Are two finite H-spaces homotopy equivalent?

Algorithmic approach

Mária Šimková

The Department of Mathematics and Statistics of the Faculty of Science, Masaryk University

MASARYK UNIVERSITY

MUNI SCI Department of Mathematics and Statistics

What is a homotopy equivalence?

Two spaces are homotopy equivalent if it is possible to transform one into another by operations: bending, shrinking and expanding.

Homeomorphism x homotopy equivalence

Let X and Y be topological spaces, and $f: X \to Y$ is a continuous map.

- The map f is a *homeomorphism* if there is a continuous map $g: Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. The spaces X and Y are said to be *homeomorphic* in such a case.
- The map f is a *homotopy equivalence* if there is a continuous map $g: Y \to X$ such that $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$. The spaces X and Y are *homotopy equivalent* in that case. For brevity, we denote $X \simeq Y$.

A historical overview of algorithms

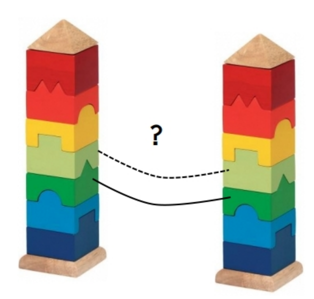
Assume that our input simplicial complexes are simply-connected and finite-dimensional. Are they homotopy equivalent?

- Brown [1] provides an algorithm under the finiteness of all higher homotopy groups. It's based on exhaustive searches as the Postnikov stages of both complexes are finite.
- Another approach presents Nabutovsky and Weinberger [3] under the weaker requirement of finitely generated homotopy groups. However, their exposition lacks implementation details and only sketches key ideas.

Data structure & Basic idea

- Simply-connected simplicial sets of dimension $d < \infty$.
- Simplicial sets with *effective homology*.
- Sergeraert et al. (e. g.[4]) introduced *effective homology framework* that mainly provides methods for computation of homology groups of even infinite simplicial sets.

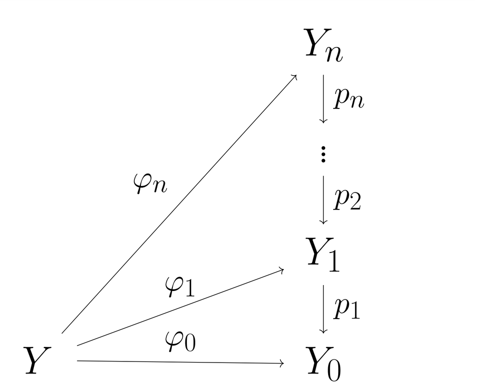
Decompose the input objects into a bunch of simple simplicial sets in a suitable sense. Inductively try to construct homotopy equivalences from the bottom to the upper stage.



Postnikov tower

Let Y be a simply-connected simplicial set. A simplicial Postnikov tower for Y is a collection of maps and spaces as per the diagram such that for each $n \geq 0$:

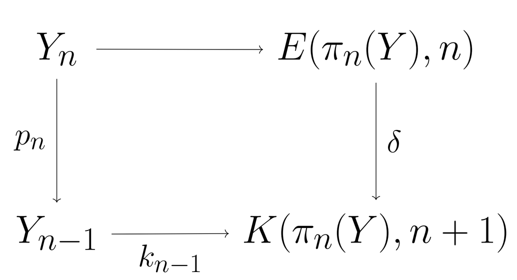
$$\begin{cases} \varphi_{n*}: \pi_k(Y) \to \pi_k(Y_n) \text{ are isomorphisms} & \text{for } 0 \leq k \leq n, \\ \pi_k(Y_n) = 0 & \text{for } k \geq n+1. \end{cases}$$



For all $n \geq 1$: Y_n is the pullback of the fibration δ along a map $k_{n-1}: Y_{n-1} \to K(\pi_n(Y), n+1)$. There is the well-known bijection

$$\mathrm{ev}: \mathrm{SMap}(Y_{n-1}, K(\pi_n, n+1)) \to Z^{n+1}(Y_{n-1}, \pi_n).$$

Denote $\kappa_{n-1} := \mathrm{ev}^{-1}(k_{n-1})$ and $[\kappa_{n-1}] \in H^{n+1}(Y_{n-1}; \pi_n(Y))$.

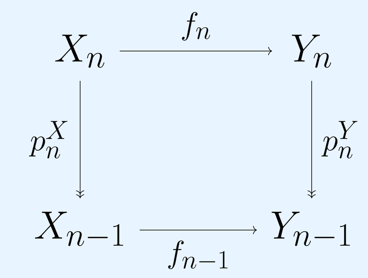


Note. In general, the simplicial set $K(\pi_n, n+1)$ is infinite, so effective homology of $K(\pi_n, n+1)$ and other derived simplicial sets need to be available for algorithmic construction of Postnikov tower (effective Postnikov tower).

Necessary condition

Let X and Y be simply connected simplicial sets with effective homology. Let $\{X_n\}$ and $\{Y_n\}$ be their effective Postnikov towers. If X and Y are homotopy equivalent then Postnikov stages X_n and Y_n are homotopy equivalent for all $n \leq d$. The right diagrams are strictly commutative. Furthermore, there is an isomorphism $\gamma: \pi_n(X) \to \pi_n(Y)$ such that the Postnikov classes satisfy the relation

$$\gamma_*[\kappa^X_{n-1}] = f^*_{n-1}[\kappa^Y_{n-1}].$$



Equivalence of Postnikov stages x equivalence of X & Y.

Let X and Y be finite simply connected simplicial sets of dimensions $\leq d$ with effective Postnikov towers $\{X_n\}$ and $\{Y_n\}$, respectively. Then $|X|$ and $|Y|$ are homotopy equivalent if and only if there is a homotopy equivalence $f_d: X_d \to Y_d$.

Necessary & sufficient condition

Let X and Y be simply connected simplicial sets with effective homology. Let $\{X_n\}$ and $\{Y_n\}$ be their effective Postnikov towers. Assume that there is a homotopy equivalence $g_{n-1}: X_{n-1} \to Y_{n-1}$. Then, X_n and Y_n are homotopy equivalent if and only if there is an isomorphism $\gamma: \pi_n(X) \to \pi_n(Y)$ and a homotopy selfequivalence $a_{n-1}: X_{n-1} \to X_{n-1}$ satisfying $\gamma_*[\kappa^X_{n-1}] = (g_{n-1}a_{n-1})^*[\kappa^Y_{n-1}]$. Moreover, if we assume that $g_{n-1} \in \mathrm{iso}^{\mathrm{ef}}(X_{n-1}, Y_{n-1})$ then $a_{n-1} \in \mathrm{aut}^{\mathrm{ef}}(X_{n-1})$.

Notation. The set $\mathrm{iso}^{\mathrm{ef}}(A, B)$ contains all effective homotopy equivalences $A \to B$, i.e., homotopy equivalences computed by a specific procedure. It is closed under composition, and every homotopy equivalence is homotopic to an effective homotopy equivalence.

Algorithmic group theory

The equation $\gamma_*[\kappa^X_{n-1}] = (g_{n-1}a)^*[\kappa^Y_{n-1}]$ indicates that we have to use algorithmic group theory. Let G be a group and let $\cdot: M \times G \to M$ be the *right action* of the group G on a set M. For a subgroup S of G, a subset $T \subseteq G$ is called a *right transversal* of S if T contains exactly one element of each coset from $G/S = \{S \cdot g \subseteq G;\ g \in G\}$.For every $y \in m^G$, denote a unique $g \in G$ such that $m^g = y$ as $g = \log y$.

Schreier's lemma. Let G be a group generated by elements from a set P, $G = \langle P \rangle$. Let S be a subgroup of G with a finite right transversal T. Then the subgroup S is generated by the set

$$W = S \cap \{rp(\overline{rp})^{-1} \in G;\ r \in T,\ p \in P, rp \neq \overline{rp}\}.$$

The uniquely determined element in $T \cap Sg$ is $\overline{g}$.

Corollary. Let a finitely generated group G act on a finite set M, $m \in M$ and let $S := \mathrm{Stab}_G(m)$. Then the subgroup S is finitely generated, and its generators are listed in the set W.

Orbit-Stabilizer algorithm.
Input: A finitely-generated group $G = \langle P \rangle$ with a right action on a finite set M, an element $m \in M$.
Output: Elements of the orbit m^G and a set Q of Schreier's generators of the stabilizer $\mathrm{Stab}_G(m)$.

Spaces of finite k-type

A simply connected set X has finite k-type if its effective Postnikov tower $\{X_n\}$ has Postnikov class $[\kappa^X_n]$ of finite order for each $n \in \mathbb{N}$. We will say that X has finite k-type through dimension d if Postnikov classes $[\kappa^X_n]$ are of finite order for all $n \leq d-1$. Example of spaces:

- H-spaces, especially Lie groups, topological groups, and simplicial groups,
- m-connected spaces with dimension at most $2m$,
- H-spaces modulo the class of finite groups,
- spaces rationally homotopy equivalent to products of Eilenberg-MacLane spaces.

Main result

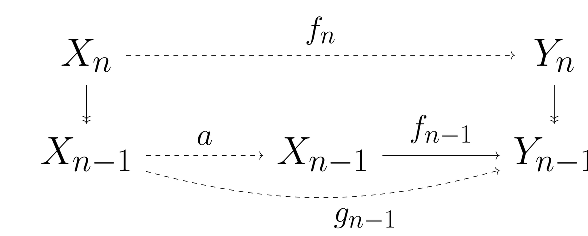
Let X and Y be simply connected finite simplicial set of dimension d. Suppose that they are of finite k-type through dimension d. Then the question of whether X and Y are homotopy equivalent is algorithmically decidable. If the spaces are homotopy equivalent, we can find a homotopy equivalence $f: X_d \to Y_d$ between their Postnikov stages in dimension d.

Proof of the main result - 1. phase: Pre-processing

- Compute effective Postnikov towers for both sets X and Y according to the instructions in [2, Section 4].
- That algorithm provides suitable isomorphism types $\sigma: \pi_n(X) \to \pi_n(Y)$ of homotopy groups.
- If the homotopy groups $\pi_n(X)$ and $\pi_n(Y)$ have different isomorphism types, then $X \not\simeq Y$.
- Using Postnikov classes, check that both spaces are of finite k-type through dimension d.
- Compute inductively generators of $\mathrm{Aut}^{\mathrm{ef}}(X_n) = \mathrm{Aut}(X_n)$ for $n \leq d$ via the algorithm:
  Input: A finite set of generators of $\mathrm{Aut}^{\mathrm{ef}}(X_{n-1})$ and $[\kappa^X_{n-1}] \in \mathrm{Tor}(H^{n+1}(X_{n-1}; \pi_n(X)))$.
  Output: A finite set of generators of the group $\mathrm{Aut}^{\mathrm{ef}}(X_n)$.

Proof of the main result - 2. phase: Processing

Base step. Since $X_1 = Y_1$ is a point, so $\mathrm{iso}^{\mathrm{ef}}(X_1, Y_1) = \{\mathrm{id}\}$.
For each $2 \leq n \leq d$ do:
Induction step. Find a certain lift $f_n \in \mathrm{iso}^{\mathrm{ef}}(X_n, Y_n)$ of $f_{n-1} \in \mathrm{iso}^{\mathrm{ef}}(X_{n-1}, Y_{n-1})$ using methods:



- Seek $a \in \mathrm{aut}^{\mathrm{ef}}(X_{n-1})$ such that g_{n-1} satisfies $\gamma_*[\kappa^X_{n-1}] = f^*_{n-1}[\kappa^Y_{n-1}]$ by the algorithm (A).
- Update all f_i to the maps $f_i \circ a_i$ for each $2 \leq i \leq n-1$ if the required a exists.

Algorithm A.
Input: $[\kappa^X_{n-1}] \in \mathrm{Tor}(H^{n+1}(X_{n-1}; \pi_n(X)))$, $[\kappa^Y_{n-1}] \in \mathrm{Tor}(H^{n+1}(Y_{n-1}; \pi_n(Y)))$ and $f_{n-1} \in \mathrm{iso}^{\mathrm{ef}}(X_{n-1}, Y_{n-1})$.
Output: A decision whether $X_n \simeq Y_n$. If yes, the algorithm provides a homotopy equivalence $f_n \in \mathrm{iso}^{\mathrm{ef}}(X_n, Y_n)$. If no, then $X \not\simeq Y$.

- Find if there is $(a, \alpha) \in \mathrm{Aut}^{\mathrm{ef}}(X_{n-1}) \times \mathrm{Aut}(\pi_n(Y)) := G$ such that
$$\sigma_*[\kappa^X_{n-1}] = \alpha_*^{-1} a^* f^*_{n-1}[\kappa^Y_{n-1}].$$
- Consider the action $\mathrm{Tor}(H^{n+1}(X_{n-1}; \pi_n(Y))) \times G \to \mathrm{Tor}(H^{n+1}(X_{n-1}; \pi_n(Y)))$
$$([\kappa], [g], \gamma) \longmapsto \gamma_*^{-1} g^*[\kappa].$$
- Apply the Orbit-Stabilizer algorithm and compute the orbit of the element $f^*_{n-1}[\kappa^Y_{n-1}]$.
- Go through the finite list of the orbit elements and decide if $\sigma_*[\kappa^X_{n-1}] \in \left(f^*_{n-1}[\kappa^Y_{n-1}]\right)^G$.
- If it is missing in the list, then $X_n \not\simeq Y_n$. It implies that $X \not\simeq Y$.
- If $\sigma_*[\kappa^X_{n-1}] \in \left(f^*_{n-1}[\kappa^Y_{n-1}]\right)^G$ then take $\log(\sigma_*[\kappa^X_{n-1}]) = ([a], \alpha)$ and f_n is a lift of $g_{n-1} = f_{n-1}a$.

Corollary

Let X and Y be finite simplicial sets. Then the question, if X and Y are stably homotopy equivalent, is algorithmically decidable.
Proof:

- X and Y of finite dimension are stably homotopy equivalent if $\exists r \in \mathbb{N}: S^r X \simeq S^r Y$.
- The dimensions of multiple suspensions $S^r X$ and $S^r Y$ of X and Y are less than twice their connectivity for r sufficiently large.
- Both $S^r X$ and $S^r Y$ are of finite k-type through their dimension.
- The previous algorithm decides if $S^r X \simeq S^r Y$, i.e., if X and Y are stably homotopy equivalent.

References

[1] E. H. Brown. Finite computability of Postnikov complexes. *The Annals of Mathematics*, 65(1):1, January 1957.

[2] M. Čadek, M. Krčál, J. Matoušek, L. Vokřínek, and U. Wagner. Polynomial-time computation of homotopy groups and Postnikov systems in fixed dimension. *SIAM Journal on Computing*, 43(5):1728–1780, January 2014.

[3] A. Nabutovsky and S. Weinberger. Algorithmic aspects of homeomorphism problems. Contemporary mathematics - American Mathematical Society, pages 245–250. American Mathematical Society, 1999.

[4] J. Rubio and F. Sergeraert. Constructive homological algebra and applications. *arXiv1208.3816*, pages 1–154, 2012.