From fluid computers to escape trajectories: Two sides of a mirror

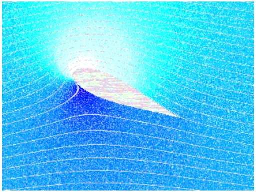
43 Winter School on Geometry and Physics, Srní

Eva Miranda

UPC-CRM

Planning for the lectures

- **Session 1: Basics in contact geometry and Euler flows. The mirror: Etnyre and Ghrist correspondence.**
- **Session 2: Constructing Fluid computers in dimension 3 via contact geometry. Existence of undecidable paths and the Navier-Stokes conjecture.**
- **Session 3: Singular symplectic and contact geometry, the (singular) Weinstein conjecture and escape trajectories in Celestial mechanics.**


Classical Euler equations on $\mathbb{R}^3$:

$$\begin{cases} \dfrac{\partial X}{\partial t} + (X \cdot \nabla) X = -\nabla P \\ \operatorname{div} X = 0 \end{cases}$$

The evolution of an **inviscid and incompressible fluid flow** on a Riemannian n-dimensional manifold (M, g) is described by the Euler equations:

$$\frac{\partial X}{\partial t} + \nabla_X X = -\nabla P \,, \qquad \operatorname{div} X = 0$$

- X is the velocity field of the fluid: a non-autonomous vector field on M.
- P is the inner pressure of the fluid: a time-dependent scalar function on M.

Incompressible fluids on Riemannian manifolds

If X does not depend on time, it is a steady or stationary Euler flow: it models a fluid flow in equilibrium. The equations can be written as:

$$\nabla_X X = -\nabla P \,, \qquad \operatorname{div} X = 0 \,,$$

$$\iff \iota_X d\alpha = -dB \,, \qquad d\iota_X \mu = 0 \,, \qquad \alpha(\cdot) := g(X, \cdot)$$

where $B := P + \frac{1}{2}||X||^2$ is the Bernoulli function.

Beltrami fields:

$\operatorname{curl} X = fX,$ with $f \in C^\infty(M) \qquad \operatorname{div} X = 0.$

Example (Hopf fields on S^3 and ABC fields on T^3)

- The Hopf fields $u_1 = (-y, x, \xi, -z)$ and $u_2 = (-y, x, -\xi, z)$ are Beltrami fields on S^3.
- The ABC flows $(\dot{x}, \dot{y}, \dot{z}) = (A \sin z + C \cos y, B \sin x + A \cos z, C \sin y + B \cos x)$, $((x, y, z) \in (\mathbb{R}/2\pi\mathbb{Z})^3)$ are Beltrami.

Hopf fields as Beltrami fields

- Observe that $\div X = 0$.
- The orbits of the Hopf field are geodesics, and hence the Hopf fields give a solution to the Euler equations with constant pressure.
- X is a Beltrami field, since $||X||^2 = 1$ and then the Beltrami function $B := P + \frac{1}{2}||X||^2$ is constant.

We can also check directly that

$$\operatorname{curl} X = 2X$$

The volume form μ form on S^3 has the property, for $r = (x_1, y_1, x_2, y_2)$ a point in S^3:

$$\mu(v_1, v_2, v_3) = \det_{\mathbb{R}^4}(r, v_1, v_2, v_3),$$

so, since $\iota_{\operatorname{curl} X}\mu = d\iota_X g$, we can check that:

$$\det(r, X, \cdot, \cdot) = 2(dx^1 \wedge dy^1 + dx^2 \wedge dy^2)$$

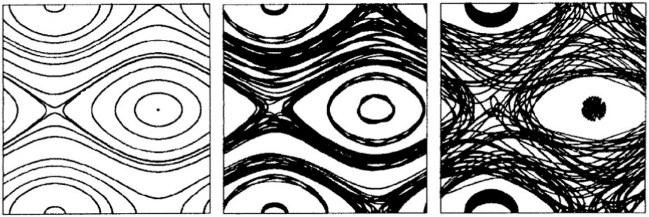
so X is a Beltrami field.

ABC flows

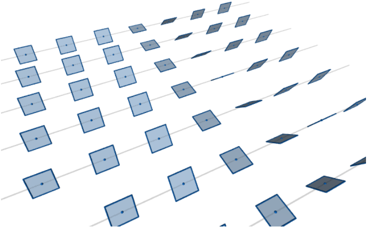
$$X_{ABC} = (A\sin z + C\cos y)\frac{\partial}{\partial x} + (B\sin x + A\cos z)\frac{\partial}{\partial y} + (C\sin y + B\cos x)\frac{\partial}{\partial z}$$

- These flows are clearly volume preserving, $\mathrm{div} X_{ABC} = \nabla \cdot X_{ABC} = 0$
- We can directly check using the usual rotational in $\mathbb{R}^3$ that these fields are eigenfields of the curl operator with eigenvalue 1:
  $\mathrm{curl}\, X_{ABC} = \nabla \times X_{ABC} = X_{ABC}$

So they are Beltrami fields.



By changing the parameters we get a variety of behaviors. For instance, if one of the parameters is zero, the flow is known to be integrable.



Symplectic	**Contact**
$\dim M = 2n$	$\dim M = 2n+1$
2-form ω, non-degenerate $d\omega = 0$	1-form α, $\alpha \wedge (d\alpha)^n \neq 0$
Darboux theorem $\omega = \sum_{i=1}^n dx_i \wedge dy_i$	$\alpha = dx_0 - \sum_{i=1}^n x_i dy_i$
Hamiltonian $\iota_{X_H}\omega = -dH$	Reeb $\alpha(R) = 1$, $\iota_R d\alpha = 0$ Ham. $\begin{cases} \iota_{X_H}\alpha = H \\ \iota_{X_H}d\alpha = -dH + R(H)\alpha. \end{cases}$

Two guiding conjectures

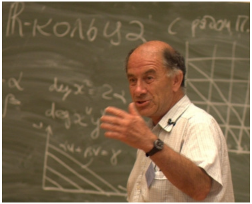
Weinstein's conjecture

The Reeb vector field of a contact compact manifold admits at least one periodic orbit.

Arnold's conjecture

Given a t-dependent Hamiltonian $H_t : \mathbb{R} \times M^{2n} \to \mathbb{R}$

$$\#\{\text{periodic orbits } X_{H_t}\} \geqslant \sum_{k=0}^{2n} \beta_k.$$



Why periodic orbits?

On peut alors avec avantage prendre [les] solutions périodiques comme première approximation, comme orbite intermédiaire [...].Ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissions essayer de pénetrer dans une place jusqu'ici reputée inabordable.

H. Poincaré. Les méthodes nouvelles de la mécanique céleste
Gauthier-Villars et fils, Paris, 1892.

An example of contact structure

The kernel of a 1-form α on M^{2n+1} is a contact structure whenever $\alpha \wedge (d\alpha)^n$ is a volume form $\Leftrightarrow d\alpha|_\xi$ is non-degenerate.

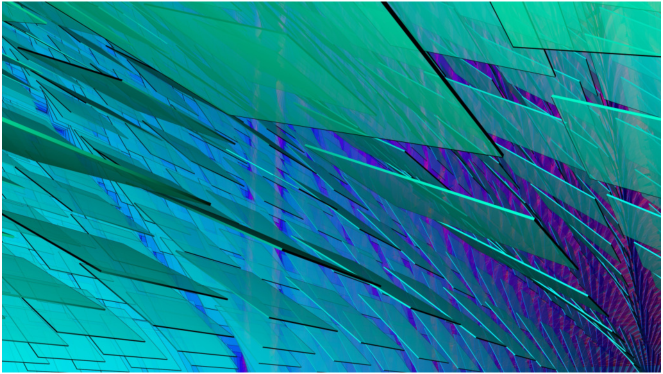
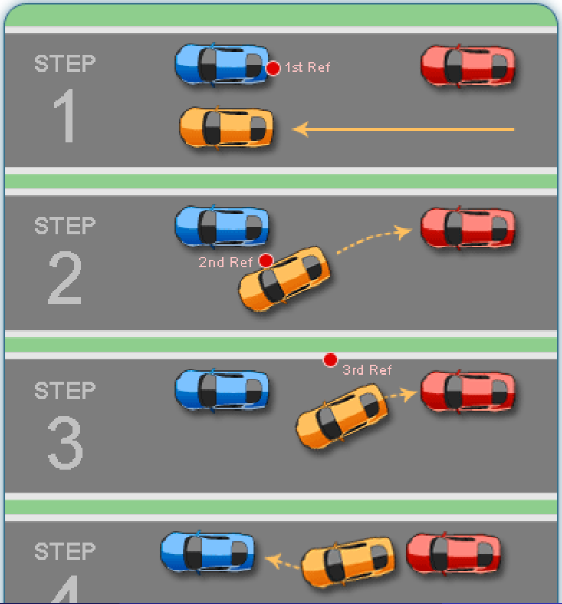


Figure: Standard contact structure on $\mathbb{R}^3$ by Robert Ghrist

$$\alpha = dz - y dx \quad \xi = \ker \alpha = \left\langle \frac{\partial}{\partial y}, y\frac{\partial}{\partial z} + \frac{\partial}{\partial x} \right\rangle d\alpha = -dy \wedge dx = dx \wedge dy$$

$$\Rightarrow \ \alpha \wedge d\alpha \ = \ dx \wedge dy \wedge dz$$

Contact geometry and parallel parking



Contact geometry and parallel parking

Theorem 17. *A car of length L can be parallel parked in any space of length $L+\epsilon$, $\epsilon > 0$.*

Proof. Let us assume that the car is on the plane $\mathbb{R}^2$. Its position can be described by a single coordinate (x, y) and the angle $\theta \in S^1$ its tires are facing, or equivalently a point in the configuration space $\mathbb{R}^2 \times S^1$, which has contact form

$$\alpha = \sin\theta dx - \cos\theta dy.$$

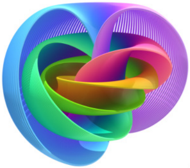
(Note that $\alpha \wedge d\alpha = -dx \wedge dy \wedge d\theta$, so we will reverse the usual orientation of S^1.) The car's path $\gamma(t) = (x(t), y(t), \theta(t))$ will satisfy $\frac{dy}{dx} = \tan\theta$, or equivalently $\frac{dx}{dt}\sin\theta - \frac{dy}{dt}\cos\theta = 0$: thus $\gamma(t)$ must be Legendrian. We now take a path through configuration space which pulls the car up parallel to the parking spot and then slides it horizontally into place without turning the wheel; this is physically impossible, but an arbitrarily close Legendrian approximation will successfully park the car. □

The Hopf fibration revisited

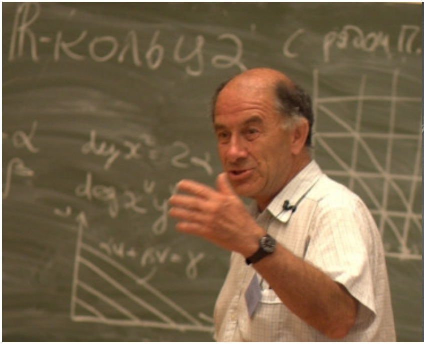
- $S^3 := \{(u,v) \in \mathbb{C}^2 \mid |u|^2 + |v|^2 = 1\}, \alpha = \frac{1}{2}(ud\overline{u} - \overline{u}du + vd\overline{v} - \overline{v}dv)$.
  The orbits of the Reeb vector field form the Hopf fibration!

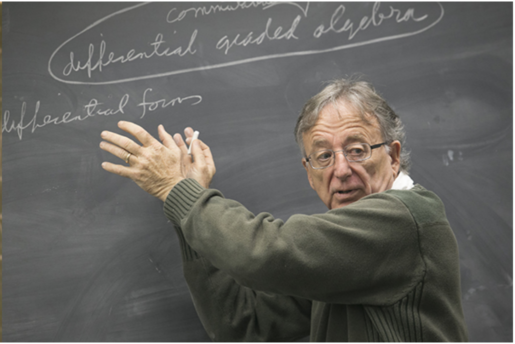
$$R_\alpha = iu\frac{\partial}{\partial u} - i\overline{u}\frac{\partial}{\partial \overline{u}} + iv\frac{\partial}{\partial v} - i\overline{v}\frac{\partial}{\partial \overline{v}}$$

- $\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ can be endowed with Hopf coordinates $(z_1, z_2) = (\cos s \exp i\phi_1, \sin s \exp i\phi_2)$, $s \in [0, \pi/2]$, $\phi_{1,2} \in [0, 2\pi)$. The Hopf field $R := \partial_{\phi_1} + \partial_{\phi_2}$ is a steady Euler flow (Beltrami) with respect to the round metric.



Geometry of Fluids

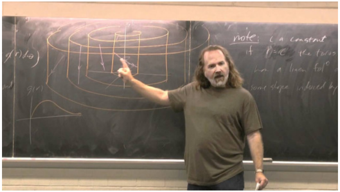




The magic mirror

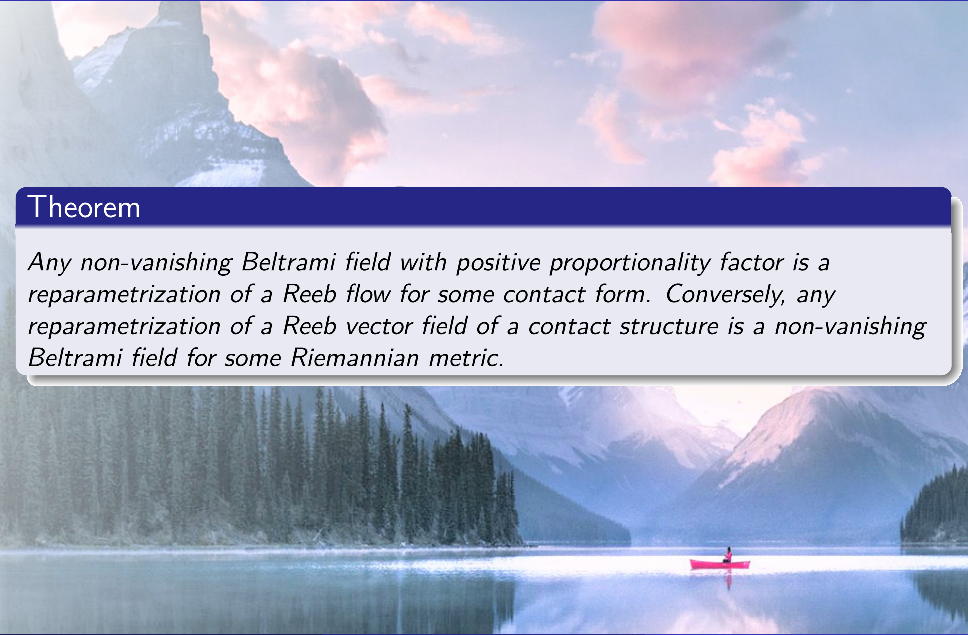
In terms of $\alpha = \iota_X g$ and μ (volume form) the **stationary Euler equations** read

$$\begin{cases} \iota_X d\alpha = -dB \\ d\iota_X \mu = 0 \end{cases}$$



- Etnyre-Ghrist:
  {Rotational non singular Beltrami v.f.} $\rightleftarrows$ {Reeb v.f. reparametrized}
- With Cardona and Peralta-Salas we have extended this picture to manifolds with cylindrical ends to get singular contact structures.
- CMPP: The Beltrami/contact correspondence works in higher dimensions.

Main theorem 1



Theorem

Any non-vanishing Beltrami field with positive proportionality factor is a reparametrization of a Reeb flow for some contact form. Conversely, any reparametrization of a Reeb vector field of a contact structure is a non-vanishing Beltrami field for some Riemannian metric.

Preliminary computations

Let us check that for $\alpha = \iota_X g$ and μ (volume form) the **stationary Euler equations** read

$$\begin{cases} \iota_X d\alpha = -dB \\ d\iota_X \mu = 0 \end{cases}$$

- The metric g induces an isomorphism $\chi(M) \leftrightsquigarrow \Omega^1(M)$. By applying it to the steady Euler equation we obtain

$$g(\nabla_X X, \cdot) = -d(P)$$

- Using the properties of the Levi-Civita connection, we will show

$$g(\nabla_X X, \cdot) = \iota_X d\alpha + \frac{1}{2} d(\iota_X \alpha)$$

- Hence, the Euler equation becomes

$$\iota_X d\alpha = -d(P + \frac{1}{2}\alpha(X)) = -dB$$

with $B := P + \frac{1}{2}||X||^2$, the Bernoulli function.

Details of the proof

Let us check the second equality:

- ∇ Levi-Civita connection: $g(\nabla_X X, Y) = X(\alpha(Y)) - g(X, \nabla_X Y)$ and $\nabla_Y X - \nabla_X Y = [X, Y] \rightsquigarrow g(\nabla_X X, Y) = X(\alpha(Y)) - \alpha([X, Y]) - \alpha(\nabla_Y X)$.
- As $Y \cdot g(X, X) = d(\alpha(X))(Y) = 2g(X, \nabla_Y X) \rightsquigarrow$ $g(\nabla_X X, Y) = X(\alpha(Y)) - \alpha([X, Y]) - \frac{1}{2} d(\alpha(X))(Y)$.

What does the apparently complicated expression on the right hand side represent? Let us check that it is exactly

$$(L_X \alpha)(Y) - \frac{1}{2} d(\alpha(X))(Y)$$

.

Let's prove it!

- From Cartan's formula $(L_X\alpha)(Y) = d\alpha(X,Y) + d(\alpha(X))(Y)$ and

$$d\alpha(X,Y) = X \cdot (\alpha(Y)) - Y \cdot (\alpha(X)) - \alpha([X,Y])$$

- So $(L_X\alpha)(Y) = X \cdot (\alpha(Y)) - \alpha([X,Y])$,
- $g(\nabla_X X, \cdot) = L_X\alpha - \frac{1}{2}d(\alpha(X))$.

So the Euler equation reads,

$$\iota_X d\alpha + d\iota_X\alpha - \frac{1}{2}d\iota_X\alpha = -dP.$$

We will write it as

$$\iota_X d\alpha = -d(P + \frac{1}{2}\alpha(X)) = -dB$$

with $B := P + \frac{1}{2}||X||^2$ the Bernoulli function.

Back to proof of Main Theorem 1!

$\Rightarrow$ Let us prove that if X is a Beltrami field it is a reparametrization of the Reeb vector field by the function $\alpha(X) = g(X, X)$

- By dualizing the Beltrami equation with the volume form we have $\mathrm{d}\iota_X g = \iota_{\operatorname{curl} X}\mu = f\iota_X\mu \leftrightsquigarrow$

$$d\alpha = f\iota_X\mu\,.$$

  Since $f > 0$ and X does not vanish $\rightsquigarrow \alpha \wedge d\alpha = f\alpha \wedge \iota_X\mu > 0$.
  Let us check this last step: Pick a frame $\{e_1, e_2, e_3\}$. Since X is nonsingular we may set $e_1 = X/\|X\|$. Let us denote by $\{e^1, e^2, e^3\}$ the corresponding dual frame; observe that $\alpha = \iota_X g = \|X\|e^1$. Since μ is a volume form $\mu = he^1 \wedge e^2 \wedge e^3$ on each chart of M, where h is some positive function, $h > 0$. Note that $\alpha \wedge d\alpha = (\iota_X g) \wedge (f\iota_X\mu) = fh\|X\|^2 e^1 \wedge e^2 \wedge e^3 \neq 0$ because$f, h, \|X\| > 0$, so α is a contact form on M.
- X satisfies

$$\iota_X(d\alpha) = f\iota_X\iota_X\mu = 0$$

  so $X \in \ker d\alpha \leftrightsquigarrow$ it is a reparametrization of the Reeb vector field by the function $\alpha(X) = g(X, X)$.
  so defining $\overline{X} = X/\alpha(X)$ completes the first part of the proof.

The other implication:

Let us prove that if X Reeb vector field then any re-scaling $Y = hX$ with h a positive function is a Beltrami field for a certain pair of metric and volume (g, μ).

Let (α, X) be the contact form and Reeb field on the 3-manifold M.

- Choose a basis $\{e_i\}_{i=1}^3$ on charts on M such that $e_i = X$ and

$$\xi = \ker\alpha = \mathrm{Span}\{e_2, e_3\}.$$

- Since α is a contact form, $d\alpha$ defines a symplectic structure on ξ. Choose $\{e_2, e_3\}$ as symplectic basis so that $d\alpha = e_2 \wedge e_3$, so $d\alpha(e_2, e_3) = 1$.
- We note by $\{e^i\}$ the dual forms, $\alpha(e_1) = \alpha(X) = 1 = e^1(e_1)$ and $\alpha(e_2) = \alpha(e_3) = 0 = e^1(e_2) = e^1(e_3), \alpha$ and e^1 act identically on a basis $\rightsquigarrow \alpha = e^1$.

$$de^1(e_i, e_j) = d\alpha(e_i, e_j) = \delta_{2i}\delta_{3j} - \delta_{2j}\delta_{3i} = e_2 \wedge e_3(e_i, e_j)$$

so

$$d\alpha = de^1 = e^2 \wedge e^3.$$

The other implication:

Now, define the following metric

$$g = \frac{1}{h} e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3$$

Set the volume form to be

$$\mu = \frac{1}{h} e^1 \wedge e^2 \wedge e^3.$$

Let us check that Y is volume preserving and parallel to its curl for the metric g and the volume μ.

- $L_Y \mu = d\iota_Y \mu = d(\iota_{he_1} \frac{1}{h} e^1 \wedge e^2 \wedge e^3) = d(e^2 \wedge e^3) = d^2 e^1 = 0$
- $\iota_{\operatorname{curl} Y} \mu = d\iota_Y g$. By construction, we have $\iota_Y g = \alpha = e^1$, so on one side we have

$$d\iota_Y g = de^1 = e^2 \wedge e^3$$

and on the other side, $\iota_Y \mu = e^2 \wedge e^3$. So $\iota_Y \mu = d\iota_Y g$, which means that $\operatorname{curl} Y = Y$.

An alternative proof that works for the Reeb-Beltrami correspondence in high dimensions

For higher dimensions (or singular structures and equivariant versions of Main Theorem 1) we can use an almost-complex structure to define the associated metric.

- Take an almost-complex structure J on $\ker \alpha = \xi$ adapted to $d\alpha$, i.e. $d\alpha(\cdot, J\cdot)$ is a positive definite quadratic form on ξ. Define the metric

$$g = \frac{1}{h}\alpha \otimes \alpha + \tilde{h} d\alpha(\cdot, J\cdot)$$

- It then follows that $\iota_Y g = \alpha$.
- For this more general proof, it is clear that the function $\tilde{h}$ can be chosen so that the Riemannian volume form is $\mu = \frac{1}{h}\alpha \wedge (d\alpha)^m$.
- Thus $\iota_Y \mu = (d\alpha)^m$ and $d\iota_Y \mu = 0$.

Therefore, Y is a Beltrami field (with factor $f = 1$) with respect to the metric g.

An equivariant mirror

This proof indeed can be adapted equivariantly to prove:

Theorem (Equivariant mirror, Fontana-M.-Peralta-Salas)

Let M be a $2n-1$-dimensional manifold and $\rho : G \times M \to M$ a compact Lie group action on M. For each non-singular ρ-invariant Beltrami field (X, g) there is a ρ-invariant contact form for which X is Reeb. Conversely, for each ρ-invariant Reeb field (X, α) there is a ρ-invariant Riemannian metric for which X is a non-singular Beltrami field.

A magic mirror

- Weinstein conjecture for Reeb vector fields $\rightsquigarrow$ **periodic orbits for Beltrami vector fields**
- h-principle $\rightsquigarrow$ Reeb embeddings $\rightsquigarrow$ **universality of Euler flows** (Cardona–M–Peralta-Salas–Presas)
- Reeb suspension of area preserving diffeomorphism of the disc $\rightsquigarrow$ **Construction of universal 3D Turing machine** (Cardona–M–Peralta-Salas–Presas)
- Uhlenbeck's genericity properties of eigenfunctions of Laplacian $\rightsquigarrow$ **existence of escape trajectories** (M–Oms–Peralta-Salas)

This mirror is equivariant and this can be useful to understand escape orbits better (work in progress with Fontana and Peralta-Salas).

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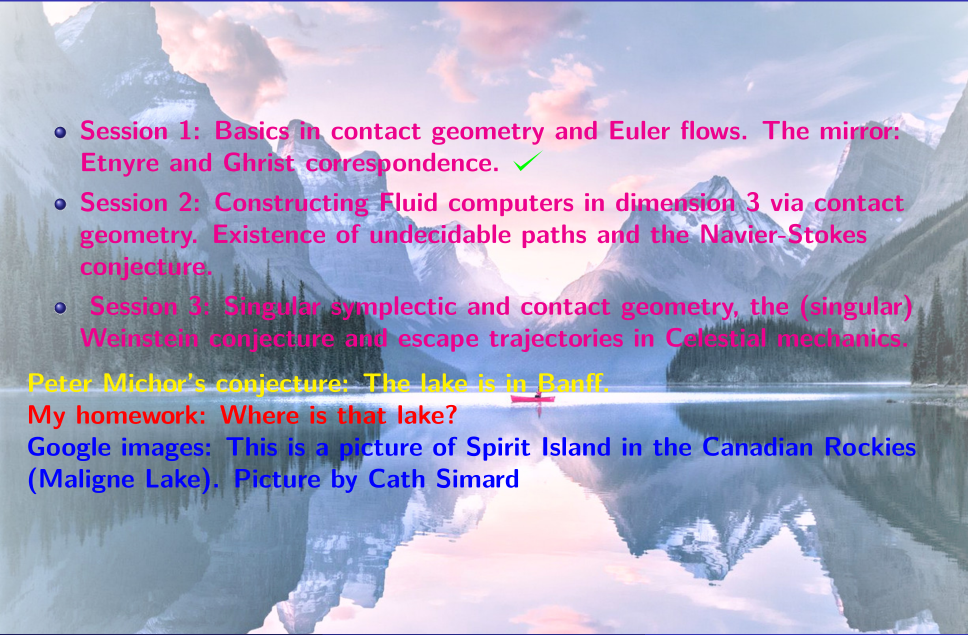

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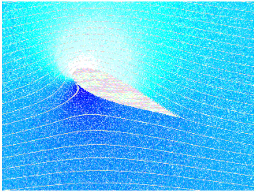
Peter Michor's conjecture: The lake is in Banff.
My homework: Where is that lake?
Google images: This is a picture of Spirit Island in the Canadian Rockies (Maligne Lake). Picture by Cath Simard

Incompressible fluids on Riemannian manifolds



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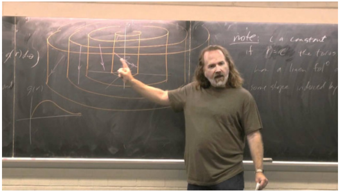

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A magic mirror

- **Weinstein conjecture for Reeb vector fields** $\rightsquigarrow$ **periodic orbits for Beltrami vector fields**
- **h-principle** $\rightsquigarrow$ Reeb embeddings $\rightsquigarrow$ **universality of Euler flows** (Cardona–M–Peralta-Salas–Presas)
- **Reeb suspension of area preserving diffeomorphism of the disc** $\rightsquigarrow$ **Construction of universal 3D Turing machine** (Cardona–M–Peralta-Salas–Presas)
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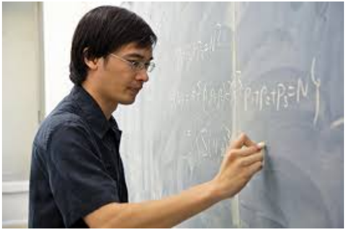
Weinstein conjecture

Weinstein's conjecture

The Reeb vector field of a contact compact manifold admits at least one periodic orbit.

- Proved in dimension 3 by Taubes.
- Every Reeb vector field has at least two periodic orbits (Cristofaro-Gardiner and Hutchings 2016).
- Examples in celestial mechanics show that Reeb vector fields generically tend to have infinite periodic orbits.
- Every nondegenerate Reeb vector field has either two or infinitely many periodic orbits (Colin, Dehornoy, Rechtman).

Tao's program for universality of Euler flows



Can any incompressible vector field can be embedded as an Euler flow by increasing the dimension of the manifold?

Tao's program and blow-up

Is it possible to construct Turing complete fluid flows? Motivation: to create an initial datum that is "programmed" to evolve to a rescaled version of itself (as a Von Neumann self-replicating machine). Can this be applied to prove blow-up of Navier-Stokes?

Can you spot 7 differences?

Figure: Barcelona and Srní in January

Maybe yes!.... but if you are far enough you will not find them....

All dynamics represented by an Euler flow

Geometrical approach to Tao's question

(Tao) Can any incompressible vector field be embedded as an Euler flow in higher dimensions?

When the Euler vector field is Beltrami we can use our magic mirror

Beltrami vector field  **Reeb vector field**

(CMPP) Can we realize a vector field on a manifold N as a Reeb vector field on a bigger compact contact manifold?

- **Necessary condition**: X geodesible in N $\iff$ preserves transverse hyperplane distribution.

  A geodesible vector field is a vector field for which there is a Riemannian metric g on M such that the orbits of X are geodesics of unit length. $\leftrightsquigarrow$ there exists a 1-form α such that $\alpha(X) = 1$ and $\iota_X d\alpha = 0$.
- **Question 1**: Is this condition sufficient?
- **Question 2**: How hard is to get a geodesible vector field?

Let us answer question 2.

Lemma

The suspension of a t-periodic vector field $X(p,t)$ is geodesible.

Consider $N \times S^1$, the vector field

$$Y(p,\theta) = (X(p,\theta), \frac{\partial}{\partial\theta}).$$

is geodesible (Take $\alpha = d\theta$. It satisfies $\iota_Y \alpha = 1$ and $\iota_Y d\alpha = 0$.)

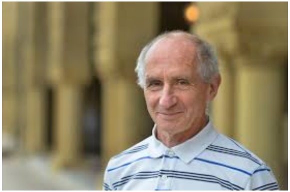
Variation in tactics: Flexibility

Inspirational: All 3-dimensional manifolds are contact (Martinet-Lutz) and in higher dimensions:

Theorem (Borman-Eliashberg-Murphy)

Any almost contact closed manifold is contact.

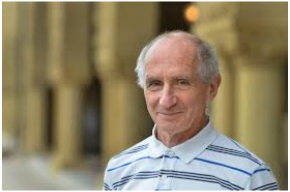
The almost contact condition is a formal condition and h-principle is the key ingredient of the proof.

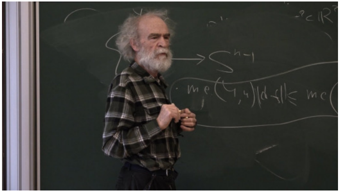


The h-principle

The philosophy of the h-principle:

- Goal: Solve an equation (PDE, partial differential relation...).
- Semigoal: Solve a linearized/formal equation.
- Miracle: Prove that there exist an homotopy that allows to deform a formal solution to a honest solution (one needs to be lucky for that: the system is undetermined or has high codimension).





Main Theorem 2: Universality

Theorem 2 (Cardona, M., Peralta-Salas & Presas)

The Euler flows are universal. The dimension of the ambient manifold $\mathbb{S}^n$ or $\mathbb{R}^n$ is the smallest odd integer $n \in \{3 \text{ dim } N + 5, 3 \text{ dim } N + 6\}$.
In the time-periodic case, the extended field u is a steady Euler flow with a metric $g = g_0 + g_P$, where g_0 is the canonical metric on $\mathbb{S}^n$ and g_P is supported in a ball that contains the invariant submanifold $e(N \times \mathbb{S}^1)$.

Idea of the construction:

By using the suspension's trick and the magic mirror the problem can be translated as universality of Reeb embeddings. This can be proved using the h-principle.

Key steps in the proof

- **Step 1:** Using the correspondence between Beltrami flows and Reeb vector fields we reduce the problem to studying the universality of high-dimensional Reeb flows.

- **Step 2:** The Reeb flows are geodesible with respect to the (adapted) metric that makes them Beltrami $\Rightarrow$ a Reeb flow restricted to any invariant submanifold is geodesible with respect to the induced metric.

Key steps in the proof II

Step 3: The converse also holds: Any geodesible flow is Reeb-extendible.

Theorem (Cardona-M.-PS-Presas)

Let N be a compact manifold and X a geodesible field. Then there is a smooth embedding $e : N \to \mathbb{S}^n$ with $n = 4\,\text{dim}\ N - 1$ and a 1-form α defining the standard contact structure ξ_{std} on $\mathbb{S}^n$ such that $e(N)$ is an invariant submanifold of the Reeb field R and $e_*X = R$. Moreover α equals α_{std} in the complement of a ball that contains $e(N)$.

Step 4: Sharpening the dimension. Using more sophisticated techniques from contact topology we prove an h-principle for Reeb embeddings and deduce:

Theorem (Reeb embeddings)

Let $e : (N, X) \to (M, \xi)$ be a embedding of N into a contact manifold (M, ξ) with X a geodesible vector field on N. Then:

- *If $\dim M \geqslant 3 \dim N + 2$, then e is isotopic to a Reeb embedding $\tilde{e}$, and $\tilde{e}$ can be taken C^0-close to e.*
- *If $\dim M \geqslant 3 \dim N$ and M is overtwisted, then e is isotopic to a Reeb embedding.*

Final step

Consider the suspension of a time-periodic v.f. on $M \times S^1$

$$Y(p,\theta) = (X(p,\theta), \frac{\partial}{\partial\theta}).$$

It is a geodesible vector field (Take $\alpha = d\theta$. It satisfies $\iota_X \alpha = 1$ and $\iota_X d\alpha = 0$.)

Proof.

- By the previous theorem applied to an $n+1$-dimensional manifold, there exist a Reeb embedding $e : N \to \mathbb{S}^n$ extending Y with n the smallest odd integer $n \in \{3\,\text{dim } N + 5, 3\,\text{dim } N + 6\}$.
- By the contact-Beltrami correspondence Theorem, there are a metric and a volume making Y a steady solution of the Euler equations.

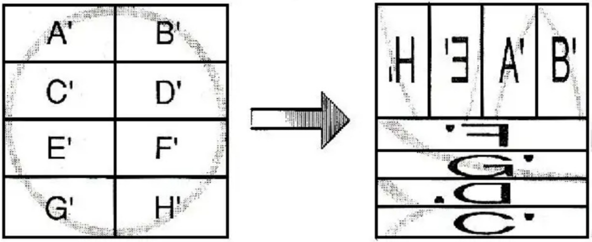
$\square$

Moore, a new form of chaos

Science: Mathematician discovers a more complex form of chaos

30 June 1990

By William Bown



Chaotic transformation: by repeatedly dividing a square into eight segments and transforming each segment separately, a scrambled mess is created which is utterly unpredictable

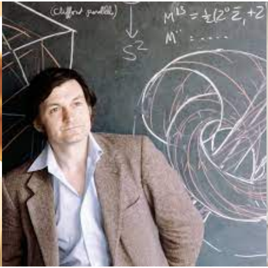
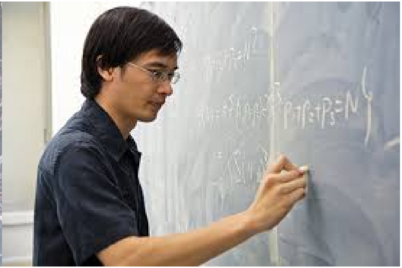
Moore generalized the notion of shift in dynamical systems and was able to simulate any Turing machine (generalized shifts). These are conjugated to maps of the square cantor set $C \times C$.

Computational complexity and Fluid Dynamics

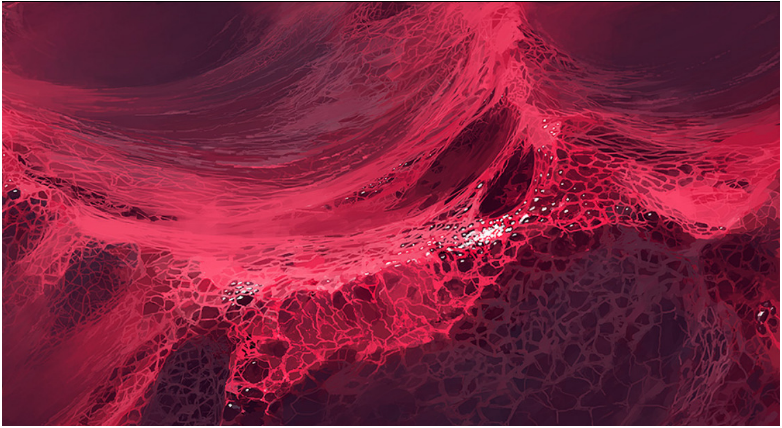
In Nature fluids (seas or volcano's lava) often rebel against what is expected....

Fluid Computers?

Are fluids *"complicated"* enough to create a Fluid Computer?

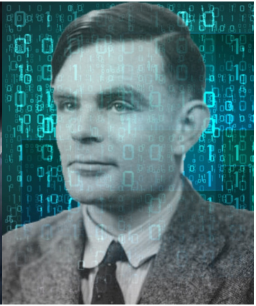
Reality or science fiction?



”For some time there was a widely held notion (zealously fostered by the daily press) to the effect that the thinking ocean of Solaris was a gigantic brain, prodigiously well-developed and several million years in advance of our own civilization, a sort of cosmic yogi, a sage, a symbol of omniscience..”
`Stanislaw Lem, Solaris.`

Levels of complexity and Alan Turing

Is the complexity of fluids enough to simulate any Turing machine?

The Millennium problem list

CMI

ABOUT PROGRAMS PEOPLE MILLENNIUM PROBLEMS PUBLICATIONS EVENTS NEWS

Millennium Problems

Yang–Mills and Mass Gap

Experiment and computer simulations suggest the existence of a "mass gap" in the solution to the quantum versions of the Yang-Mills equations. But no proof of this property is known.

Riemann Hypothesis

The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2.

P vs NP Problem

If it is easy to check that a solution to a problem is correct, is it also easy to solve the problem? This is the essence of the P vs NP question. Typical of the NP problems is that of the Hamiltonian Path Problem: given N cities to visit, how can one do this without visiting a city twice? If you give me a solution, I can easily check that it is correct. But I cannot so easily find a solution.

Navier–Stokes Equation

This is the equation which governs the flow of fluids such as water and air. However, there is no proof for the most basic questions one can ask: do solutions exist, and are they unique? Why ask for a proof? Because a proof gives not only certitude, but also understanding.

Hodge Conjecture

The answer to this conjecture determines how much of the topology of the solution set of a system of algebraic equations can be defined in terms of further algebraic equations. The Hodge conjecture is known in certain special cases, e.g., when the solution set has dimension less than four. But in dimension four it is unknown.

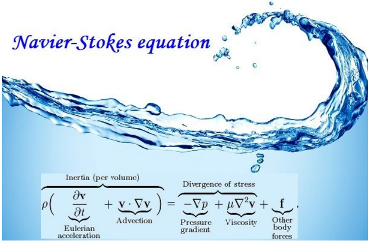
Poincaré Conjecture

In 1904 the French mathematician Henri Poincaré asked if the three dimensional sphere is characterized as the unique simply connected three manifold. This question, the Poincaré conjecture, was a special case of Thurston's geometrization conjecture. Perelman's proof tells us that every three manifold is built from a set of standard pieces, each with one of eight well-understood geometries.

Birch and Swinnerton-Dyer Conjecture

Supported by much experimental evidence, this conjecture relates the number of points on an elliptic curve mod p to the rank of the group of rational points. Elliptic curves, defined by cubic equations in two variables, are fundamental mathematical objects that arise in many areas: Wiles' proof of the Fermat Conjecture, factorization of numbers into primes, and cryptography, to name three.

The Navier-Stokes problem: Existence of global smooth solutions



The *Navier–Stokes* equations are then given by

$$(1)\qquad \frac{\partial}{\partial t}u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x,t) \qquad (x \in \mathbb{R}^n, t \geq 0),$$

$$(2)\qquad \operatorname{div} u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0 \qquad (x \in \mathbb{R}^n, t \geq 0)$$

with initial conditions

$$(3)\qquad u(x,0) = u^\circ(x) \qquad (x \in \mathbb{R}^n).$$

Here, $u^\circ(x)$ is a given, C^∞ divergence-free vector field on $\mathbb{R}^n$, $f_i(x,t)$ are the components of a given, externally applied force (e.g. gravity), ν is a positive coefficient (the viscosity), and $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian in the space variables. The *Euler equations* are equations (1), (2), (3) with ν set equal to zero.

$$(4)\qquad |\partial_x^\alpha u^\circ(x)| \leq C_{\alpha K}(1+|x|)^{-K} \quad \text{on } \mathbb{R}^n, \text{ for any } \alpha \text{ and } K$$

and

$$(5)\qquad |\partial_x^\alpha \partial_t^m f(x,t)| \leq C_{\alpha m K}(1+|x|+t)^{-K} \quad \text{on } \mathbb{R}^n \times [0,\infty), \text{ for any } \alpha, m, K$$

We accept a solution of (1), (2), (3) as physically reasonable only if it satisfies

$$(6)\qquad p, u \in C^\infty(\mathbb{R}^n \times [0,\infty))$$

and

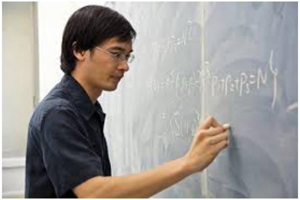
$$(7)\qquad \int_{\mathbb{R}^n} |u(x,t)|^2 dx < C \quad \text{for all } t \geq 0 \quad \text{(bounded energy).}$$

The Navier-Stokes problem

(A) Existence and smoothness of Navier–Stokes solutions on $\mathbb{R}^3$. Take $\nu > 0$ and $n = 3$. Let $u^\circ(x)$ be any smooth, divergence-free vector field satisfying (4). Take $f(x,t)$ to be identically zero. Then there exist smooth functions $p(x,t), u_i(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ that satisfy (1), (2), (3), (6), (7).

(C) Breakdown of Navier–Stokes solutions on $\mathbb{R}^3$. Take $\nu > 0$ and $n = 3$. Then there exist a smooth, divergence-free vector field $u^\circ(x)$ on $\mathbb{R}^3$ and a smooth $f(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$, satisfying (4), (5), for which there exist no solutions (p,u) of (1), (2), (3), (6), (7) on $\mathbb{R}^3 \times [0,\infty)$.

Tao and Turing Machines




One could hope to design logic gates entirely out of ideal fluid. If these gates were sufficiently "Turing complete", and also "noise-tolerant" one could then hope to combine enough of these gates together to "program" a self-replicating von Neumann machine.

Tao, JAMS, 2016

Tao's programme and the Navier-Stokes conjecture

Is it possible to construct Turing complete fluid flows? Motivation: to create an initial datum that is "programmed" to evolve to a rescaled version of itself (as a Von Neumann self-replicating machine). Can this be applied to prove blow-up of Navier-Stokes?

Turing machines and the halting problem

In computability theory, the halting problem is the problem of determining, from a description of an arbitrary computer program and an input, whether the program will finish running (halting state), or continue to run forever.

Turing, 1936: The halting problem is undecidable.

Alan Turing proved in 1936 that a general algorithm to solve the halting problem for all possible program-input pairs cannot exist.

Mathematical framework: an interdisciplinary problem

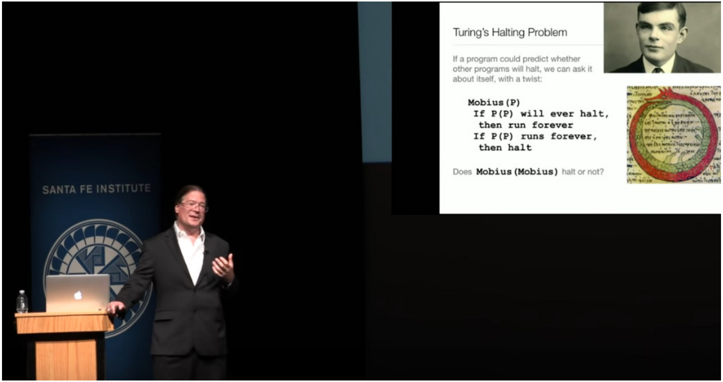
- Hydrodynamics $\rightsquigarrow$ **Euler or Navier-Stokes equations (PDEs).**
- Computation: $\rightsquigarrow$ **Turing machines (Computer Science).**


A explosive combo

- **If a universal fluid computer exists $\rightsquigarrow$ existence of undecidable fluid particle paths, i.e., no general algorithm exists to decide whether the trajectories of the flow starting at certain points will reach a certain (explicit) open set.**
- **Computational complexity in hydrodynamics: very different from chaos theory complexity.**

Our view point

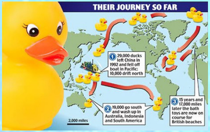
Can we construct an Euler flow from Moore's construction?

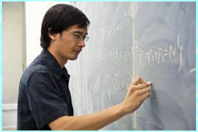


Logical chaos from 2D to 3D

The idea is to promote Moore's construction to a 3D construction as a Poincaré section of a "physical" system. Such a vector field has properties of a "Reeb vector field". What's the connection to Euler's equations?

Chronology



- 1991, Moore: Is hydrodynamics capable of performing computations?
- January 10, 1992: 29000 rubber ducks were lost in the ocean.
- July 2007: One rubber duck show ups in Scotland.
- July 2017: Tao asks about universality of Euler flows.
- November 2019: (Cardona--M.--Peralta-Salas --Presas) Steady Euler flows are universal.
- December 2020: (Cardona--M.--Peralta-Salas-- Presas) There exist stationary Turing complete Euler flows in dimension 3.
- April 2021: (Cardona--M.--Peralta-Salas)There exist time-dependent Euler flows which are Turing complete in high dimension.
- November 2021:(Cardona--M.--Peralta-Salas) Euclidean case.

Turing complete Euler flows

Turing completeness

A vector field on M is Turing complete if it can simulate any Turing machine $\leftrightsquigarrow$ The halting of any Turing machine with a given input is equivalent to a certain bounded trajectory of the field entering a certain open set of M (precise definition later).

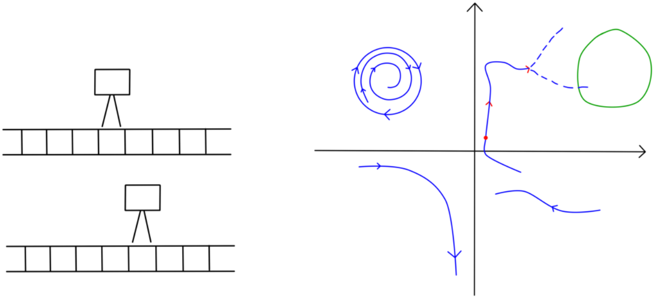
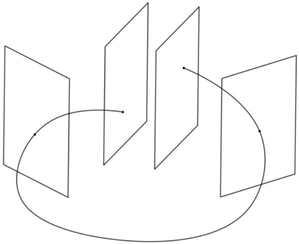


Figure: Turing machine and Turing complete vector field associated to a point and an open set.

Turing completeness of the Euler flows

Theorem 1 (Cardona, M., Peralta-Salas, Presas)

There exists a Turing complete Eulerisable flow on $\mathbb{S}^{17}$. This flow is Beltrami with constant proportionality factor.



Key ingredients of the proof: There exists an orientation-preserving diffeomorphism ϕ of $\mathbb{T}^4$ encoding a universal Turing machine (Tao). The use of an h-principle gives the construction. As it is given by holonomic approximation, it is algorithmic.

A fluid computer in dimension 3

Theorem 2 (Cardona, M., Peralta-Salas & Presas)

There exists an Eulerisable flow X in $\mathbb{S}^3$ that is Turing complete. The metric g that makes X a stationary solution of the Euler equations can be assumed to be the round metric in the complement of an embedded solid torus.

Turing, 1936: The halting problem is **undecidable**.

Corollary

There exist undecidable fluid particle paths: there is no algorithm to decide whether a trajectory will enter an open set or not in finite time.

Does this give finite-time blow-up for Navier-Stokes?

Short answer: No

Long answer: On a Riemannian 3-manifold (M,g) the Navier-Stokes read as

$$\begin{cases} \frac{\partial u}{\partial t} + \nabla_u u - \nu\Delta u = -\nabla p\,, \\ \operatorname{div} u = 0\,, \\ u(t=0) = u_0\,, \end{cases} \tag{1}$$

where $\nu > 0$ is the viscosity.

- Δ is the Hodge Laplacian (whose action on a vector field is defined as $\Delta u := (\Delta u^\flat)^\sharp$).
- The vector field X is of Beltrami type (with constant factor 1). When considered as an initial datum of NS, we obtain:

$$X(t) = X\,e^{-\nu t}$$

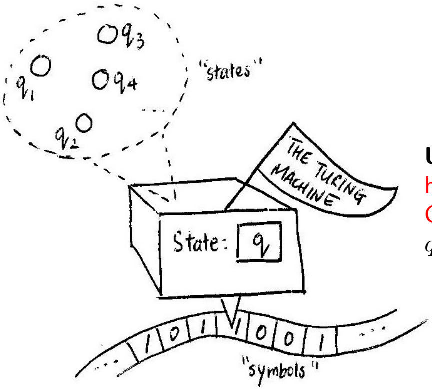
$\Longrightarrow$ it exists for all time.

- The exponential decay implies that it can simulate just a finite number of steps of any Turing machine.

Turing machines

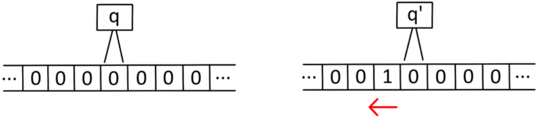
Turing machine

A Turing machine is defined as $T = (Q, q_0, q_{halt}, \Sigma, \delta)$, where Q is a finite set of states, including an initial state q_0 and a halting state q_{halt}, Σ is the alphabet, and $\delta : (Q \times \Sigma) \longrightarrow (Q \times \Sigma \times \{-1, 0, 1\})$ is the transition function. The input of a Turing machine is the current state $q \in Q$ and the current tape $t = (t_n)_{n \in \mathbb{Z}} \in \Sigma^{\mathbb{Z}}$.

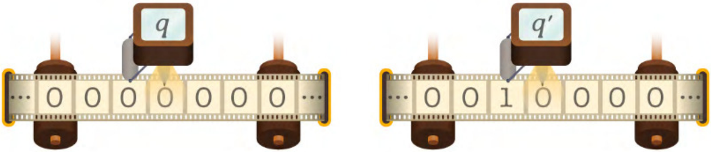


User's guide: If the current state is q_{halt} then halt the algorithm and return t as output. Otherwise compute $\delta(q, t_0) = (q', t_0', \varepsilon)$, replace q with q', t_0 with t_0' and t by the ε-shifted tape.

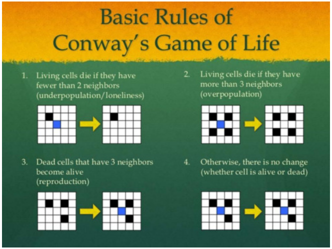
Example



Example: $\delta(q, 0) = (q', 1, +1)$, we replace 0 by 1, the new state is q' and we shift the tape to the left.



Conway's game of life



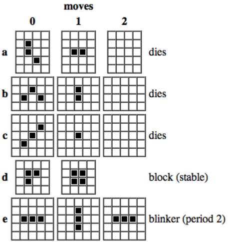
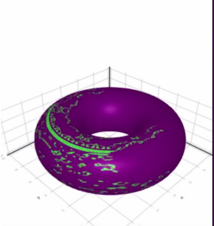
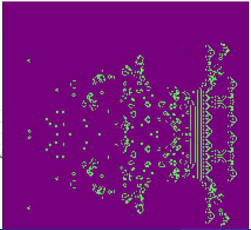


Figure: John von Neumann: every Turing machine has a cellular automaton which simulates it.

Conway's game of life on a torus:





A simple dynamical system: The cantor set

Dynamical systems simulating Turing machines

A dynamical system (vector field, map, diffeomorphism) on X is Turing complete if

for any integer $k \geqslant 0$, given a Turing machine T, an input tape t, and a finite string $(t^*_{-k}, ..., t^*_k)$ of symbols of the alphabet, there exist an explicitly constructible point $p \in X$ and an open set $U \subset X$ such that the orbit of the system through p intersects U if and only if T halts with an output tape whose positions $-k, ..., k$ correspond to the symbols $t^*_{-k}, ..., t^*_k$.

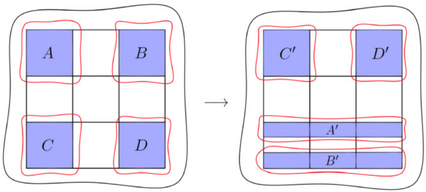
- Moore generalized the notion of shift to simulate any Turing machine (Generalized shifts).
- Generalized shifts are conjugated to maps of the square Cantor set $C^2 \subset I^2$. **Point assignment**: take $A = \{0, 1\}$. Given $s = (...s_{-1}.s_0 s_1...) \in A^{\mathbb{Z}}$, we can associate to it an explicitly constructible point in the square Cantor set. Express the coordinates of the assigned point in base 3: the coordinate y corresponds to the expansion $(y_0, y_1, ...)$ where $y_i = 0$ if $s_i = 0$ and $y_i = 2$ if $s_i = 1$. Analogously, the coordinate x corresponds to the expansion $(x_1, x_2, ...)$ in base 3 where $x_i = 0$ if $s_{-i} = 0$ and $x_i = 2$ if $s_{-i} = 1$.

A Turing complete diffeomorphism of the disk

Proposition (Cardona, Miranda, Peralta-Salas & Presas)

For each bijective generalized shift and its associated map of the square Cantor set ϕ, there exists an area-preserving diffeomorphism of the disk $\varphi : D \to D$ which is the identity in a neighborhood of ∂D and whose restriction to the square Cantor set is conjugated to ϕ.

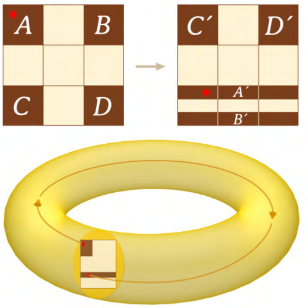
Idea of the proof: Extend a construction by Moore which is a piecewise linear map using disjoint blocks containing all the Cantor set:



The magic mirror strikes again...

Theorem (Cardona, M., Peralta-Salas & Presas)

This (Turing complete) diffeomorphism of the disk can be realized as the time-one-map of a Reeb vector field.



The contact form α can be used on the complementary set of a dense torus. This together with the magic mirror proves Theorem 2.

From fluid computers to escape trajectories: Two sides of a mirror

43 Winter School on Geometry and Physics, Srní

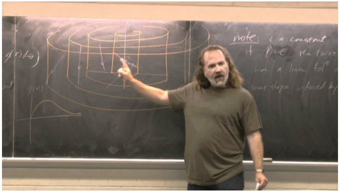
Eva Miranda

UPC-CRM

The magic mirror

In terms of $\alpha = \iota_X g$ and μ (volume form) the **stationary Euler equations** read

$$\begin{cases} \iota_X d\alpha = -dB \\ d\iota_X \mu = 0 \end{cases}$$

- Etnyre-Ghrist:
  {Rotational non singular Beltrami v.f.} $\rightleftarrows$ {Reeb v.f. reparametrized}

A magic mirror

- **Weinstein conjecture for Reeb vector fields** ⇝ **periodic orbits for Beltrami vector fields**
- **h-principle** ⇝ Reeb embeddings ⇝ **universality of Euler flows** (Cardona–M–Peralta-Salas–Presas)
- **Reeb suspension of area preserving diffeomorphism of the disc** ⇝ **Construction of universal 3D Turing machine** (Cardona–M–Peralta-Salas–Presas)
- Uhlenbeck's genericity properties of eigenfunctions of Laplacian ⇝ existence of escape trajectories (M–Oms–Peralta-Salas)

A fluid computer in dimension 3

Theorem 2 (Cardona, M., Peralta-Salas & Presas)

There exists an Eulerisable flow X in $\mathbb{S}^3$ that is Turing complete. The metric g that makes X a stationary solution of the Euler equations can be assumed to be the round metric in the complement of an embedded solid torus.

Theorem (Cardona, M., Peralta-Salas & Presas)

A (Turing complete) diffeomorphism of the disk which is the identity close to the boundary can be realized as the time-one-map of a Reeb vector field.

My homework

Stefan Nemirovski: Can you get a tight contact structure on S^3?

Yes!

Indeed we proved that

Theorem

Let (M, ξ) be a contact 3-manifold and $\varphi : D \to D$ an area-preserving diffeomorphism of the disk which is the identity (in a neighborhood of) the boundary. Then there exists a defining contact form α whose associated Reeb vector field R exhibits a Poincaré section with first return map conjugated to φ.

Key point: We build a contact form α such the contact structure defined by $\ker \alpha$ is homotopic through contact structures to ξ.

We can choose the defining contact form β in the complement of the toroidal set U.

More precisely, given any contact form β defining the contact structure ξ, there is another defining contact form α such that $\alpha = \widetilde{\alpha}'$ on U and $\alpha = \beta$ in the complement of a neighborhood of U.

Celestial mechanics and Fluid Dynamics

Arnold's dream of establishing a connection between the dynamical complexity of celestial mechanics and of stationary solutions of hydrodynamics: *"Car les écoulements avec* $\operatorname{curl} v = \lambda v$ *admettent, probablement, des lignes de courant avec une topologie aussi compliquée que celle des orbites en mécanique céleste."*

The Kepler problem

- Motion of a planet orbiting a star.
- Write $q = (q^1, q^2) \subset \mathbb{R}^2$ for the position coordinates of the planet and $(q, p) = (q^1, q^2, p^1, p^2)$ for the corresponding natural momentum coordinates on $T^*\mathbb{R}^2$. The Hamiltonian is

$$H : T^*(\mathbb{R}^2 \setminus \{0\}) \to \mathbb{R}, \quad (q, p) \mapsto \frac{|p|^2}{2} - \frac{1}{|q|}.$$

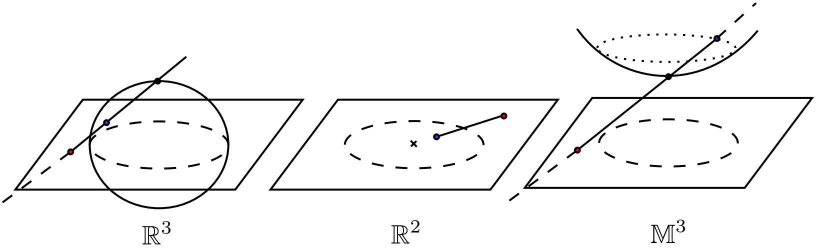
A contact structure appears when we restrict to an arbitrary energy level $H = c$.

The geometry of the Kepler problem after regularization

Theorem (Moser-Osipov-Belbruno)

The dynamics of the Kepler problem on the energy level $H = c$ are equivalent to the cogeodesic flow on

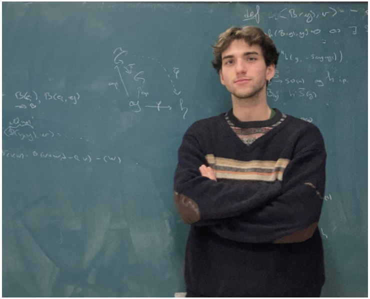
- $S^*\mathbb{S}^2$ *for* $c < 0$,
- $S^*\mathbb{R}^2$ *for* $c = 0$ *and*
- $S^*\mathbb{H}^2$ *for* $c > 0$



Reeb and Beltrami fields on spherical cotangent bundles

Spherical cotangent bundles from two perspectives

Let (M, g) be a Riemannian manifold and S^*M its spherical cotangent bundle. The Reeb field of the Liouville form restricted to S^*M is Beltrami with respect to the canonical cotangent lift of g restricted to S^*M.



The Kepler problem as an Euler flow

Theorem (Fontana-M.-Peralta)

The regularized Kepler flow on the c-energy level is a stationary Beltrami solution to the Euler equations on

- $S^*\mathbb{S}^2$ *if* $c < 0$,
- $S^*\mathbb{R}^2$ *if* $c = 0$ *and*
- $S^*\mathbb{H}^2$ *if* $c > 0$

with the lifts to the spherical cotangent bundles of the respective constant curvature metrics. The flow lines are lifted geodesics. The Kepler flow on the plane is recovered from the natural stereographic projections of the respective surfaces, or from the involution $p \mapsto \frac{p}{|p|^2}$ *when* $c = 0$.

Restricted planar circular 3-body problem

- Time-dependent potential: $U(q,t) = \frac{1-\mu}{|q-q_E(t)|} + \frac{\mu}{|q-q_M(t)|}$
- Time-dependent Hamiltonian:
  $H(q,p,t) = \frac{|p|^2}{2} - U(q,t), \quad (q,p) \in^2 \setminus \{q_E, q_M\} \times \mathbb{R}^2$
- Rotating coordinates $\rightsquigarrow$ Time independent Hamiltonian
  $H(q,p) = \frac{p^2}{2} - \frac{1-\mu}{|q-q_E|} + \frac{\mu}{|q-q_M|} + p_1 q_2 - p_2 q_1$

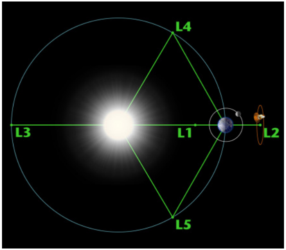


Figure: Lagrange points (Source: NASA/WMAP Science Team)

Revisiting the Planar restricted 3-body problem

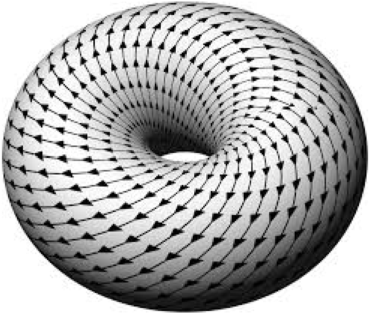
- Consider the canonical change $(X, Y, P_X, P_Y) \mapsto (r, \alpha, P_r =: y, P_\alpha =: G)$.
- Introduce **McGehee coordinates** (x, α, y, G), where $r = \frac{2}{x^2}, \quad x \in \mathbb{R}^+$, can be then extended to infinity $(x = 0)$.
- The symplectic structure becomes a singular object

$$-\frac{4}{x^3} dx \wedge dy + d\alpha \wedge dG.$$

which extends to a b^3-symplectic structure on $\mathbb{R} \times \mathbb{T} \times \mathbb{R}^2$.

- A vector field v is a b-**vector field** if $v_p \in T_pZ$ for all $p \in Z$. The b-**tangent bundle** ${}^bTM$ is defined by

$$\Gamma(U, {}^bTM) = \left\{ \begin{array}{c} \text{b-vector fields} \\ \text{on } (U, U \cap Z) \end{array} \right\}$$



b-forms

- The **b-cotangent bundle** ${}^bT^*M$ is $({}^bTM)^*$. Sections of $\Lambda^p({}^bT^*M)$ are b-**forms**, ${}^b\Omega^p(M)$.The standard differential extends to

$$d : {}^b\Omega^p(M) \to {}^b\Omega^{p+1}(M)$$

- This defines the b-cohomology groups. Mazzeo-Melrose

$${}^bH^*(M) \cong H^*(M) \oplus H^{*-1}(Z).$$

- A **b-symplectic form** is a closed, nondegenerate, b-form of degree 2.
- We can introduce **b-contact structures on a manifold** M^{2n+1} as b-forms of degree 1 for which $\alpha \wedge (d\alpha)^n \neq 0$.
- The b-cotangent bundle can be replaced by other algebroids (E-symplectic) known to Nest and Tsygan.

Other examples this week...

- Edge structure: $(M, \partial M)$ such that ∂M is a fiber bundle over Y. Edge vector fields are tangent to these fibers at points on ∂M.
- Edge tangent bundle ${}^{e}TM$ is a bundle whose sections are the edge vector fields.

Attacking the b^m-Weinstein's conjecture

Theorem (M.-Oms)

Let (M, α) be a 3-dimensional b^m-contact manifold and assume the critical hypersurface Z to be closed. Then there exists infinitely many periodic Reeb orbits on Z.

Proof.

1. $\alpha = u\frac{dz}{z^m} + \beta$
2. The restriction on Z of the 2-form $\Theta = ud\beta + \beta \wedge du$ is symplectic and the Reeb vector field is Hamiltonian.
3. u is non-constant on Z.
4. R_α is Hamiltonian on Z for $-u$,
5. $u^{-1}(p)$ where p regular is a circle,
6. R_α periodic on $u^{-1}(p)$.

□

Contact geometry of RPC3BP revisited

In rotating coordinates: $H(q,p)=\frac{|p|^2}{2}-\frac{1-\mu}{|q-q_E|}+\frac{\mu}{|q-q_M|}+p_1q_2-p_2q_1$

- Symplectic polar coordinates: (r,α,P_r,P_α).
- McGehee change of coordinates: $r=\frac{2}{x^2}$.

b^3-symplectic form: $-4\frac{dx}{x^3}\wedge dP_r+d\alpha\wedge dP_\alpha$.

Lemma

The vector field $Y=p\frac{\partial}{\partial p}$ is a Liouville vector field and is transverse to Σ_c for $c>0$.

Singular contact

Is Σ_c b^3-contact after McGehee? Can we apply the results on periodic orbits?

b^3-contact form in the RPC3BP

Theorem

[M.-Oms] After the McGehee change, the Liouville vector field $Y = p\frac{\partial}{\partial p}$ is a b^3-vector field that is everywhere transverse to Σ_c for $c > 0$ and the level-sets $(\Sigma_c, \iota_Y\omega)$ for $c > 0$ are b^3-contact manifolds.

1. *The critical set is a cylinder.*
2. *The Reeb vector field admits infinitely many non-trivial periodic orbits on the critical set.*

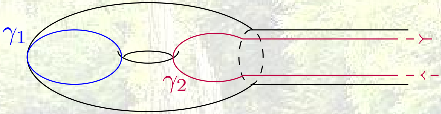
Proof.

- On the critical set, Hamiltonian $H = \frac{1}{2}P_r^2 - P_\alpha$, so that $Y(H) = P_r^2 - P_\alpha = \frac{1}{2}\frac{P_r^2}{2} + c > 0$;
- b^3-contact form $\alpha = (P_r\frac{dx}{x^3} + P_\alpha d\alpha)|_{H=c}$ with $Z = \{(x, \alpha, P_r, P_\alpha)|x = 0, \frac{1}{2}P_r^2 - P_\alpha = c\}$;
- $R_\alpha|_Z = X_{P_r}$ and the cylinder is foliated by periodic orbits.

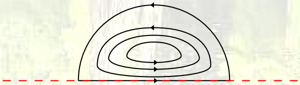
$\square$

The singular Weinstein conjecture re-loaded

A true singular Weinstein structures should also admit singular orbits as below:



Or,



Singular Weinstein conjecture

Let (M, α) be a compact b-contact manifold with critical hypersurface Z. Then there exists always a Reeb orbit $\gamma : \mathbb{R} \to M \setminus Z$ such that $\lim_{t \to \pm\infty} \gamma(t) = p_\pm \in Z$ and $R_\alpha(p_\pm) = 0$ (singular periodic orbit).

Escape orbits and Singular orbits

Singular periodic orbits are a particular case of *escape orbits* γ, $\gamma \subset M \setminus Z$ such that $\lim_{t\to\infty} \gamma(t) = p$ where p is an equilibrium point in Z (respectively $\lim_{t\to-\infty} \gamma(t) = p$).

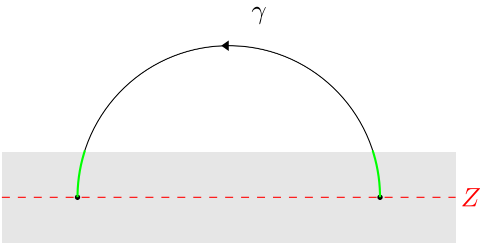


Figure: Singular periodic orbit vs. Escape orbits (in green)

A magic mirror

b-Beltrami vector fields to the rescue

Theorem (Cardona-M.-Peralta-Salas)

- *Any rotational Beltrami field non-vanishing as a section of* ${}^bTM$ *on* M *is a Reeb vector field (up to rescaling) for some* b*-contact form on* M*.*
- *Given a* b*-contact form* α *with Reeb vector field* X *then any nonzero rescaling of* X *is a rotational Beltrami field for some* b*-metric and* b*-volume form on* M*.*

Practical tip

X is a Beltrami vector field on (M, g) $\leftrightsquigarrow$ the Reeb vector field associated to the b-contact form $\alpha = g(X, \cdot)$ is given by $\frac{1}{||X||^2}X$.

True inspiration comes in a hat...

For regular Beltrami fields, there cannot exist surfaces invariant by Hamiltonian vector fields. However, for singular Beltrami vector fields....

Escape orbits and Singular orbits

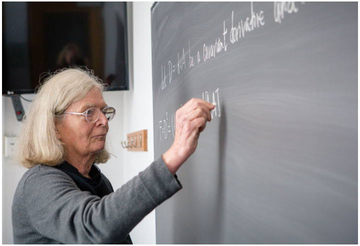
Exact b-metric $\leftrightsquigarrow$ Melrose b-contact forms:

$$g = \frac{dz^2}{z^2} + \pi^* h \tag{1}$$

with h Riemannian metric on Z.

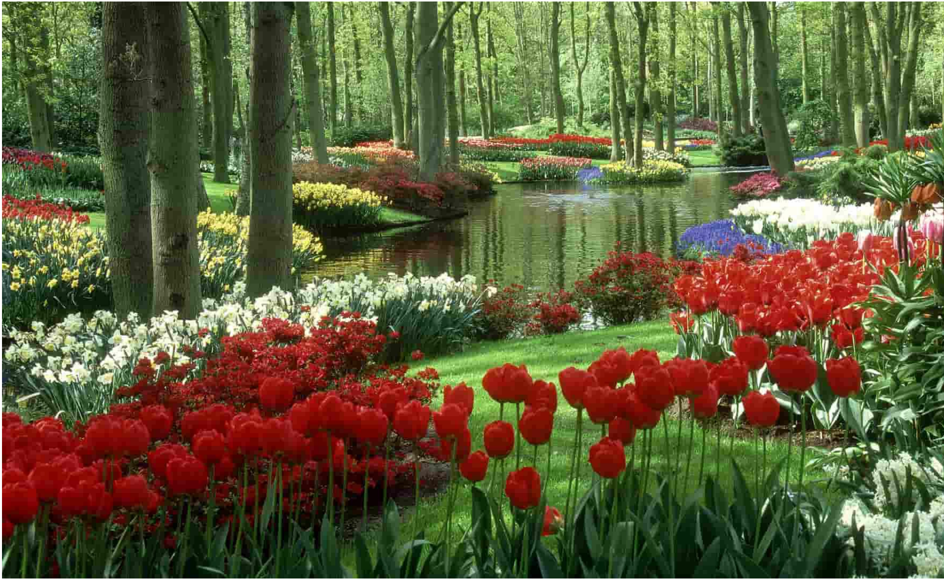
Theorem (M-Oms-Peralta, "lockdown theorem")

There exists at least $2 + b_1(Z)$ escape orbits for Reeb vector fields of generic Melrose b-contact forms on (M, Z).



Proof: The Beltrami equation $\rightsquigarrow$ the Hamiltonian function associated to (R, Z) is an eigenfunction of the induced Laplacian on Z $\rightsquigarrow$ (Uhlenbeck) generically Morse and has regular zero set

A garden of singular orbits



A garden of singular orbits

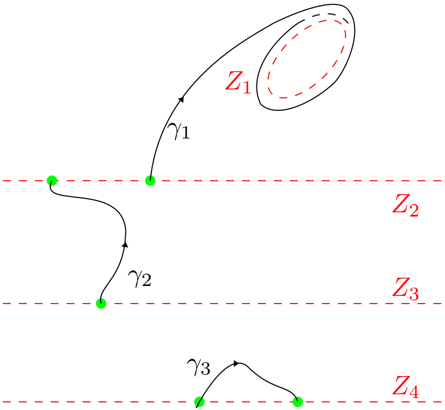


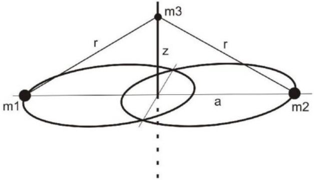
Figure: Different types of escape and singular periodic orbits: γ_1 is a generalized singular periodic orbit, γ_2, γ_3 are singular periodic orbits

Generalized singular periodic orbits

Definition

An orbit $\gamma :\to M \setminus Z$ of a b-Beltrami field X is a generalized singular periodic orbit if there exist $t_1 < t_2 < \cdots < t_k \to \infty$ such that $\gamma(t_k) \to p_+ \in Z$ and $t_{-1} > t_{-2} > \cdots > t_{-k} \to -\infty$ such that $\gamma(t_{-k}) \to p_- \in Z$, as $k \to \infty$.

p_+ and p_- may be contained in different components and are not necessarily zeros of X.



This includes oscillatory motions:orbits $(q(t), p(t))$ in the phase space $T^*\mathbb{R}^n$ such that $\overline{\lim}_{t\to\pm\infty} q(t) = \infty$ and $\underline{\lim}_{t\to\pm\infty} q(t) < \infty$.

A more symmetric case

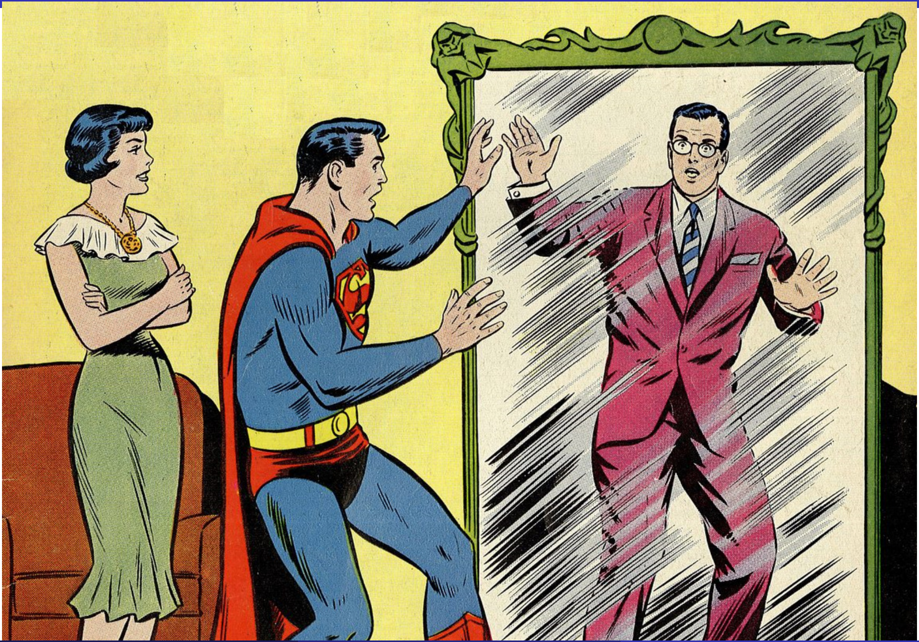
For $g = \frac{dz^2}{z^2} + dx^2 + dy^2$, we can prove more.

Theorem (M-Oms-Peralta Salas)

When g is semi-locally as above and X a generic asymptotically symmetric b-Beltrami vector field, X has a generalized singular periodic orbit. Moreover, it has a singular periodic orbit or at least 4 escape orbits.

In the case of $(\mathbb{T}^3, \alpha = C \cos y dx + B \sin x dy + (C \sin y + B \cos x)\frac{dz}{\sin z})$ for $|B| \neq |C|$, the singular Weinstein conjecture is satisfied.
2 or infinity? Indeed when the genus of the surface is greater than zero from the argument of our proof we get infinite orbits!

What about the restricted three body problem?



The Euclidean case

Theorem (Cardona, M., Peralta-Salas, 2021)

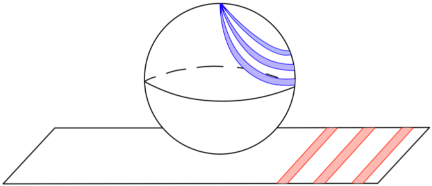
There exists a Beltrami vector field on $\mathbb{R}^3$ which is Turing complete.

- This vector field does not have finite energy.
- The vector field has an invariant plane where all the computations of the machine take place.
- The computational power of this machine is weakly robust. It persists when the vector field is perturbed with an error with exponential decay.
- The proof is not geometrical: It requires a Cauchy-Kovalevskaya theorem and techniques of the theory of dynamical systems of gradient type.
- This construction has compact approximations which are Turing complete (at $\mathbb{T}^3$) with tapes of finite length.
- It is generic: The Turing completeness occurs with probability 1 for arbitrary Beltrami vector fields.

Outside the Beltrami box

Outside the Beltrami box

The Euler equations on (M,g) are Turing complete if: for any integer $k \geqslant 0$, given a Turing machine T, an input tape t, and a finite string $(t^*_{-k},...,t^*_k)$ of symbols of the alphabet, there exist an explicitly constructible vector field $X_0 \in \mathfrak{X}^\infty_{vol}(M)$ and an open set $U \subset \mathfrak{X}^\infty_{vol}(M)$ such that the solution to the Euler equations with initial datum X_0 is defined for all time and intersects U if and only if T halts with an output tape whose positions $-k,...,k$ correspond to the symbols $t^*_{-k},...,t^*_k$.



The manifold

The manifold M is diffeomorphic to $SO(N) \times \mathbb{T}^{N+1}$ and $\dim(M) \lesssim 10^{35}$.

Theorem 4 (Cardona, M., & Peralta-Salas)

There exists a smooth compact Riemannian manifold (M,g) such that the Euler equations on (M,g) are Turing complete. In particular, the problem of whether the solution to the Euler equations with an initial datum X_0 will reach a certain open set $U \subset \mathfrak{X}^\infty_{vol}(M)$ or not is undecidable.

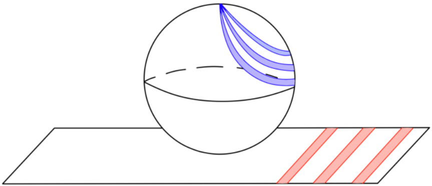
Proof

- There exist polynomial vector fields which are Turing complete on a sphere. Idea: *We compactify* a proof by *Graça et al* on $\mathbb{R}^n$ and we regularize it to get global smooth vector fields.
- Recall:

Theorem (Torres de Lizaur)

Given a polynomial vector field Y on $\mathbb{S}^n$. There exists a Riemannian manifold (M, g) such that $(\mathbb{S}^n, Y)$ can be embedded as Euler equations on (M, g).

- Combine to conclude.



The manifold

The manifold M is diffeomorphic to $SO(N) \times \mathbb{T}^{N+1}$ and $\dim(M) \lesssim 10^{35}$.

What's next?

- Other mirrors, other worlds: Cosymplectic, confoliations....
- Computational complexity versus dynamical complexity.

Theorem (Cardona, M., Peralta-Salas)

There exists Turing complete vector fields on S^2 with zero topological entropy.

- What about the general case?
- Tao's embeddings and representation theory. Blow-up with other geometries?


- Can we get Turing complete constructions from examples in Celestial mechanics exhibiting chaos? (Such at the 3BP?).