

preserves $\Delta_{\bar 0}$ and $\Delta_{\bar 1}$. The Weyl group W generated by S_α, $\alpha \in \Delta_{\bar 0}$, does not act transitively on the set of of all simple root systems. Given simple root system Δ^0 the odd reflection (Serganova) at an odd isotropic root $\alpha \in \Delta^0_{\bar 1}$ is

$$S_\alpha(\beta) = \begin{cases} \beta+\alpha, & \langle\alpha,\beta\rangle \neq 0 \\ \beta, & \langle\alpha,\beta\rangle = 0,\ \beta\neq\alpha\ , \\ -\alpha, & \beta = \alpha \end{cases} \qquad \beta \in \Delta^0.$$

Such odd reflection does not extend to the entire Δ, but acts only on Δ^0. Nevertheless, the obtained Weyl groupoid $\hat W$, generated by all S_α, acts transitively on the set of of all simple root systems.

Example: $\mathfrak{sl}(2|1)$ of $\dim = (4|4)$

Here $\mathfrak{h} = \mathbb{R}^2$, $\mathfrak{h}^* = \langle \epsilon_1 - \epsilon_2, \epsilon_2 - \delta \rangle \subset \mathbb{R}^3(\epsilon_1, \epsilon_2, \delta)$ and

$$\begin{array}{l} \Delta_{\bar 0} = \{\pm(\epsilon_1 - \epsilon_2)\} \\ \Delta_{\bar 1} = \{\pm(\epsilon_i - \delta)\} \end{array} \qquad \mathfrak{g} = \begin{pmatrix} \times & \epsilon_1 - \epsilon_2 & \epsilon_1 - \delta \\ \epsilon_2 - \epsilon_1 & \times & \epsilon_2 - \delta \\ \delta - \epsilon_2 & \delta - \epsilon_2 & \times \end{pmatrix}$$

with $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$, $\langle \delta, \delta \rangle = -1$, $\langle \epsilon_i, \delta \rangle = 0$.
Different system of positive roots and corresp Dynkin diagrams:

	Even	Odd	Simple	Dynkin	Cartan
(1)	$\epsilon_1 - \epsilon_2$	$\delta - \epsilon_1, \delta - \epsilon_2$	$\epsilon_1 - \epsilon_2, \delta - \epsilon_1$	$\bigcirc\!\!-\!\!\!-\!\!\otimes$	$\begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$
(2)	$\epsilon_1 - \epsilon_2$	$\epsilon_1 - \delta, \delta - \epsilon_2$	$\epsilon_1 - \delta, \delta - \epsilon_2$	$\otimes\!\!-\!\!\!-\!\!\otimes$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
(3)	$\epsilon_1 - \epsilon_2$	$\epsilon_1 - \delta, \epsilon_2 - \delta$	$\epsilon_2 - \delta, \epsilon_1 - \epsilon_2$	$\otimes\!\!-\!\!\!-\!\!\bigcirc$	$\begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$

The odd reflections mapping DDs are $S_{\epsilon_1 - \delta}(\epsilon_1 - \delta) = \delta - \epsilon_1$, $S_{\epsilon_1 - \delta}(\delta - \epsilon_2) = \epsilon_1 - \epsilon_2$ and similar.

Parabolics and $\mathbb{Z}$-gradings

Let Δ^+ be a choice of positive roots, Δ^0 the corresponding simple root system $\{\alpha_1, \dots, \alpha_r\}$ and $\{Z_i\} \subset \mathfrak{h}$ be the dual basis to $\{\alpha_i\} \subset \mathfrak{h}^*$. Let $\chi \subset \{1, \dots, r\}$ be a choice of crosses on nodes of the corresponding DD. Then $Z = \sum_{i \in \chi} Z_i$ is a grading element defining a $\mathbb{Z}$-grading of $\mathfrak{g}$ with non-negative part being parabolic

$$\mathfrak{g} = \underbrace{\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}}_{\mathfrak{m}} \oplus \underbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1 \cdots \oplus \mathfrak{g}_k}_{\mathfrak{p}}, \quad \mathfrak{g}_i = \{v \in \mathfrak{g} : [Z, v] = iv\}.$$

There is a bijective correspondence between (equivalence classes of) $\mathbb{Z}$-gradings and parabolics. The latter $\mathfrak{p} = \mathfrak{p}_\chi^\Xi$ are given by a choice of DD Ξ and a subset of simple roots χ.

Weyl reflection groupoid: Assume that two system of simple roots are related by an odd reflection S_{α_i}, corresponding to grey node i of the Dynkin diagrams Ξ, Ξ' with nodes N, N', and remaining nodes permuted by the bijection $z : N \to N'$, $z(i) = i'$. Then for a subset $N \setminus \{i\} \supset \chi \overset{z}{\longleftrightarrow} \chi' = z(\chi) \subset N \setminus \{i'\}$

$$\mathfrak{p}_\chi^\Xi \simeq \mathfrak{p}_{\chi'}^{\Xi'}.$$

BGG and BBW

If $\mathfrak{g}$ is of type I or II, then typical simple modules have a finite BGG resolution. For type II atypical Kac modules never have finite BGG.

Recently Koulembier advanced in solving BBW for basic LSA: for distinguished Borel subgroup $B \subset G$ denote λ-highest weight representation by $L_\lambda(\mathfrak{b})$, and similarly for a parabolic $P \supset B$. For a P-module V and denote $\Gamma_k(G/P, V) = H^k(G/P, G \times_P V^*)^*$. Then

$$\Gamma_k(G/P, L_\lambda(\mathfrak{p})) = \Gamma_k(G/B, L_\lambda(\mathfrak{b})).$$

If the weight λ is regular, there exists a unique $w \in W$ such that $\Lambda = w \cdot \lambda \in \mathcal{P}^+$ and

$$\Gamma_k(G/B, L_\lambda(\mathfrak{b})) = \begin{cases} K_\Lambda & \text{if } \ell(w) = k, \\ 0 & \text{if } \ell(w) \neq k. \end{cases}$$

Here K_Λ is the maximal finite-dimensional quotient of the integral dominant Verma module $M_\Lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} L_\Lambda(\mathfrak{b})$.

However this almost never applicable to adjoint representations, and Kostant's cohomology remains not computed for general LSA.

A supermanifold in the sense of Berezin–Kostant–Leites is a ringed space $M = (M_o, \mathcal{A}_M)$ such that $\mathcal{A}_M|_{\mathcal{U}_o} \cong C^\infty_{M_o}|_{\mathcal{U}_o} \otimes \Lambda^\bullet \mathbb{S}^*$ as sheaves of superalgebras for any sufficiently small open subset $\mathcal{U}_o \subset M_o$. Here $\mathbb{S}$ is a vector space of fixed dimension. We set $\dim(M) = (m|n) = (\dim M_o | \dim \mathbb{S})$, call M_o the reduced manifold and $\mathcal{A}_M = (\mathcal{A}_M)_{\bar{0}} \oplus (\mathcal{A}_M)_{\bar{1}}$ the structure sheaf.

Let $\mathcal{J} = \langle \mathcal{A}_{\bar{1}} \rangle = \mathcal{J}_{\bar{0}} \oplus \mathcal{J}_{\bar{1}}$ be the subsheaf generated by nilpotents: $\mathcal{J}_{\bar{1}} = \mathcal{A}_{\bar{1}}$ and $\mathcal{J}_{\bar{0}} = \mathcal{A}_{\bar{1}}^2$. For any sheaf $\mathcal{E}$ of $\mathcal{A}_M$-modules on M_o the evaluation is $\mathrm{ev} : \mathcal{E} \to \mathcal{E}/(\mathcal{J} \cdot \mathcal{E})$. Thus $\mathrm{ev} : \mathcal{A}_M \to C^\infty_{M_o}$, $f \mapsto \mathrm{ev}(f)$, yields the canonical morphism $\imath : M_o \hookrightarrow M$, with evaluation $\mathrm{ev}(f)$ at $x \in M_o$ being $\mathrm{ev}_x(f)$. We stress, however, that there is no canonical morphism from M to M_o.

A Lie supergroup is a supermanifold $G = (G_o, \mathcal{A}_G)$ that is also a group object in the category of supermanifolds. (The reduced manifold G_o is a Lie group.) It can be represented by a Harish-Chandra pair $(G_o, \mathfrak{g})$, $\mathrm{Lie}(G_o) = \mathfrak{g}_{\bar{0}}$.

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Actions of Lie supergroups

A Lie supergroup is a supermanifold $G = (G_o, \mathcal{A}_G)$ that is also a group object in the category of supermanifolds. This means there exist morphisms (where $e = (\mathrm{pt}, \mathbf{k})$ is a superpoint, $\mathbf{k} = \mathbb{R} \vee \mathbb{C}$)

$$\mu : G \times G \to G, \quad \iota : G \to G, \quad \epsilon : e \to G, \quad \upsilon : G \to e$$

satisfying (below $\mathrm{diag} : G \to G \times G$)

$$\mu \circ (\mu \times \mathrm{id}) = \mu \circ (\mathrm{id} \times \mu), \quad \mu \circ (\epsilon \times \mathrm{id}) = \mathrm{id} = \mu \circ (\mathrm{id} \times \epsilon),$$
$$\mu \circ (\iota \times \mathrm{id}) \circ \mathrm{diag} = \epsilon \circ \upsilon = \mu \circ (\mathrm{id} \times \iota) \circ \mathrm{diag}, \quad \upsilon \circ \epsilon = \mathrm{id}$$

An action of G on M is a morphism $\varphi : G \times M \to M$ such that

$$\varphi \circ (\mu \times \mathrm{id}) = \varphi \circ (\mathrm{id} \times \varphi), \quad \varphi \circ (\epsilon \times \mathrm{id}) = \mathrm{id}\,.$$

This induces a homomorphism of LSAs $\bar{\varphi} : \mathfrak{g} \to \mathrm{Vect}(M)$. Note the evaluation map $\mathrm{ev}_x : \mathrm{Vect}(M) \to T_x M$.

Definition

The action φ is transitive if its reduction $\varphi_o : G_o \times M_o \to M_o$ is such and the map $\mathrm{ev}_x \circ \bar{\varphi} : \mathfrak{g} \to T_x M$ is surjective $\forall x \in M_o$.

Homogeneous supergeometry

Definition

A supermanifold M is homogeneous if a Lie supergroup G transitively acts on it.

In this case one can define the stabilizer of a point $a \in M$: it is a sub(super)group $H \subset G$, given by the pair $(H_o, \mathcal{A}_H)$, where H_o is the stabilizer of $a \in M_o$ and $\mathcal{A}_H = \mathcal{A}_G/(\varphi \circ (\mathrm{id} \times a))^*(\mathfrak{m}_a)$, where $\mathfrak{m}_a \subset \mathcal{A}_M$ is the maximal ideal of a.

Equivalently if G is given as a Harish-Chandra pair $(G_o, \mathfrak{g})$, then H is the Harish-Chandra pair $(H_o, \mathfrak{h})$ with $\mathfrak{h} = \mathrm{Ker}(\mathrm{ev}_x \circ \bar{\varphi})$.
In this case the algebraic data encoding the homogeneous manifold $M = G/H$ is $(G_o/H_o, \mathfrak{g}/\mathfrak{h})$.

Definition

A generalized flag supermanifold is the homogeneous space G/P with G a (semi)simple Lie supergroup and P a parabolic subgroup.

Global vs Local Geometry

It is a classical result that all holomorphic vector fields on a flag manifold in $\mathbb{C}^n$ are fundamental for the natural action of $SL(n, \mathbb{C})$, i.e. their Lie algebra of holomorphic vector fields is $\mathfrak{sl}(n, \mathbb{C})$. This was extended to generalized flag manifold G/P by Onishchik:

$$\mathrm{Vect}_{\mathfrak{hol}}(G/P) = \mathfrak{g}.$$

Recently, Vishnyakova extended this results further to generalized flag supermanifold in several cases (for some homogeneous superspaces introduced by Manin). It is remarkable that one of her main tools is the classical BBW theorem, as is also our case below.

We want to get a local result, namely to specify geometries for which $\mathfrak{g} = \mathrm{Lie}(G)$ is the symmetry Lie superalgebra. The geometric structures responsible for reduction from the sheaf of all vector fields $\mathcal{T}M$ on M are non-holonomic and we introduce them next.

Super distributions

A distribution on a supermanifold M is a graded $\mathcal{A}_M$-subsheaf $\mathcal{D} = \mathcal{D}_{\bar{0}} \oplus \mathcal{D}_{\bar{1}} \subset \mathcal{T}M$ that is locally a direct factor. Any such sheaf is locally free, associating the VB $D = \mathrm{ev}(\mathcal{D}) \subset TM$. This induces a reduced subbundle $D|_{M_o} \subset TM|_{M_o}$ that does not determine $\mathcal{D}$.

The weak derived flag of (bracket-generating) $\mathcal{D}$ is defined so:

$$\mathcal{D}^1 = \mathcal{D} \subset \mathcal{D}^2 \subset \cdots \subset \mathcal{D}^i \subset \cdots, \qquad \mathcal{D}^i = [\mathcal{D}, \mathcal{D}^{i-1}],$$

where each $\mathcal{D}^i \subset \mathcal{T}M$ is a graded $\mathcal{A}_M$-subsheaf, also assumed locally direct factor (regularity).

Example (non-regular superextension of Hilbert–Cartan equation)

Let $M = \mathbb{R}^{5|2}(x, u, p, q, z \,|\, \theta, \nu)$ be endowed with superdistribution

$$\mathcal{D} = \langle D_x = \partial_x + p\partial_u + q\partial_p + q^2\partial_z,\ \partial_q \,|\, D_\theta = \partial_\theta + q\partial_\nu + \theta\partial_p + 2\nu\partial_z \rangle$$

of rank $(2|1)$. We directly compute

$$\mathcal{D}^2 = \langle D_x,\ \partial_q,\ \partial_p + 2q\partial_z \,|\, D_\theta,\ \partial_\nu,\ \theta\partial_u \rangle.$$

This is not a superdistribution, due to the presence of a nilpotent.

Tanaka-Weisfeiler prolongation

For regular $\mathcal{D}$ we get filtration $\mathcal{D}^i$ of $\mathcal{T}M$, compatible with brackets of supervector fields: $[\mathcal{D}^i(\mathcal{U}), \mathcal{D}^j(\mathcal{U})] \subset \mathcal{D}^{i+j}(\mathcal{U})$. Setting $\mathrm{gr}(\mathcal{T}M)_{-i} = \mathcal{D}^i/\mathcal{D}^{i-1}$ for $i > 0$, we get a locally free sheaf of $\mathcal{A}_M$-modules and Lie superalgebras over M_o:

$$\mathrm{gr}(\mathcal{T}M) = \bigoplus_{i<0} \mathrm{gr}(\mathcal{T}M)_i.$$

It is strongly regular if there exists Lie superalgebra $\mathfrak{m} = \bigoplus_{-\mu \leq i < 0} \mathfrak{g}_i$ such that $\mathrm{gr}(\mathcal{T}_x M) \cong (\mathcal{A}_M)_x \otimes \mathfrak{m} \ \forall \ x \in M_o$.

Assuming strong regularity, non-degeneracy (no center in $\mathfrak{g}_{-1}$) and fundamental property ($\mathfrak{g}_{-1}$ generates $\mathfrak{m}$) define Tanaka–Weisfeiler prolongation of $\mathfrak{m}$ as the maximal $\mathbb{Z}$-graded LSA $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ s.t:

- extension: $\mathfrak{g}_- = \mathfrak{m}$,
- transitivity: $[X, \mathfrak{g}_{-1}] \neq 0$ for $0 \neq X \in \mathfrak{g}_{\geq 0}$.

It exists and unique, and is denoted $\mathfrak{g} = \mathrm{pr}(\mathfrak{m})$. There is prolongation version $\mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0)$ and also other reductions.

$G(3)$ supergeometries: regular extension of HC equation

There are 19 parabolic supergeometries associated to the simple exceptional LSA $G(3)$. Consider Hilbert-Cartan type supergeometry $G(3)/P_2^{IV}$, which is equivalent to a gerneric $(2|4)$ superdistribution on a $(5|6)$-dimensional supermanifold. (Similarly a generic rank 2 distribution in 5D gives a $G(2)/P_1$ geometry.)

Tanaka-Weisfeiler prolongation of the symbol of this distribution has the following dimensions of the components:

$$(2|0, 1|2, 2|4, 7|2, 2|4, 1|2, 2|0).$$

Therefore $\dim \mathfrak{s} \leq (17|14)$ and the maximal symmetry is $G(3)$. The corresponding distribution super-extends the Hilbert-Cartan distribution; on $M = \mathbb{R}^{5|6}(x, u, u_x, u_{xx}, z|\nu, \tau, u_\nu, u_\tau, u_{x\nu}, u_{x\tau})$ it is

$$\mathcal{D}_{\bar{0}} = \langle D_x = \partial_x + u_x \partial_u + u_{xx} \partial_{u_x} + \left(\tfrac{u_{xx}^2}{2} + u_{x\nu} u_{x\tau}\right) \partial_z + u_{x\tau} \partial_{u_\tau} + u_{x\nu} \partial_{u_\nu}, \partial_{u_{xx}} \rangle,$$

$$\mathcal{D}_{\bar{1}} = \langle D_\tau = \partial_\tau + u_\tau \partial_u + u_{x\tau} \partial_{u_x} + u_{xx} u_{x\tau} \partial_z + u_{xx} \partial_{u_\nu},\ \partial_{u_{x\tau}},$$

$$D_\nu = \partial_\nu + u_\nu \partial_u + u_{x\nu} \partial_{u_x} + u_{xx} u_{x\nu} \partial_z - u_{xx} \partial_{u_\tau},\ \partial_{u_{x\nu}} \rangle.$$

Encoding as differential equations

The SHC (super-Hilbert–Cartan) differential equation is

$$z_x = \frac{1}{2}u_{xx}^2 + u_{x\nu}u_{x\tau},\ z_\tau = u_{xx}u_{x\tau},\ z_\nu = u_{xx}u_{x\nu},\ u_{\tau\nu} = u_{xx}$$

Another way to encode this is via super PDE

$$\begin{cases} u_{xx} = \frac{1}{2}u_{yy}^3 + 2u_{yy}u_{y\nu}u_{y\tau}u_{yy}, & u_{xy} = \frac{1}{2}u_{yy}^2 + u_{y\nu}u_{y\tau}, \\ u_{x\nu} = u_{yy}u_{y\nu}, \quad u_{x\tau} = u_{yy}u_{y\tau}, & u_{\nu\tau} = -u_{yy}, \end{cases}$$

Here x, y, u even, ν, τ odd; e.g. $u_{\nu\tau} = -u_{\tau\nu}$, $u_{\nu\nu} = u_{\tau\tau} = 0$.

Theorem (BK, A. Santi, D. The ⋄ 2019)

The internal (EDS) symmetry of the SHC and external (contact) symmetry of the super-PDE is $G(3)$.

On the proof: jets and Spencer complex

If $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{A}_M$ at $x \in M$ (this contains the subideal generated by odds: $\mathcal{J} = \langle (\mathcal{A}_M)_{\bar{0}} \rangle \subset \mathcal{A}_M$) then $J^k_x(M) = \mathcal{A}_M / \mathfrak{m}_x^{k+1}$ is the space of k-jets at x, $k = 0, 1, \dots, \infty$.

Similarly if $\mathcal{V}_M$ is sheaf of sections of a bundle E over M with typical fiber V, then $J^k_x(E) = \mathcal{V}_M / (\mathfrak{m}_x^{k+1} \cdot \mathcal{V}_M)$.

One further makes those jet-spaces J^k into a supermanifold and bundle over M (not union over points!) with equations $\mathcal{E}_k \subset J^k$.

The symbols $g_k(x) \subset S^k T^*_x M \otimes V$ are defined as typical (tangent) fibers of projections $\mathcal{E}_k \to \mathcal{E}_{k-1}$ yielding the Spencer complex:

$$\dots \to \Lambda^{i-1} T^* M \otimes g_{j+1} \to \Lambda^i T^* M \otimes g_j \to \Lambda^{i+1} T^* M \otimes g_{j-1} \to \dots$$

In nonholonomic situation (weighted jets) the tangent bundle TM is filtered with the associated graded $\mathfrak{m}$ represented on V, whence the generalized Spencer complex = Chevalley–Eilenberg complex:

$$\dots \to \Lambda^{i-1} \mathfrak{m}^* \otimes \mathfrak{g}_{j+1} \to \Lambda^i \mathfrak{m}^* \otimes \mathfrak{g}_j \to \Lambda^{i+1} \mathfrak{m}^* \otimes \mathfrak{g}_{j-1} \to \dots$$

On the proof: Hochschild–Serre spectral sequence

If the parabolic and nilradical of the opposite parabolic are given by

$$\mathfrak{g} = \underbrace{\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}}_{\mathfrak{m}} \oplus \underbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1 \cdots \oplus \mathfrak{g}_k}_{\mathfrak{p}}$$

then the claims about symmetry algebra are equivalent to:
$\mathfrak{g} = \mathrm{pr}(\mathfrak{m}) \Leftrightarrow H^1(\mathfrak{m}, \mathfrak{g})_{\geq 0} = 0$, $\mathfrak{g} = \mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0) \Leftrightarrow H^1(\mathfrak{m}, \mathfrak{g})_+ = 0$.

To prove this we use filtation $0 \subset \mathfrak{m}_{\bar{0}} \subset \mathfrak{m}$ and the Hochschild–Serre spectral sequence $E_r^{p,q} \Rightarrow H^n(\mathfrak{m}, \mathfrak{g})$. We have:

$$E_0^{p,q} = \mathfrak{g} \otimes \Lambda^p \mathfrak{m}_{\bar{1}}^* \otimes \Lambda^q \mathfrak{m}_{\bar{0}}^*, \quad E_1^{p,q} = H^q(\mathfrak{m}_{\bar{0}}, \mathfrak{g} \otimes \Lambda^p \mathfrak{m}_{\bar{1}}^*).$$

For cohomology H^n the sequence degenerates on $(n+2)$nd page:

$$H^0(\mathfrak{m}, \mathfrak{g}) = E_2^{0,0},\ H^1(\mathfrak{m}, \mathfrak{g}) = E_2^{1,0} \oplus E_3^{0,1},\ H^2(\mathfrak{m}, \mathfrak{g}) = E_2^{2,0} \oplus E_3^{1,1} \oplus E_4^{0,2}.$$

Strategy: describe $H^{d,n}(\mathfrak{m}, V)$ using Kostant's version of BBW theorem, use $(\mathfrak{g}_0)_{\bar{0}}$ equivariance to restrict and compute the differentials, apply long exact sequence in cohomology to proceed.

$G(3)$ supergeometries: odd contact structure

Consider $M = \mathbb{R}^{1|7}(u|\xi_1, \dots, \xi_7)$ with odd contact structure $\mathcal{D} = \mathrm{Ker}(\alpha)$, $\alpha = du - \sum_1^7 \epsilon_i \xi_i \, d\xi_i$. Then $d\alpha|_{\mathcal{D}}$ is a super conformal symplectic form, or in classical term a conformal metric structure $[g]$, which we assume of signature $(7,0)$ or $(3,4)$. In the latter case it is convenient to change coordinates to have

$$g = d\xi_1 \wedge d\xi_4 + d\xi_2 \wedge d\xi_5 + d\xi_3 \wedge d\xi_6 + d\xi_7 \wedge d\xi_7.$$

The symmetry algebra of this nonholonomic distribution is infinite-dimensional: $K(1|7)$ or $K(1|3,4)$ resp. It is isomorphic to $\mathcal{A}_M$ equipped with the Lagrange bracket, $[X_f, X_g] = X_{\{f,g\}}$.

Let us fix a supersymmetric cubic, or classically a 3-form on $\mathcal{D}$

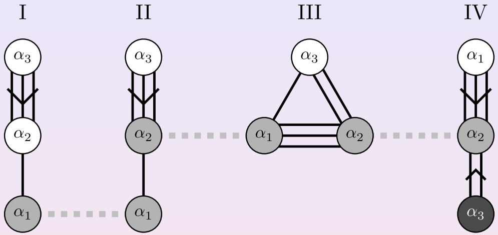
$$q = d\xi_1 d\xi_4 d\xi_7 + d\xi_2 d\xi_5 d\xi_7 + d\xi_3 d\xi_6 d\xi_7 - d\xi_1 d\xi_2 d\xi_3 + d\xi_4 d\xi_5 d\xi_6.$$

Define $\mathfrak{g}_0$ to be a subalgebra of $\mathfrak{gl}(\mathfrak{g}_{-1})$ conformally preserving q. Then $\mathrm{pr}(\mathfrak{g}_-, \mathfrak{g}_0) = \mathrm{sym}(\mathcal{D}, [q]) = G(3)$ with odd contact gradation

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2, \qquad \mathfrak{g}_0 = \mathbb{R} \oplus G(2).$$

Other $G(3)$ supergeometries

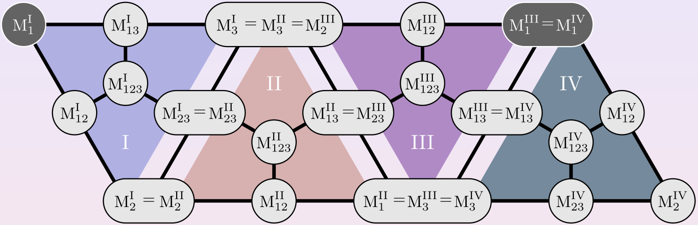
The Lie superalgebra $G(3)$ has 4 different root systems up W-action (one oribit of Weyl groupoid: odd reflections indicated).



For each Dynkin diagram label $\Xi \in \{\text{I}, \text{II}, \text{III}, \text{IV}\}$, the corresponding simple root system $\Pi_\Xi = \{\alpha_1, \alpha_2, \alpha_3\}$ is defined in the following table:

	I	II	III	IV
α_1	$\delta - \varepsilon_1 - \varepsilon_2$	$\varepsilon_1 + \varepsilon_2 - \delta$	$\varepsilon_2 - \delta$	$\varepsilon_2 - \varepsilon_1$
α_2	ε_1	$\delta - \varepsilon_2$	$\delta - \varepsilon_1$	$\varepsilon_1 - \delta$
α_3	$\varepsilon_2 - \varepsilon_1$	$\varepsilon_2 - \varepsilon_1$	ε_1	δ

A choice of root system type Ξ together with a choice of a parabolic subgroup P_χ^Ξ, with $\chi \in \mathcal{P}(\{1,2,3\}) \setminus \{\emptyset\}$, gives one of 19 possible supergeometries $G(3)/P_\chi^\Xi$ with the following twistor correspondences:



Theorem (BK, A. Llabres ⋄ 2022)

For every 17 non-special geometries $\mathrm{pr}(\mathfrak{m}) = G(3)$, *so local (and hence global) symmetries of the vector superdistributions induced on* G/P *are equal to* $G(3)$. *For 2 special contact geometries* $\mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0) = G(3)$ *and this is the symmetry of reduced structures.*

Realizing $F(4)$ as symmetry of super-PDE

Consider the following scalar super-PDE on an even function $u = u(x)$, with $u_{ij} = \partial_x^i \partial_{x^j} u$, etc (sign rule!)

- 2nd order system, with x^0, x^1, x^2 even, and x^3, x^4 odd:

$$
\begin{aligned}
&u_{00} = u_{12}^2 u_{22} + 2u_{12}u_{23}u_{24},\ u_{01} = \tfrac{1}{2}u_{12}^2,\\
&u_{02} = u_{12}u_{22} + u_{23}u_{24},\ u_{03} = u_{12}u_{23},\ u_{04} = u_{12}u_{24},\\
&u_{11} = 0,\ u_{12} = -u_{34},\ u_{13} = 0,\ u_{14} = 0.
\end{aligned}
$$

- 3rd order system, with all x_0, x_1, x_2, x_3 odd:

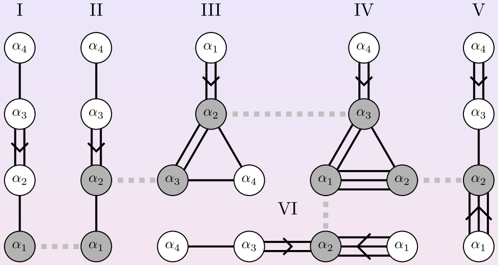
$$u_{0ab} = u_{ab}u_{123}, \quad 1 \le a < b \le 3.$$

Theorem (A. Santi, D. The ◇ 2022)

The contact symmetry superalgebra of these super-PDEs is $F(4)$.

Other $F(4)$-supergeometries

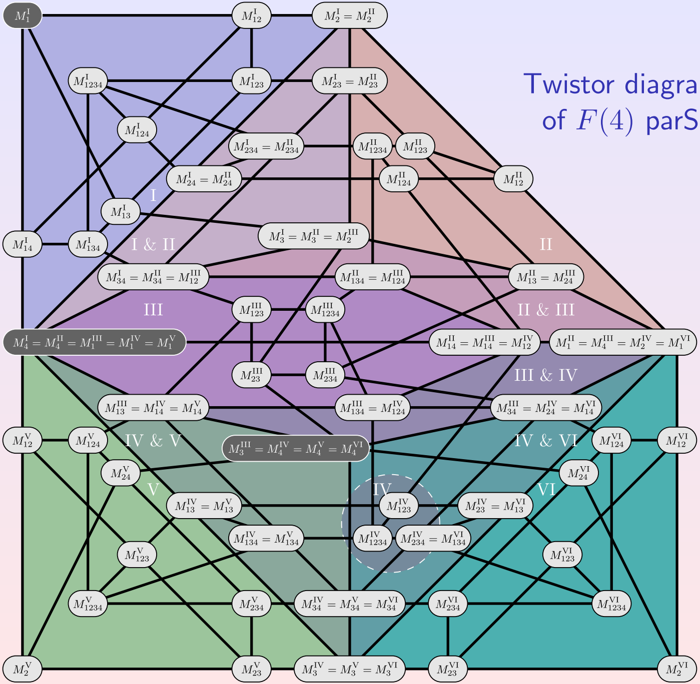
Very similar is the situation with $F(4)$-supergeometries that has 6 different root systems up W-action (we indicate Weyl groupoid):



Theorem (BK, A. Llabres ⋄ 2022)

For every 52 non-special geometries $\mathrm{pr}(\mathfrak{m}) = F(4)$*, so local (and hence global) symmetries of the vector superdistributions induced on* G/P *are equal to* $F(4)$*. For 2 special contact geometries and for 1 irreducible supergeometry* $\mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0) = F(4)$ *and this is the symmetry of the 3 reduced structures.*

Twistor diagram of $F(4)$ parSG



Realization as symmetries of differential equations

From the above theorem we deduce for $\mathfrak{g}$ being either $G(3)$ or $F(4)$

Corollary

Let $\mathcal{D}$ be a distribution with the same symbol as any of 17 distributions on $\exp(\mathfrak{m}) \subset G/P$ in the case of $G(3)$ or 52 distributions in the case of $F(4)$. Then dimension of the symmetry algebra $\dim \mathfrak{s}$ is bounded by $(17|14)$ or $(24|16)$ respectively.

Contrary to the classical case, in the supersetting the integral $(1|0)$ or $(0|1)$ curves are insufficient to recover the distribution. For most canonical distributions $\mathfrak{g}^{-1} \pmod{\mathfrak{p}}$ on G/P, the span of the tangent vectors of $(1|1)$ integral curves (for odd part these are null vectors, which square to zero) gives the distribution.

Yet also these curves are sufficient to recover $D = \mathrm{ev}(\mathcal{D})$ but not $\mathcal{D}$. This happens for distributions of depth> 2. Thus higher dimensional integral submanifolds are required to realize $\mathfrak{g}$ as symmetry of differential equation.

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Supersymmetries of geometric structures III

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Based on joint works with
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Superbundles: FB

A fiber bundle (FB) is a submersion $\pi : E \to M$ with typical fiber F. This means a collection of compatible trivializations

$$\varphi_{\mathcal{U}} : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times F, \quad \mathrm{pr}_{\mathcal{U}} \circ \varphi_{\mathcal{U}} = \pi|_{\mathcal{U}}$$

over superdomains $\mathcal{U} = (U, \mathcal{A}_M|_U)$ for domains $U \subset M_o$.

For an open cover $\{\mathcal{U}_i : i \in I\}$ of M the family of fibered isomorphisms $\varphi_{ij} : \mathcal{U}_{ij} \times F \to \mathcal{U}_{ij} \times F$ (over identity) is called a cocycle if $\varphi_{ii} = \mathbf{1}_{\mathcal{U}_{ij}}$ and $\varphi_{ij}\varphi_{jk} = \varphi_{ik}$ on $\mathcal{U}_{ijk}$.

Proposition

A fiber bundle (E, M, π) with trivializations $(\mathcal{U}_i, \varphi_i)$ defines the cocycle $\{\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}\}_{i,j \in I}$, and any cocycle determines a FB.

By abuse of notations we write a cocycle as $\varphi_{ij} : \mathcal{U}_{ij} \to \mathrm{Aut}(F)$, $i, j \in I$. (The rhs is an infinite-dimensional supermanifold.)

Superbundles: VB

A geometric vector bundle (gVB) is a FB with vector fiber F, given via a cocycle $\varphi_{ij} : \mathcal{U}_{ij} \to \mathrm{GL}(F)$, where $\mathcal{U}_{ij} = \mathcal{U}_i \cap \mathcal{U}_j$ for an open cover $\mathcal{U}_i$ of M.

A section of $\pi : E \to M$ is defined as a morphism $\sigma : M \to E$ such that $\pi \circ \sigma = \mathbf{1}_M$. The set of all even sections $\Gamma_E(\mathcal{U})_{\bar{0}}$ over $\mathcal{U} \subset M$ yields a sheaf of right $\mathcal{O}_M$-modules, but it is not locally free. We extend it to a locally free sheaf $\Gamma_E(\mathcal{U}) = \Gamma_E(\mathcal{U})_{\bar{0}} \oplus \Gamma_E(\mathcal{U})_{\bar{1}}$.

An algebraic vector bundle (aVB) over M is a locally free sheaf $\mathcal{E}$ on M_o of $\mathcal{O}_M$-modules of finite rank. The above association $\pi \rightsquigarrow \Gamma_E$ gives a functor from gVB to aVB.

Theorem

The category of the geometric VBs with morphisms of VBs is equivalent to the category of algebraic VBs with morphisms of locally free coherent sheaves, provided the bases are connected.

Superbundles: PB

A geometric principal bundle (gPB) with structure group G is a FB $\pi : P \to M$ with typical fiber G with the transition cocycle: $\varphi_{ij} : \mathcal{U}_{ij} \to G \subset \mathrm{Aut}(G)$.

Let $\pi_1 : P_1 \to M_1$, $\pi_2 : P_2 \to M_2$ be gPB with structure groups G_1, G_2. Let $\gamma : G_1 \to G_2$ be a homomorphism of Lie supergroups. A γ-morphism of principal bundles is a γ-equivariant fiber bundle morphism $(\psi_P, \psi_M) : (P_1, M_1) \to (P_2, M_2)$: if $\alpha_i : P_i \times G_i \to P_i$ are actions then

$$\alpha_2 \circ (\psi_P \times \gamma) = \psi_M \circ \alpha_1.$$

This, in particular, gives reduction of the structure group.

An algebraic principal bundle (aPB) over M is a sheaf $\mathcal{P}$ of right $\mathcal{G}_M$-sets that is locally simply transitive; $\mathcal{G}_M(\mathcal{U}) = G[\mathcal{U}]$.

Theorem

The categories of geometric PBs and algebraic PBs with γ-morphisms are equivalent, for homomorphisms of supergroups γ.

Frame bundles

Let $\dim(M) = (m|n)$, $V = \mathbb{R}^{m|n}$. Consider the trivial VB over M

$$V_M = M \times V \to M.$$

Let $\mathcal{V}_M$ be the associated locally free sheaf on M_o. Frame bundle $\pi : Fr_M \to M$ is defined via the geometric-algebraic correspondence as the sheaf of $\mathcal{A}_M$-linear isomorphisms from $\mathcal{V}_M$ to $\mathcal{T}M$:

$$\mathcal{F}r_M(\mathcal{U}_o) = \left\{\mathcal{A}|_{\mathcal{U}_o}\text{-linear isomorphism } F : \mathcal{V}_M|_{\mathcal{U}_o} \to \mathcal{T}M|_{\mathcal{U}_o}\right\}.$$

We embed $\mathcal{A}_M$-sheaves $\mathcal{F}r_M \hookrightarrow (\mathcal{T}_M^{m|n}) = \mathcal{T}M^{\oplus m} \bigoplus \Pi\mathcal{T}M^{\oplus n}$. Vector fields $X_i \in \mathfrak{X}(\mathcal{U})_{\bar{0}}$ $(1 \le i \le m)$, $Y_j \in \mathfrak{X}(\mathcal{U})_{\bar{1}}$ $(1 \le j \le n)$ with reduction defining a basis of $T_xM = (T_xM)_{\bar{0}} \oplus (T_xM)_{\bar{1}}$ at each $x \in M_o$, give the frame $F \in \mathcal{F}r_M(\mathcal{U}_o)$ so

$$F : \mathcal{V}_M|_{\mathcal{U}_0} \to \mathcal{T}M|_{\mathcal{U}_0}\,, \qquad (a_i|b_j) \mapsto \sum_{i=1}^m a_i X_i + \sum_{j=1}^n b_j Y_j\,.$$

The sheaf of groups $\mathcal{GL}_M : \mathcal{U}_o \mapsto GL(V)[\mathcal{U}]$ acts naturally on the set of frame fields and this locally simply transitive action makes $\mathcal{F}r_M$ into aPB over M with structure group $GL(V)$.

G-structures

For $G \subset GL(V)$, a G-structure is a reduction of the frame bundle $Fr_M \supset F_G \xrightarrow{\pi} M$. It is a subsheaf $\mathcal{F}_G \subset \mathcal{F}r_M$ on which the subsheaf $\mathcal{G}_M \subset \mathcal{GL}_M$ acts locally simply transitively. Soldering form

$$\vartheta \in \Omega^1(F_G, V) \text{ is given by } \vartheta_F(\xi) = F^{-1}(\pi_* \xi),\ \xi \in \mathrm{Vect}(F_G).$$

Definition

- A horizontal distribution is a subsheaf $\mathcal{H} \subset \mathcal{T}F_G$ on $(F_G)_o$ of $\mathcal{A}_{F_G}$-modules that is complementary to $\mathrm{Ker}(\pi_*) \subset \mathcal{T}F_G$.
- A normalization is a supervector space $N \subset V \otimes \Lambda^2 V^*$ that is complementary to $\mathrm{Im}(\delta)$ in the Spencer complex

$$0 \to \mathfrak{g}^{(1)} \longrightarrow \mathfrak{g} \otimes V^* \xrightarrow{\delta} V \otimes \Lambda^2 V^* \to 0.$$

Any horizontal distribution gives an isomorphism $\mathcal{H} \cong \pi^* \mathcal{T}M$, whence a morphism $\phi_{\mathcal{H}} : \pi^* \mathcal{T}M \to \mathcal{T}F_G$. The torsion of $\mathcal{H}$ is

$$T_{\mathcal{H}}(X_1, X_2) = d\vartheta(\phi_{\mathcal{H}} X_1, \phi_{\mathcal{H}} X_2).$$

Prolongations

Let $Fr_0 = F_G$, $\mathcal{F}r_0 = \mathcal{F}_G$. Define $\mathcal{F}r_1 : (F_G)_o \supset \mathcal{V}_o \mapsto \mathcal{F}r_1(\mathcal{V}_o)$ to be the sheaf on $(Fr_0)_o$ given by

$$\mathcal{F}r_1(\mathcal{V}_o) = \left\{ \mathcal{H}(\mathcal{V}_o) \mid \mathcal{H} \subset \mathcal{T}Fr_0|_{\mathcal{V}_o} \text{ such that } T_{\mathcal{H}} \in \mathcal{N}|_{\mathcal{V}_o} \right\},$$

for any superdomain $\mathcal{V}_o \subset (Fr_0)_o$. The sheaf of Abelian groups $\mathcal{G}^{(1)}_{F_G} : \mathcal{V}_o \mapsto \mathfrak{g}^{(1)}[\mathcal{V}]$ on $(Fr_0)_o$ acts simply-transitively on $\mathcal{F}r_1$ whence an affine bundle $Fr_1 \to Fr_0$ by the geometric-algebraic correspondence.

Further prolongations follow the same scheme and yield the tower

$$M \leftarrow Fr_0 \leftarrow Fr_1 \leftarrow Fr_2 \leftarrow \dots.$$

The structure group of $Fr_k \to Fr_{k-1}$ is Abelian $\mathfrak{g}_k = \mathfrak{g}^{(k-1)}$:

$$0 \to \mathfrak{g}_k \to \mathfrak{g}_{k-1} \otimes V^* \to \mathfrak{g}_{k-2} \otimes \Lambda^2 V^* \to \dots$$

A G-structure F_G is called of finite type if this tower stabilizes. This happens at the level k when $\mathfrak{g}_k = 0$.

Automorphisms of G-structures and generalizations

We introduce the automorphism supergroup of G-structures as a Harish-Chandra pair $\mathrm{Aut}(\mathcal{F}_G) = \big(\mathrm{Aut}(\mathcal{F}_G)_{\bar{0}}, \mathfrak{aut}(\mathcal{F}_G)\big)$.

Definition

- An automorphism of $\mathcal{F}_G$ is such a $\varphi = (\varphi_o, \varphi^*) \in \mathrm{Aut}(M)_{\bar{0}}$ that
$$\varphi_*(\mathcal{F}_G) \subset (\varphi_o)_*^{-1}\mathcal{F}_G.$$
- An infinitesimal automorphism of $\mathcal{F}_G$ on a superdomain $\mathcal{U} \subset M$ is a supervector field $X \in \mathrm{Vect}(\mathcal{U})$ such that
$$\mathcal{L}_X\big(\mathcal{F}_G(\mathcal{U}_o)\big) \subset \mathcal{F}_G(\mathcal{U}_o) \cdot \big(\mathfrak{g} \otimes \mathcal{A}_M(\mathcal{U}_o)\big) \subset \mathcal{T}_M^{m|n}(\mathcal{U}_o).$$

For nonholonomic geometric structures $(M, \mathcal{D}, q)$ given via distributuion $\mathcal{D}$ and possible auxillary structure q we generalize the above to graded frames, introduce normalization via generalized Spencer complex and then construct prolongation bundles.

The automorphism supergroup is defined correspondingly.

Filtered geometric super structures, in particular G-structures are defined through succesive frame bundles reductions. In particular, super tensors, connections and differential equations are such.

Example (nondegenerate even super-symmetric form)

The supermanifold $M = \mathbb{R}^{m|2n}(x,\xi)$ with the metic $g = (1+k\|x\|^2)^{-2} \cdot \sum_{i=1}^m dx_i^2 + \sum_{i=1}^n d\xi_i d\xi_{i+n}$ has symmetry:

$$\mathfrak{g} = \begin{cases} \mathfrak{osp}(m+1|2n) & k>0 \\ \mathfrak{osp}(m|2n) \ltimes \mathbb{R}^{m|2n} & k=0 \\ \mathfrak{osp}(m,1|2n) & k<0. \end{cases}$$

Example (nondegenerate even super-skew-symmetric form)

The supermanifold $M = \mathbb{R}^{2n|m}(x,\xi)$ with the symplectic form $\omega = \sum_{i=1}^n dx_i \wedge dx_{i+n} + \sum_{i=1}^m d\xi_i \wedge d\xi_i$ has infinite-dim symmetry $\mathfrak{symp}(\omega) \simeq \mathcal{O}_M/\mathbb{R}$ – prolongation of $\mathfrak{spo}(2n|m)$ (see below).

Symmetry dimension bound

Theorem (BK, A.Santi, D.The ⋄ 2021)

Let $\mathfrak{s}$ be the symmetry superalgebra of a bracket-generating, strongly regular filtered geom.structure $(M, \mathcal{D}, q)$, with the Tanaka–Weisfeiler prolongation $\mathfrak{g} = \mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0)$. If the reduced manifold M_o is connected, then $\dim \mathfrak{s} \leq \dim \mathfrak{g}$ in the strong sense: the inequality applies to both even and odd dimensions.

The LSA $\mathfrak{s}$ can be considered as a superalgebra of vector fields localized in a fixed neighborhood $U_o \subset M_o$ or as germs of those.

Assuming $\dim \mathfrak{g}$ is finite, the above bound is sharp: there exists a standard model G/P with the required symmetry dimension.

Theorem (—)

With the above assumptions $\mathrm{Aut}(M, \mathcal{D}, q)$ is a Lie supergroup. If M_o is connected, then $\dim \mathrm{Aut}(M, \mathcal{D}, q) \leq \dim \mathfrak{g}$ in strong sense.

Sketch of the proof

As in Cartan method we construct a tower of PB with successive structure groups G_0 for $k=0$ and $\mathfrak{g}_k$ for $k>0$:

$$M \leftarrow \mathcal{P}_0 \leftarrow \mathcal{P}_1 \leftarrow \dots$$

consisting of partial frames. (No functor of points.)

We adapt the Tanaka construction revised by I.Zelenko to the super-context, using a uniform normalization via the generalized Spencer complex

$$0 \to \mathfrak{g} \xrightarrow{\delta} \mathfrak{m}^* \otimes \mathfrak{g} \xrightarrow{\delta} \Lambda^2\mathfrak{m}^* \otimes \mathfrak{g} \to \dots$$

The structure functions, used in normalizations, are as follows:

$$c^-_{\mathcal{H}_\ell} \in \mathcal{A}_{F_\ell} \otimes (\Lambda^2\mathfrak{m}^* \otimes \mathfrak{g}_{<\ell}),\ c^+_{\mathcal{H}_\ell} \in \mathcal{A}_{F_\ell} \otimes (\mathfrak{g}^+_{\leq \ell-1})^* \otimes (\mathfrak{m}^* \otimes \mathfrak{g})_\ell.$$

The final bundle $\mathcal{P} \to M$ has a canonical connection $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$, whence the claim.

Absolute parallelism: infinitesimal automorphisms

By fixing a basis of $\mathfrak{g}$, the absolute parallelism is a coframe $\{\omega^\beta\}$ on $\mathcal{P}$. Let $\{e_\alpha\}$ be the dual frame: $\langle e_\alpha, \omega^\beta \rangle = (-1)^{|\alpha||\beta|}\omega^\beta(e_\alpha) = \delta^\beta_\alpha$.

Lemma

Let $\{e_\alpha\}$ be a frame on a supermanifold $P = (P_o, \mathcal{A}_P)$ with connected reduced manifold. Fix $x \in P_o$. Then any symmetry $v \in \mathrm{Vect}(P)$ of the frame is determined by its value at x.

Indeed, for the ideal $\mathcal{J} = (\mathcal{A}_P)^2_{\bar 1} \oplus (\mathcal{A}_P)_{\bar 1} \subset \mathcal{A}_P$ generated by nilpotents and the map $\mathcal{J}^k/\mathcal{J}^{k+1} \to \mathcal{T}^*P \otimes \mathcal{J}^{k-1}/\mathcal{J}^k$ is injective for any $k > 0$. Hence evaluation is injective on symmetries:

$$\mathrm{ev} : [\mathrm{Vect}(P) \supset \mathfrak{s}] \hookrightarrow \Gamma(TP|_{P_o})$$

The condition that $v = a^\gamma e_\gamma \in \mathrm{Vect}(P)$ preserves the coframe is:

$$0 = L_v\omega^\gamma = d\imath_v\omega^\gamma + \imath_v d\omega^\gamma = da^\gamma - \frac{1}{2}a^\delta(-1)^{|\alpha||\beta|}\imath_{e_\delta}(\omega^\alpha \wedge \omega^\beta)c^\gamma_{\alpha\beta},$$

equivalent to complete PDE system $da^\gamma = (-1)^{|\beta||v|}(\omega^\beta)a^\alpha c^\gamma_{\alpha\beta}$.

Holonomic examples

• Super-Riemann structures (M, g) are G_0-structures with $G_0 = \mathrm{OSp}(m|2n)$. For $\mathfrak{g}_0 = \mathrm{Lie}(G_0) = \mathfrak{osp}(m|2n)$ we have $\mathfrak{g}_1 = \mathfrak{g}_0^{(1)} = 0$. Hence the Lie superalgebra of Killing supervector fields satisfies

$$\dim \mathfrak{s} \le \dim \mathfrak{g}_{-1} + \dim \mathfrak{g}_0 = \Bigl(\tbinom{m+1}{2} + \tbinom{2n+1}{2} \,\big|\, 2n + 2mn\Bigr).$$

• Almost super-symplectic structures (M, ω) are G_0-structures with $G_0 = \mathrm{SpO}(2n|m)$. For $\mathfrak{g}_0 = \mathrm{Lie}(G_0) = \mathfrak{spo}(2n|m)$ we have: $\mathfrak{g}_i = \mathfrak{g}_0^{(i)} = S^{i+2}V^*$, $V = TM$ (in the super-sense), so $\mathfrak{g}_0 \subset \mathfrak{gl}(V)$ is of infinite type unless M is purely odd:

$$\mathfrak{g} = \mathrm{pr}(\mathfrak{g}_0) \simeq \oplus_{i=1}^{\infty} S^i V^*.$$

In the case M is purely odd $(n = 0)$, the Lie superalgebra of symplectic supervector fields satisfies:

$$\dim \mathfrak{s} \le \sum_{i=-1}^{m-2} \dim \mathfrak{g}_i = (2^{m-1} - 1|2^{m-1}).$$

Holonomic examples

• Periplectic structures (M, q) with q odd ndg bilinear form on TM are irreducible G_0-structures with $G_0 = \mathrm{Pe}(n)$. For $\mathfrak{g} = \mathrm{Lie}(G_0) = \mathfrak{pe}(n)$ we have $\mathfrak{g}^{(1)} = 0$. Hence the Lie superalgebra of symmetries satisfies:

$$\dim \mathfrak{s} \leq (n^2 + n|n^2 + n).$$

(There are some other periplectic-related structures for which the prolongations are different/longer.)

• Projective structures on supermanifolds of $\dim M = (m|n)$ are equivalence classes of torsion-free connections: $\nabla \sim \nabla'$ iff $\nabla - \nabla' = \mathrm{Id} \circ \omega \in \Gamma(S^2\mathcal{T}^*M \otimes \mathcal{T}M)$ for an even $\omega \in \Omega^1(M)$. We have $\mathfrak{g}_0 = \mathfrak{gl}(m|n)$ and its prolongation if $(m|n) \neq (1|0)$ is $\mathfrak{g}_1 = S^2V^* \otimes V = V^* \oplus (S^2V^* \otimes V)_0 = \mathfrak{g}_1' \oplus \mathfrak{g}_1''$. Projective connection reduces this to the first component, further prolongations are trivial. Whence the bound for symmetries:

$$\dim \mathfrak{s} \leq \dim V + \dim \mathfrak{gl}(V) + \dim \mathfrak{g}_1' = \big(2m + n^2 + m^2 \,|\, 2n + 2mn\big).$$

Nonholonomic examples: distributions with structures

• Let $\mathfrak{m} = \mathfrak{heis}(1|7) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} = \mathbb{R}^{1|0} \oplus \mathbb{R}^{0|7}$ and q a generic field of cubics on $\mathcal{D} = \exp(\mathfrak{g}_{-1})$ in $M = \exp(\mathfrak{m})$. Then $\dim \mathfrak{s} \leq (17|14)$. For a left-invariant cubic q we have:

$$\mathfrak{s} = \mathrm{sym}(M, \mathcal{D}, [q]) = G(3) \quad \Leftrightarrow \quad H^1(\mathfrak{m}_1^I, G(3))_+ = 0.$$

• Let $\mathfrak{m} = \mathfrak{heis}(1|8) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} = \mathbb{R}^{1|0} \oplus \mathbb{R}^{0|8}$ and Q a generic field of quartics on $\mathcal{D} = \exp(\mathfrak{g}_{-1})$ in $M = \exp(\mathfrak{m})$. Then $\dim \mathfrak{s} \leq (24|16)$. For a left-invariant quartic Q we have:

$$\mathfrak{s} = \mathrm{sym}(M, \mathcal{D}, [Q]) = F(4) \quad \Leftrightarrow \quad H^1(\mathfrak{m}_1^I, F(4))_+ = 0.$$

• Let $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} = \mathbb{R}^{m|n} \oplus \mathbb{R}^{m+1|n}$, $\mathcal{D} = \exp(\mathfrak{g}_{-1})$ split as $\mathbb{R}^{1|0} \oplus \mathbb{R}^{m|n}$ in $M = \exp(\mathfrak{m})$, and p is the projector corresponding to splitting. Then $\dim \mathfrak{s} \leq (m^2 + 2m + n^2 | 2n + 2mn)$. The most symmetric case corresponds to the trivial ODE system $Y_{xx} = 0$:

$$\mathfrak{s} = \mathrm{sym}(M, \mathcal{D}, p) = \mathfrak{sl}(m+1|n).$$

Odd second & third order ODEs

Recall at first the classical story ($y = y(x)$ even):

- 2nd ord ODEs $y'' = f(x, y, y')$ mod point transformations have at most 8-dim symmetry algebra and max symmetry $\mathfrak{sl}(3)$ corresponds to $y'' = 0$;
- 3rd ord ODEs $y''' = f(x, y, y', y'')$ mod contact transformations have at most 10-dim symmetry algebra and max symmetry $\mathfrak{sp}(4, \mathbb{R})$ corresponds to $y''' = 0$.

Now let us look to super analogs ($\xi = \xi(x)$ odd, x even):

- 2nd ord ODEs $\xi'' = f(x, \xi, \xi')$ mod point transformations have $(4|4)$-dim symmetry algebra and always trivialize to $\xi'' = 0$ with symmetry $\mathfrak{sl}(2|1)$;
- 3rd ord ODEs $\xi''' = f(x, \xi, \xi', \xi'')$ mod contact transformations have at most $(4|4)$-dim symmetry algebra and max symmetry corresponds to $\xi''' = 0$.

Details on third order ODEs

	Even part	Odd part
$+2$	$\cdot$	$-\xi\partial_x + \xi'\xi''\partial_{\xi''} + 2\xi'\xi'''\partial_{\xi'''}$
$+1$	$\frac{x^2}{2}\partial_x + x\xi\partial_\xi + \xi\partial_{\xi'}$ $+(\xi' - x\xi'')\partial_{\xi''} - 2x\xi'''\partial_{\xi'''}$	$\cdot$
0	$x\partial_x + \xi\partial_\xi - \xi''\partial_{\xi''} - 2\xi'''\partial_{\xi'''}$ $\xi\partial_\xi + \xi'\partial_{\xi'} + \xi''\partial_{\xi''} + \xi'''\partial_{\xi'''}$	$\cdot$
-1	$-\partial_x$	$\frac{x^2}{2}\partial_\xi + x\partial_{\xi'} + \partial_{\xi''}$
-2	$\cdot$	$x\partial_\xi + \partial_{\xi'}$
-3	$\cdot$	∂_ξ

These are all *point* symmetries. The derived superalgebras of $\mathfrak{g}$:

$$\mathfrak{g}^{(1)} = \mathbb{R}^{0|1} \ltimes \mathfrak{g}^{(2)}, \quad \mathfrak{g}^{(2)} \cong \mathfrak{sl}(2,\mathbb{R})_{\bar{0}} \ltimes (S^2\mathbb{R}^2)_{\bar{1}}.$$

We also have for non-flat cases:

$$\dim \operatorname{sym}(\xi''' = \xi'') = (2|3), \quad \dim \operatorname{sym}(\xi''' = \xi\xi'\xi'') = (2|2).$$

N-extended Poincaré superstructures

Let $(\mathbb{V}, g)$ be a metric vector space and $\mathbb{S}$ be a spin module. Let $\mathfrak{g}_{-2} = \mathbb{V}$, $\mathfrak{g}_{-1} = \underbrace{\mathbb{S} \oplus \cdots \oplus \mathbb{S}}_{N}$ and $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ be a LSA with consistent gradation: $\mathfrak{m}_{\bar{0}} = \mathfrak{g}_{-2}$, $\mathfrak{m}_{\bar{1}} = \mathfrak{g}_{-1}$. Then $\mathfrak{m} \oplus \mathfrak{so}(\mathbb{V})$ is the N-extended Poincaré superalgebra. Brackets $\Lambda^2\mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ were classified by D.Alekseevsky-V.Cortes (Λ^2 in super sense).

The prolongation $\mathfrak{g} = \mathrm{pr}(\mathfrak{m})$ was computed by A.Altomani-A.Santi. It equals $\mathfrak{m} \oplus \mathfrak{g}_0$, $\mathfrak{g}_0 = \mathfrak{so}(\mathbb{V}) \oplus \mathbb{R} \oplus \mathfrak{g}_0^\dagger$, except for the cases $A(m|3)/P_{2,m+2}$, $B(m|2)/P_2$, $D(m|2)/P_2$, $F(3|1)/P_2$, where the prolongation is the corresponding semisimple Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \cdots \oplus \mathfrak{g}_2$. This gives the symmetry bound

$$\dim \mathfrak{s} \leq \left(\binom{d+1}{2} + 1 + \dim \mathfrak{g}_0^\dagger \,\middle|\, N \cdot 2^{[d/2]} \right),$$

where $d = \dim \mathbb{V}$ (achieved for the homogeneous model).

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Thanks for your attention!