Lorentz Geometry and Contact Topology

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Lecture 1

“Techniques of differential topology in relativity” $50 + \varepsilon$ years after Roger Penrose

Lorentz manifolds

$\mathcal{X}$ connected smooth manifold (henceforth usually of dim ≥ 3)

A *Lorentz metric* $\langle\ ,\ \rangle$ on $\mathcal{X}$ is a bilinear symmetric form of signature $(+,-,\ldots,-)$ on $T\mathcal{X}$.

A non-zero tangent vector is

timelike if $\langle v, v\rangle > 0$
null or *lightlike* if $\langle v, v\rangle = 0$
spacelike if $\langle v, v\rangle < 0$

Null and timelike vectors together are called *non-spacelike*.
The zero vector is traditionally in a category by itself.

A Lorentz metric on $\mathcal{X}$ exists
$\Leftrightarrow$ $\mathcal{X}$ admits a line field (canonical up to homotopy)
$\Leftrightarrow$ $\mathcal{X}$ is either open or has Euler characteristic zero
$\Leftrightarrow$ $\mathcal{X}$ admits a non-vanishing vector field

Spacetimes

A *time-orientation* is a continuous choice of a *future hemicone*

$$C_x^{\uparrow} = \text{connected component of } \{v \in T_x\mathcal{X} \mid \langle v, v \rangle \geq 0, v \neq 0\}$$

in the cone of non-spacelike vectors at each point $x \in \mathcal{X}$.

Time orientation $\Leftrightarrow$ orientation on the line field associated to the Lorentz metric. $\exists$ up to passing to a double cover of $\mathcal{X}$.

Definition

A *spacetime* is a connected time-oriented Lorentz manifold.

The vectors in $C_x^{\uparrow}$ are called *future-pointing*.

A piecewise smooth curve is *future-directed* (abbreviated *f.-d.*) if all its tangent vectors are future-pointing.

Causality and chronology

$x, y \in \mathcal{X}$ two points (a.k.a. *events*) in a spacetime $\mathcal{X}$

Causality relation:

$x \leq y$ if either $x = y$ or there is a f.-d. curve connecting x to y

Chronology relation:

$x \ll y$ if there is a f.-d. timelike curve connecting x to y

$\mathcal{X}$ is *causal* if there are no closed f.-d. curves

$\mathcal{X}$ is *chronological* if there are no closed f.-d. timelike curves

(Non)example. A compact spacetime contains closed f.-d. timelike curves and so is never causal (or even chronological).

$\mathcal{X}$ is causal $\Leftrightarrow$ $\leq$ is a partial order. ($\leq$ is always reflexive and transitive. Causality means that it is also anti-symmetric, i.e. $x \leq y$ and $y \leq x$ implies $x = y$.)

Strong causality and Alexandrov topology

$\mathcal{X}$ is a *strongly causal* spacetime if every point in $\mathcal{X}$ has an arbitrarily small neighbourhood such that every f.-d. curve enters it at most once.

The Alexandrov topology on $\mathcal{X}$ is the interval topology associated to $\ll$, i.e. generated by the temporal intervals

$$I_{x,y} := \{z \in \mathcal{X} \mid x \ll z \ll y\}$$

This topology is named after Alexander D. Alexandrov and must not be confused with the Alexandrov topology on posets named after Pavel S. Alexandrov.

Theorem (Kronheimer & Penrose 1967)
$\mathcal{X}$ is strongly causal $\Leftrightarrow$ Alexandrov topology is Hausdorff
$\Leftrightarrow$ Alexandrov topology is the manifold topology on $\mathcal{X}$.

Global hyperbolicity

$\mathcal{X}$ is called *globally hyperbolic* if

(i) $\mathcal{X}$ is strongly causal

(ii) the causal intervals

$$J_{x,y} := \{z \in \mathcal{X} \mid x \leq z \leq y\}$$

are compact for all $x, y \in \mathcal{X}$.

Name $\leftsquigarrow$ $\exists$ *global* solutions for the *hyperbolic* wave equation

Bernal & Sánchez 2005:
(i) can be replaced by $\mathcal{X}$ being causal.

Hounnonkpe & Minguzzi 2019:
If $\dim \mathcal{X} \geq 3$, (i) can be replaced by $\mathcal{X}$ being non-compact.

The classical definition can be formulated purely in terms of $\ll$ and $\leq$: The Alexandrov topology is Hausdorff and causal intervals are compact with respect to it.

Strong cosmic censorship hypothesis

Penrose 1996:

"Physically reasonable" spacetimes are globally hyperbolic.

Examples:

- Minkowski spacetime
- Lorentz products

$$\mathcal{X} = (\mathbb{R} \times Y, dt^2 - g)$$

  where (Y, g) is a *complete* Riemann manifold

- Fridman–Lemaître–Robertson–Walker spacetimes (cosmological models)
- *Outer* parts of black hole models (Schwarzschild, Kerr, ...)
- Maximally extended Schwarzschild is globally hyperbolic. Extended Kerr is not even chronological.

Cauchy surfaces

A f.-d. curve $\gamma : (a, b) \to \mathcal{X}$ is *inextensible* if it does not have limits as $t \to a+$ and $t \to b-$.

A *Cauchy surface* in $\mathcal{X}$ is a subset such that every inextensible f.-d. curve intersects it exactly once.

A *Cauchy function* $f : \mathcal{X} \to \mathbb{R}$ is a (continuous) function increasing from $-\infty$ to $+\infty$ on every inextensible f.-d. curve.
Level sets of f are Cauchy surfaces in $\mathcal{X}$.

Example
$\mathcal{X} = (\mathbb{R} \times Y, dt^2 - g)$ with (Y, g) complete.
Each $\{t\} \times Y$ is a Cauchy surface.

Theorem (Geroch 1970)
$\mathcal{X}$ is globally hyperbolic $\Leftrightarrow \exists$ Cauchy function on $\mathcal{X}$

Corollary
$\mathcal{X}$ is *homeomorphic* to $\mathbb{R} \times \{\text{Cauchy surface}\}$.

Constructing Cauchy functions (after Geroch)

$\mathcal{X}$ globally hyperbolic spacetime

μ smooth probability measure on $\mathcal{X}$

$I^+(x) := \{z \in \mathcal{X} \mid x \ll z\}$ open in $\mathcal{X}$ (always)

$J^+(x) := \{z \in \mathcal{X} \mid x \leq z\}$ closed in $\mathcal{X}$ ($\Leftarrow$ g.h.)

$f^+(x) := \mu(J^+(x)) \stackrel{\text{Sard}}{=} \mu(I^+(x))$

If $\gamma : (a, b) \to \mathcal{X}$ is a f.-d. curve, then

(i) $f^+ \circ \gamma$ is continuous and decreasing

(ii) $\lim_{t \to b-} f^+(\gamma(t)) = 0$ or $\exists \lim_{t \to b-} \gamma(t)$ (main use of g.h.)

$f := -\log f^+ + \log f^-$ is a *continuous* Cauchy function,
where f^- is defined by reversing the time orientation on $\mathcal{X}$.

Smooth splitting of globally hyperbolic spacetimes

Theorem (Bernal & Sánchez 2005)
A globally hyperbolic spacetime admits a *smooth* Cauchy function with everywhere *timelike* gradient.

Corollary
There exists a *diffeomorphism* $\phi : \mathbb{R} \times M \longrightarrow \mathcal{X}$ such that
(i) $\phi(\mathbb{R} \times \{x\})$ is a timelike f.-d. curve $\forall x \in M$
(ii) $\phi(\{t\} \times M)$ is a *smooth spacelike* Cauchy surface $\forall t \in \mathbb{R}$

The set of smooth Cauchy time functions is convex, so all such splittings are isotopic. The smooth manifold M depends only on the causal structure of $\mathcal{X}$.

Other proofs and generalisations (to cone fields):
Fathi & Siconolfi 2012, Chruściel & Grant & Minguzzi 2016, Bernard & Suhr 2018

Smooth structures I

For every $n \geq 3$, there exist uncountably many contractible smooth n-manifolds not homeomorphic to $\mathbb{R}^n$ (McMillan 1962, Curtis & Kwun 1965, Glaser 1966)

For $n = 4$, there exist uncountably many smooth 4-manifolds homeomorphic but not diffeomorphic to $\mathbb{R}^n$ (Taubes 1987)

Theorem (Stallings 1962; McMillan 1962 + Perelman 2003) Let X be a contractible n-manifold diffeomorphic to $\mathbb{R} \times M$. Then X is diffeomorphic to $\mathbb{R}^n$.

Corollary (Newman & Clarke 1987; Chernov & N. 2013) A contractible globally hyperbolic spacetime is diffeomorphic to $\mathbb{R}^n$ but can have any contractible manifold as its Cauchy surface.

Global hyperbolicity 'censors out' a single smooth structure.

Smooth structures II

What if $\mathcal{X}$ is not contractible?

Theorem (Chernov & N. 2013)
Suppose that a 4-dimensional globally hyperbolic spacetime $\mathcal{X}$ is homeomorphic to $\mathbb{R} \times M$ with M closed and orientable. Then $\mathcal{X}$ is diffeomorphic to $\mathbb{R} \times M$.

(Proof uses Pereleman's geometrisation theorem and Turaev's topological h-cobordism theorem for geometric 3-manifolds.)

Question
Is it true that at most one smooth structure on a 4-manifold can underlie a globally hyperbolic spacetime structure?

Example
Not true in (some) higher dimensions. Let Σ be an exotic 7-sphere. Then $\mathbb{R} \times S^7$ and $\mathbb{R} \times \Sigma$ are homeomorphic but not diffeomorphic ($\Leftarrow$ h-cobordism theorem) and both are globally hyperbolic products.

Literature

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Lecture 2

“Spaces of null geodesics”

35 years after Robert Low

Levi-Civita connection and geodesics

$\mathcal{X}$ spacetime with Lorentz metric $\langle\ ,\ \rangle$

$\nabla : T\mathcal{X} \to T\mathcal{X} \otimes T^*\mathcal{X}$ Levi-Civita connection:

(i) $d\langle v, w\rangle = \langle \nabla v, w\rangle + \langle v, \nabla w\rangle \Leftrightarrow \nabla$ preserves $\langle\ ,\ \rangle$

(ii) $\nabla_v w - \nabla_w v = [v, w] \Leftrightarrow \nabla$ has zero torsion

$\gamma : (a, b) \to \mathcal{X}$ is a *geodesic* if $\nabla_{\frac{\partial}{\partial t}} \dot{\gamma} = 0$

$\langle \dot{\gamma}, \dot{\gamma}\rangle =$ const by (i) $\Rightarrow$ timelike, null, and spacelike geodesics

Gauss Lemma

Let $\gamma_s : (a, b) \to \mathcal{X}$, $0 \leq s < \epsilon$, be a family of curves such that γ_0 *is a geodesic* and $\langle \dot{\gamma}_s, \dot{\gamma}_s\rangle$ *is independent of* s. Then

$$\langle \dot{\gamma}_0(t), J(t)\rangle = \text{const}$$

where $J(t) := \frac{d}{ds}\Big|_{s=0} \gamma_s(t)$ is the vector field along γ_0 tangent to this family.

Space of light rays I

$\mathfrak{N}_{\mathcal{X}} := \{\text{inextensible f.-d. null geodesics}\}/\sim$

$\gamma(t) \sim \gamma(\lambda t + \tau), \lambda > 0, \tau \in \mathbb{R}$, affine reparametrisation

$\mathfrak{N}_{\mathcal{X}}$ can be a rather wild topological space in general.

$\mathcal{X}$ globally hyperbolic, $M \subset \mathcal{X}$ spacelike Cauchy surface

$$\rho_M : \mathfrak{N}_{\mathcal{X}} \longrightarrow ST^*M$$

$$\rho_M(\boldsymbol{\gamma}) := [\langle \dot{\gamma}(t_0), \cdot \rangle|_{TM}] \in ST^*_{\gamma(t_0)}M, \quad \gamma(t_0) \in M$$

ODE theory & linear algebra:

- ρ_M is well-defined and bijective
- $\rho_{M'} \circ (\rho_M)^{-1} : ST^*M \overset{\text{diffeo}}{\longrightarrow} ST^*M'$ for any other M'

$\mathfrak{N}_{\mathcal{X}}$ is a smooth manifold — but more is true

Contact structure on contact elements

M smooth manifold

$$
\begin{aligned}
ST^*M &= \{\text{non-zero 1-forms on } M \text{ mod positive scalars}\}\\
&= \{\text{co-oriented hyperplanes in } TM\}\\
&= \{\text{contact elements on M}\}
\end{aligned}
$$

$\pi_M : ST^*M \to M$ projection to M (sphere bundle)

$ST^*M \ni \xi \longmapsto$ co-oriented hyperplane $H_\xi \subset T_\xi ST^*M$

$H_\xi = \ker \pi_M^* \xi$ viewing ξ as a 1-form up to positive scalar

$H \subset T(ST^*M)$ distribution of co-oriented hyperplanes

This distribution is 'maximally nonintegrable', i.e. a *contact structure* on ST^*M.

Contact forms on contact elements

$\lambda = \sum p_i dq_i$ canonical 1-form on T^*M: $\lambda_\xi = \pi_M^* \xi$

$\omega = d\lambda = \sum dp_i \wedge dq_i$ canonical symplectic form on T^*M

$\iota : ST^*M \hookrightarrow T^*M$ any *fibrewise starshaped* embedding. Then

(i) $\ker \iota^*\lambda = H$ (respecting co-orientation)

(ii) $\iota^*\lambda \wedge d(\iota^*\lambda) \wedge \cdots \wedge d(\iota^*\lambda) = \iota^*(\epsilon \lrcorner \omega^n) \neq 0$

$\iota \leftrightarrow \iota^*\lambda$ gives us all *contact forms* defining H

$(T^*M - 0, \omega = d\lambda)$ is the *symplectisation* of (ST^*M, H)

Example

g Riemann metric on M

$\iota_g : ST^*M \xrightarrow{\cong} S_g^*M \subset T^*M$ unit length 1-forms w.r.t. g

$\alpha_g := \iota_g^*\lambda$ contact form on ST^*M associated to g

$\Omega_g := \alpha_g \wedge d\alpha_g \wedge \cdots \wedge d\alpha_g$ Liouville measure of g

Space of light rays II

$\mathcal{X}$ globally hyperbolic, $M \subset \mathcal{X}$ spacelike Cauchy surface

$\iota_M := \iota_g \circ \rho_M : \mathfrak{N}_{\mathcal{X}} \to ST^*M \to S^*_g M$ where $g = -\langle\ ,\ \rangle|_M$

$\alpha_M := \rho_M^* \alpha_g = \iota_M^* \lambda$ contact form on $\mathfrak{N}_{\mathcal{X}}$ associated to M

$\boldsymbol{\gamma} \in \mathfrak{N}_{\mathcal{X}}$, $x = \gamma(t_0) \in M$, $n_M(x)$ f.-p. unit normal to M at x

$$\iota_M(\boldsymbol{\gamma}) = \frac{\langle \dot\gamma(t_0), \cdot \rangle|_M}{\langle \dot\gamma(t_0), n_M(x)\rangle}$$

$\mathbf{v} = \frac{d}{ds}\big|_{s=0} \boldsymbol{\gamma}_s$ tangent vector to $\mathfrak{N}_{\mathcal{X}}$ at $\boldsymbol{\gamma} = \boldsymbol{\gamma}_0$

$J(t) = \frac{d}{ds}\big|_{s=0} \gamma_s(t)$ Jacobi field on γ for a family γ_s

Lemma $\quad \alpha_M(\mathbf{v}) = \dfrac{\langle \dot\gamma(t_0), J(t_0)\rangle}{\langle \dot\gamma(t_0), n_M(x)\rangle}$

Gauss Lemma $\Rightarrow \alpha_{M'}(\mathbf{v}) = f_{MM'}(\boldsymbol{\gamma}) \cdot \alpha_M(\mathbf{v}),\ \ f_{MM'} > 0$

Theorem (Low 1988)
$\mathfrak{N}_{\mathcal{X}}$ is a contact manifold and each ρ_M is a contactomorphism

Redshift

$M, M' \subset \mathcal{X}$ spacelike Cauchy surfaces

$\boldsymbol{\gamma} \in \mathfrak{N}_{\mathcal{X}}$, $x = \gamma(t_0) \in M$, $x' = \gamma(t_0') \in M'$

$$\alpha_{M'} = \frac{\langle \dot\gamma(t_0), n_M(x)\rangle}{\langle \dot\gamma(t_0'), n_{M'}(x')\rangle}\alpha_M$$

$$\begin{aligned}\langle \dot\gamma, n\rangle &= \text{energy of the photon } \gamma(t) \text{ measured by } n\\ &\sim \text{frequency of the photon } \gamma(t) \text{ measured by } n\end{aligned}$$

$\dfrac{\langle \dot\gamma, n_E\rangle}{\langle \dot\gamma, n_R\rangle}$ = *redshift* from n_E to n_R along the light ray $\boldsymbol{\gamma} \in \mathfrak{N}_{\mathcal{X}}$

Corollary (Chernov & N. 2018)
The ratio of the contact forms on $\mathfrak{N}_{\mathcal{X}}$ associated to Cauchy surfaces M and M' is the redshift between them.

Remark
$(\mathfrak{N}_{\mathcal{X}}, \text{contact structure})$ is a *conformal* invariant of $\mathcal{X}$.
Contact forms on $\mathfrak{N}_{\mathcal{X}}$ reflect the *metric* properties of $\mathcal{X}$.

Space of light rays III

What if $\mathcal{X}$ is *not* globally hyperbolic?

Observation (Penrose 1980s, Khesin & Tabachnikov 2009)
If $\mathfrak{N}_{\mathcal{X}}$ is a smooth manifold, then it is contact.

Examples

- $\mathcal{X}$ is strongly causal
  $\Rightarrow \mathfrak{N}_{\mathcal{X}}$ is smooth but *not* necessarily Hausdorff (Low 1988)
- $\mathcal{X}$ is causally simple (strongly causal and $\leq$ is closed)
  $\Rightarrow \mathfrak{N}_{\mathcal{X}}$ is Hausdorff *if* $\mathcal{X}$ is conformally equivalent to an open subset of a globally hyperbolic spacetime; not true in general (Hedicke & Suhr 2020)
- $\exists$ *compact* $\mathcal{X}$ such that $\mathfrak{N}_{\mathcal{X}}$ is a manifold (Guillemin 1989, Suhr 2013, Marin-Salvador 2021)

Question
Which contact manifolds may occur?

Skies (a.k.a. celestial spheres)

$\mathfrak{S}_x := \{\boldsymbol{\gamma} \in \mathfrak{N}_{\mathcal{X}} \mid x \in \gamma\}$ the *sky* of $x \in \mathcal{X}$

If M is a Cauchy surface with $x \in M$ ($\exists$ by Bernal–Sánchez),
$\rho_M(\mathfrak{S}_x) = ST^*_x M =$ fibre of ST^*M at $x \in M$

A submanifold in a $(2n+1)$-dimensional contact manifold is *Legendrian* if it is tangent to the contact distribution and has dimension n (maximal possible).

Example

Fibres of ST^*M are Legendrian $\Leftarrow H \supset \ker d\pi_M$

Corollary

$\mathfrak{S}_x$ is a Legendrian sphere in $\mathfrak{N}_{\mathcal{X}}$

Properties

(i) $\mathfrak{S}_x \cap \mathfrak{S}_y \neq \varnothing \Leftrightarrow \exists$ null geodesic through x and y

(ii) $\mathfrak{S}_x$ and $\mathfrak{S}_y$ tangent at $\boldsymbol{\gamma} \Leftrightarrow x$ and y *conjugate* along γ

Wavefronts

$\Lambda \subset ST^*M$ Legendrian, $W(\Lambda) := \pi_M(\Lambda)$ wavefront of Λ

Generically, $W(\Lambda)$ is a *singular* co-oriented hypersurface in M.

$\mathfrak{W}_{x,M} := \pi_M \circ \rho_M(\mathfrak{S}_x) \subset M$ wavefront of x on M

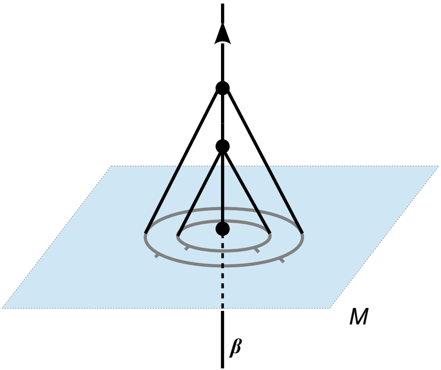

Figure: Expanding wavefronts for a f.-d. timelike curve near $\beta \cap M$

Legendrian isotopies

A *Legendrian isotopy* in a contact manifold (Y, H) is an equivalence class of *parametrised Legendrian isotopies*:

$$j : L \times [0,1] \longrightarrow Y$$

where $j_t = j|_{L\times\{t\}} : L \hookrightarrow Y$ is Legendrian $\forall t \in [0,1]$. Parametrised isotopies are equivalent if they differ by a fibrewise diffeomorphism of $L \times [0,1]$.

Definition (Eliashberg & Polterovich 2000, Bhupal 2001)
A Legendrian isotopy $j : L \times [0,1] \to Y$ in $(Y, H = \ker\alpha)$ is $\begin{matrix} \textit{non-negative} \\ \textit{positive} \end{matrix}$ if $\alpha(\frac{d}{dt} j_t(x)) \begin{matrix} \geq 0 \\ > 0 \end{matrix} \quad \forall (x,t) \in L \times [0,1]$.

Independent of parametrisation and contact form defining H.

Example
Legendrian isotopy in ST^*M is non-negative $\iff$ wavefronts in M move in the direction of their co-orientation.

F.-d. curves and Legendrian isotopies

$\beta : (a, b) \to \mathcal{X}$ smooth curve (not necessarily f.-d.)

$\mathfrak{S}_{\beta(t)}$ Legendrian isotopy in $\mathfrak{N}_{\mathcal{X}}$

$\boldsymbol{\gamma} \in \mathfrak{S}_{\beta(t_0)}$ and $\mathbf{v} = \frac{d}{dt}\Big|_{\boldsymbol{\gamma}} \mathfrak{S}_{\beta(t)}$ for *some* parametrisation

$\alpha_M(\mathbf{v}) = \dfrac{\langle \dot\gamma, \dot\beta(t_0) \rangle}{\langle \dot\gamma, n_M(x) \rangle}$ for a Cauchy surface $M \ni x = \beta(t_0)$

A vector in a time-oriented Lorentz vector space is f.-p. (resp., f.-p. timelike) $\Longleftrightarrow$ its scalar product with every f.-p. null vector is non-negative (resp., positive).

β is f.-d. (timelike) $\Longleftrightarrow \alpha_M(\frac{d}{dt}\mathfrak{S}_{\beta(t)}) \geq 0$ (> 0)

Proposition (Chernov & N. 2010, 2020)

β is f.-d. / f.-d. timelike $\Longleftrightarrow$ $\mathfrak{S}_{\beta(t)}$ is non-negative / positive

Literature

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Lecture 3

Causality and contact topology

Causality and orderability I

Λ, Λ' closed Legendrian submanifolds

$\begin{matrix} \Lambda \preccurlyeq \Lambda' \\ \Lambda \lll \Lambda' \end{matrix}$ if $\exists$ $\begin{matrix} \text{non-negative} \\ \text{positive} \end{matrix}$ Legendrian isotopy from Λ to Λ'

Proposition A (Chernov & N. 2010, 2020)

$x \leq y \Longrightarrow \mathfrak{S}_x \preccurlyeq \mathfrak{S}_y$ and $x \ll y \Longrightarrow \mathfrak{S}_x \lll \mathfrak{S}_y$

Definition (Eliashberg & Polterovich 2000)

A Legendrian isotopy class $\mathcal{L}$ is *orderable* if $\preccurlyeq$ is a partial order on it ($\Leftrightarrow \nexists$ non-constant non-negative loops in $\mathcal{L}$).

Proposition B (Chernov & N. 2010, 2020)

If $\mathcal{X}$ is globally hyperbolic and *the Legendrian isotopy class of skies in $\mathfrak{N}_{\mathcal{X}}$ is orderable*, the converse implications hold:

$x \leq y \Longleftarrow \mathfrak{S}_x \preccurlyeq \mathfrak{S}_y$ and $x \ll y \Longleftarrow \mathfrak{S}_x \lll \mathfrak{S}_y$

Recall that ρ_M maps the class of skies to the class of the fibre of ST^*M for any Cauchy surface $M \subset \mathcal{X}$.

Causality and orderability II

Legendrian link = *ordered* pair of disjoint Legendrians

Observation (Low 1988, Chernov & Rudyak 2008)
$\{(x, y) \in \mathcal{X} \times \mathcal{X} \mid x \text{ and } y \text{ causally } \textit{un}\text{related}\}$ is *connected* for any globally hyperbolic $\mathcal{X}$, $\dim \mathcal{X} \geq 3$. Hence $\mathfrak{S}_x \sqcup \mathfrak{S}_y$ is in the same isotopy class of Legendrian *links* for any such x, y.

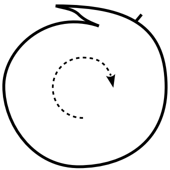
- $\mathcal{X} \cong \mathbb{R} \times M$ smooth splitting, $t : \mathcal{X} \to \mathbb{R}$ Cauchy function
- $\{(x, y) \mid x \neq y, t(x) = t(y)\}$ connected, x and y unrelated
- If $t(y) \neq t(x)$, move y into $\{t(x)\} \times M$ along $\mathbb{R} \times \{\underline{y}\}$.

Legendrian isotopy extension theorem $\Rightarrow$
$\Lambda \preccurlyeq \Lambda'$ and $\Lambda \lll \Lambda'$ are preserved under Legendrian link isotopy

Proof of Proposition B (for $\leq$ and $\preccurlyeq$)
Assume that the sky class is orderable and $\mathfrak{S}_x \preccurlyeq \mathfrak{S}_y$.
- x and y must be causally related, as otherwise $\mathfrak{S}_y \preccurlyeq \mathfrak{S}_x$ ↯
- $y \leq x$ is impossible, as this also implies $\mathfrak{S}_y \preccurlyeq \mathfrak{S}_x$ by Prop. A

Do orderable Legendrian isotopy classes exist?

Example (Colin & Ferrand & Pushkar' 2007-2017)
Positive Legendrian loop in $ST^*\mathbb{R}^2$:



Not in the Legendrian isotopy class of the fibre of $ST^*\mathbb{R}^2$.

Theorem (Liu 2020, Pancholi & Pérez & Presas 2018)
There is a positive Legendrian loop based at any *loose* Legendrian ('containing' a zigzag $\times$ closed submanifold).

Theorem (Laudenbach 2008)
There is a positive loop of Legendrian *immersions* based at any Legendrian submanifold.

Refocussing spacetimes

Definition (Besse 1978)
A Riemannian manifold (Y, g) is called a Y^x_ℓ-*manifold* if all unit speed geodesics from $x \in Y$ return to x at time $\ell > 0$.

Examples: CROSSes (S^n, $\mathbb{R}\mathrm{P}^n$, $\mathbb{C}\mathrm{P}^n$, $\mathbb{H}\mathrm{P}^n$, CaP^2) and their isometric quotients; certain exotic spheres

The fibre class in ST^*Y is *not* orderable:
(Co-)geodesic flow is contact $\Rightarrow$ positive loop based at $ST^*_x Y$

Proposition B is false for $(\mathbb{R} \times Y, dt^2 - g)$:
- null geodesics: $(s, \underline{\gamma}(s))$ with $\underline{\gamma}$ a unit speed geodesic in Y
- $\mathfrak{S}_{(0,x)} = \mathfrak{S}_{(\ell,x)}$ light rays through $(0, x)$ *refocus* at (ℓ, x)
- $(0, x') \ll (\ell, x)$ if x' is close to $x \Rightarrow \mathfrak{S}_{(0,x')} \lll \mathfrak{S}_{(\ell,x)}$
- $\mathfrak{S}_{(0,x')} \lll \mathfrak{S}_{(0,x)}$ but $(0, x')$ is causally unrelated to $(0, x)$

Y is compact, $\pi_1(Y)$ is finite, and its rational cohomology ring is generated by one element (Bérard-Bergery 1977)

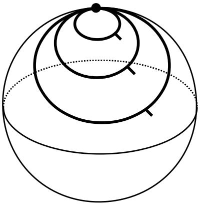
Positive *vs* non-negative Legendrian loops

Proposition (Chernov & N. 2016)
$\exists$ non-constant non-negative loop $\Longleftrightarrow$ $\exists$ positive loop

Different from the Lorentz case: causality $\neq$ chronology

Caveat Method of proof (Eliashberg & Polterovich 2000)
$\rightsquigarrow$ positive loop homotopic to an *iterate* of non-negative one

Question Is the following non-negative loop in ST^*S^2 homotopic to a positive one? (Its double iterate is.)



Lorentz origin: Skies of points on a null geodesic in $\mathbb{R} \times S^2$.

Contact rigidity I

Theorem (Chernov & N. 2010)
The fibre class in ST^*M is orderable if the universal cover $\widetilde{M}$ of M is non-compact.

Method of proof: Spectral invariants of generating functions for Lagrangians in $T^*\widetilde{M}$ ($\leftsquigarrow$ Sikorav 1980s, Viterbo 1992)

Another proof (Guillermou & Kashiwara & Schapira 2012): microlocal sheaf theory ($\leftsquigarrow$ Tamarkin 2008)

Sharp in low dimensions:

- $\dim M = 2 \Rightarrow M \neq S^2, \mathbb{RP}^2 = S^2/\mathbb{Z}_2$
- $\dim M = 3 \Rightarrow M \neq S^3/\Gamma$, $\Gamma \subset \mathrm{O}_4$ (Perelman)

Theorem (Frauenfelder & Labrousse & Schlenk 2015)
The fibre class is orderable if the integral cohomology ring of $\widetilde{M}$ is not isomorphic to that of a CROSS.

Proof: $\exists$ positive loop $\rightsquigarrow$ 'slow growth' of Floer homology

Linking of skies: Low's conjectures

$x, y \in \mathcal{X}$ with disjoint skies (not on the same null geodesic)

$\mathcal{U} :=$ Legendrian isotopy class $\ni \mathfrak{S}_x \sqcup \mathfrak{S}_y$ for *un*related x, y

Corollary ('Legendrian Low Conjecture')
If the Cauchy surface M is as in the two theorems above,
x and y are causally related $\iff \mathfrak{S}_x \sqcup \mathfrak{S}_y \notin \mathcal{U}$.

$\mathcal{TU} :=$ *smooth* isotopy class of links containing $\mathcal{U}$

Example (Low 1988, Natário & Tod 2004)
$\exists\ \mathcal{X}$ with $M \cong \mathbb{R}^3$ and $x \ll y \in \mathcal{X}$ with $\mathfrak{S}_x \sqcup \mathfrak{S}_y \in \mathcal{TU}$
$\leftrightsquigarrow$ Linking of 2-spheres reduces to homotopy (Haefliger 1961)

Theorem (Low's conjecture 1988; Chernov & N. 2010)
If $M^2 \neq S^2, \mathbb{R}\mathrm{P}^2$, x and y causally related $\Leftrightarrow \mathfrak{S}_x \sqcup \mathfrak{S}_y \notin \mathcal{TU}$.

$\leftrightsquigarrow$ description of Legendrian cable links (Ding & Geiges 2010)

Legendrian linking distinguishes between $x \leq y$ and $y \leq x$.
Smooth linking in dim 2 does not ($\leftrightsquigarrow$ Traynor 1997).

Twistor map

$\mathcal{X}$ globally hyperbolic with 'nice' M, $\dim \mathcal{X} = n + 1$

$\mathcal{S}$ = Legendrian isotopy class of skies in $\mathfrak{N}_{\mathcal{X}}$

$\mathcal{S}$ is an infinite dimensional manifold modelled on $C^\infty(S^{n-1})$
(A neighbourhood of a Legendrian Λ is contactomorphic to a neighbourhood of the zero section in the 1-jet bundle $\mathcal{J}^1(\Lambda)$.
Nearby Legendrians $\longleftrightarrow$ graphs of 1-jets of functions on Λ.)

$\mathfrak{S} : \mathcal{X} \longrightarrow \mathcal{S}$, $x \mapsto \mathfrak{S}_x$, 'ur-twistor map' of Penrose

Properties

(i) $\mathfrak{S}$ is a smooth embedding (not proper in general)
(ii) $\leq$ and $\ll$ are the pull-backs of $\preccurlyeq$ and $\lll$ by $\mathfrak{S}$

Remark/Question

Our proof of (ii) was by contradiction. Is there a direct way of producing a f.-d. curve connecting x to y from a non-negative Legendrian isotopy connecting $\mathfrak{S}_x$ to $\mathfrak{S}_y$?

Interval topology on Legendrians

$\lll$ defines the *interval topology* on a Legendrian isotopy class

Corollary If M is nice, the interval topology on $\mathcal{S}$ induces the manifold topology on $\mathcal{X}$ via the twistor embedding $\mathfrak{S}$.

If the interval topology on $\mathcal{S}$ is *Hausdorff*, one can define the *interval completion* $\widehat{\mathcal{X}} := \overline{\mathfrak{S}(\mathcal{X})}$ with good properties.

Question
Is the interval topology on an orderable class Hausdorff?
If not, is there a useful notion of 'strong causality'?

Proposition (Chernov & N. 2020)
If M is smoothly covered by an open subset in $\mathbb{R}^n$, then the interval topology on the fibre class in ST^*M is Hausdorff.

Proof: Spectral invariants for $\mathcal{J}^1(S^{n-1}) \cong ST^*\mathbb{R}^n$.

Proposition (Chernov & N. 2020)
Nonintersecting Legendrians in an orderable class have disjoint interval neighbourhoods.

Contact rigidity II

$\widetilde{\mathcal{L}} \to \mathcal{L}$ universal cover of a Legendrian isotopy class

$\preccurlyeq$ lift of $\preccurlyeq$ to $\widetilde{\mathcal{L}}$ ($\leftsquigarrow$ curves with non-negative projections to $\mathcal{L}$)

Definition (Chernov & N. 2016)

$\mathcal{L}$ is *universally orderable* if $\preccurlyeq$ is a partial order on $\widetilde{\mathcal{L}}$

Theorem (Chernov & N. 2016)

The fibre class in ST^*M is always *universally* orderable.

Proof: Generating hypersurfaces for Legendrians in ST^*M
(Eliashberg & Gromov 1998, Pushkar' 2016)

Generalisation: Chantraine & Colin & Dimitroglou Rizell 2019

$\widetilde{\mathcal{X}}$ globally hyperbolic with Cauchy surface $\widetilde{M}$

$\Rightarrow$ if $\mathcal{X}$ is globally hyperbolic with 'exceptional' M,
passing to finite covering may assume $\mathcal{X}$ simply connected

Corollary (Chernov & N. 2016)

$x \leq y \iff \widetilde{\mathfrak{S}}_x \preccurlyeq \widetilde{\mathfrak{S}}_y$ where $\widetilde{\mathfrak{S}} : \mathcal{X} \to \widetilde{\mathcal{S}}$ is a lift of $\mathfrak{S}$

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