# Killing Tensors in Koutras–McIntosh Spacetimes

Boris Kruglikov and Wijnand Steneker

The Koutras–McIntosh family of metrics include conformally flat pp-waves and the Wils metric. It appeared in their paper of 1996 as an example of a spacetime without scalar curvature invariants and infinitesimal symmetries. Here we demonstrate that these metrics have no "hidden symmetries", by which we mean Killing tensors of low degrees. The technique we use is the geometric theory of overdetermined PDEs and the Cartan prolongation-projection method. Application of those allows to prove non-existence of polynomial in momenta integrals for the equation of geodesics in a mathematical rigorous way. Using the same technique we can classify all lower degree Killing tensors and prove in several examples of pp-waves that all Killing tensors of degree 3 and 4 are reducible.

## Koutras–McIntosh Spacetimes

The following is the Koutras–McIntosh family of spacetimes for $(a,b)\neq(0,0)$:

$$g = 2(ax+b)\,du\,dw - 2aw\,dx\,du + \big(f(u)(ax+b)(x^2+y^2) - a^2w^2\big)\,du^2 - dx^2 - dy^2. \tag{0.1}$$

These metrics were shown in [1] to possess neither invariants nor symmetries. The first property means that all polynomial curvature invariants, i.e., complete contractions of tensor products of the Riemann tensor and its covariant derivatives $\nabla_{i_1}\cdots\nabla_{i_s}R_{abcd}$, vanish and so cannot be used to distinguished $g$ from the Minkowski metric. The second property above means there are no Killing vectors, or linear integrals, for (0.1). In the paper we show that it also does not possess "hidden symmetries" (i.e., Killing tensors of low degrees).

## Hamiltonian formalism

The energy function $H=\frac{1}{2}\|p\|_g^2$ of the geodesic flow writes in local coordinates

$$H(x,p)=\frac{1}{2}g^{ij}(x)p_ip_j \qquad [g^{ij}]=[g_{ij}]^{-1}.$$

It is well-known that geodesics of $g$ are projections to the base $M$ of trajectories of the corresponding Hamiltonian vector field $X_H=\omega^{-1}dH$ on $T^*M\overset{g}{\simeq}TM$.

Integrating the equations of geodesics requires conserved quantities for this Hamiltonian system. A function $I:T^*M\to\mathbb{R}$ is an *integral* (of motion) $X_H(I)=0$ if it Poisson commutes with the Hamiltonian:

$$\{H,I\}=\sum_{i=1}^{n}\left(\frac{\partial H}{\partial p_i}\frac{\partial I}{\partial x^i}-\frac{\partial H}{\partial x^i}\frac{\partial I}{\partial p_i}\right)=0.$$

By Noether's theorem, a Killing field $X=X^i(x)\partial_{x^i}$ corresponds to a *linear in momenta* integral of motion $I(x,p)=\langle p,X\rangle=X^i(x)p_i$.

More generally, a *Killing tensor* of degree $d$ corresponds to a *homogeneous in momenta* polynomial

$$I_d := a^{i_1\cdots i_d}(x)\,p_{i_1}\cdots p_{i_d}, \tag{0.2}$$

which Poisson commutes with $H$, and is thus a polynomial integral.

## Geometric Theory of PDEs

### I: Killing Equation as a PDE

Since the Hamiltonian is quadratic in momenta, for any (0.2) the Poisson bracket $\{H,I_d\}$ is of degree $d+1$ in momenta. Consequently, Killing $d$-tensors correspond to solutions of a system of differential equations formed by vanishing of $p$-coefficients of the Poisson bracket, which we call the *Killing equation*,

$$\mathcal{E}_d := \{F=0 : F\in \mathrm{coeffs}_p(\{H,I_d\})\}. \tag{0.3}$$

This is an overdetermined system of linear first order PDEs on the coefficients $a^{i_1\cdots i_d}(x)$ of the Killing tensor. Actually, there are $\binom{n+d}{d+1}$ equations on $\binom{n+d-1}{d}$ unknown functions. Denote solutions to this system – the linear space of all Killing $d$-tensors – by $K_d$.

### II: Jet Formalism

The notion of jet-space formalizes the computational device of truncated Taylor polynomials. If $x^i$ are local coordinates on $N$ then the jet-space $J^kN$ of $k$-jet of functions $u:N\to\mathbb{R}$ has local coordinates $(x^i,u_\sigma)$ for multi-indices $\sigma=(i_1,\dots,i_n)$, $i_s\geq 0$, $|\sigma|=\sum i_s\leq k$. The space of $k$-jets of maps $u:\mathbb{R}^n\to\mathbb{R}^m$ will be denoted by $J^k(n,m)$. Any map $u=(u^j):\mathbb{R}^n\to\mathbb{R}^m$ lifts to the jet-section $j^ku:\mathbb{R}^n\to J^k(n,m)$ given by $x^i\mapsto u^j_\sigma=\partial u^j(x)/\partial x^\sigma$.

**Definition 1 (Geometric PDE).** A partial differential equation of order $k$ is a submanifold $\mathcal{E}\subseteq J^k(n,m)$. A *solution* of the PDE is defined to be a function $u:\mathbb{R}^n\to\mathbb{R}^m$ such that its $k$-jet $j^ku$ takes values in $\mathcal{E}$. We denote by $\mathrm{Sol}(\mathcal{E})$ the space of all (local) solutions of $\mathcal{E}$.

Elements of a $k$'th order geometric PDE $\mathcal{E}\subseteq J^k(n,m)$ are solutions *up to order* $k$ (at a point).

### III: Prolongation

To find the solutions of the PDE $\mathcal{E}=\{F(x^i,u^j_\sigma)=0\}$ up to order $k+1$ and higher, we have to differentiate the defining equations. To encode the chain rule, we define the $q$'th *total derivative* of $F:J^k\to\mathbb{R}^s$ to be a vector-function on $J^{k+1}$ given by

$$D_qF := \frac{\partial F}{\partial x^q}+\sum_{j=1}^{m}\sum_{|\sigma|\leq k}\frac{\partial F}{\partial u^j_\sigma}\cdot u^j_{\sigma+1_q}. \tag{0.4}$$

(Here we use the notation $\sigma+1_q$ for the multi-index obtained by adding 1 to the $q$'th entry of $\sigma$.) Now, a point $(x^i,u^j_\sigma)\in J^{k+1}$ is said to be a solution of $\mathcal{E}$ up to order $k+1$ if it satisfies the following system of equations:

$$\mathcal{E}^{(1)} := \Big\{F(x^i,u^j_\sigma)=0,\ (D_qF)(x^i,u^j_\sigma)=0\ \forall\, q=1,\dots,n\Big\}.$$

The resulting system of equations is called the *first prolongation* of $\mathcal{E}$. By construction, a solution of the prolongation $\mathcal{E}^{(1)}$ is still a solution of $\mathcal{E}$. We inductively define the $l$'th prolongation by $\mathcal{E}^{(l)}=(\mathcal{E}^{(l-1)})^{(1)}\subset J^{k+l}$. It corresponds to solutions *up to order* $k+l$ (at a point).

**Definition 2 (Finite Type).** A PDE $\mathcal{E}\subseteq J^k(n,m)$ is called of *finite type* $l$ if after $l$ prolongations all the highest order derivatives of the dependent variables can be expressed algebraically in terms of the lower order derivatives. A PDE is called of *Frobenius type* if it is of finite type $0$.

**Theorem 3 (Killing PDE is of Finite Type).** *The PDE $\mathcal{E}_d$ defining a Killing $d$-tensor is a first order linear PDE of finite type $d$ with $\mathrm{Sol}(\mathcal{E}_d)=K_d$. This equation and its prolongations possess no compatibility conditions before achieving Frobenius type.*

### IV: Projection

Let $\mathcal{E}=\{F(x^i,u^j_\sigma)=0\}\subseteq J^k(n,m)$ be a PDE of order $k$. Its solution up to order $k$ can be extended to order $(k+l)$ if and only if it belongs to the projection of the prolongation $\pi_{k+l,k}(\mathcal{E}^{(l)})\subseteq\mathcal{E}$, where $\pi_{k+l,k}(j^{k+l}u):=j^ku$. In the case of equality here, every $k$-jet solution can be extended to a $(k+l)$-jet solution. In the opposite case, there is a linear combination of iterated total derivatives up to order $l$, $\square(F)=\sum_{|\tau|\leq l}a^\tau D_\tau F$, which has order $k$.

**Definition 4 (Compatibility).** A *compatibility condition* of $\mathcal{E}$ is an equation defining $\pi_{k+l,k}(\mathcal{E})$ that is algebraically independent of $F$ and that is satisfied by all formal solutions.

Associated to a PDE is the Cartan distribution. Solutions arise as integral manifolds of this distribution. Therefore, in the PDE setting, Frobenius theorem implies:

**Theorem 5 (Frobenius Theorem).** *Solutions of a PDE $\mathcal{E}\subseteq J^k(n,m)$ of finite type $l$ are determined uniquely by their $(k+l-1)$-jets. If in addition $\mathcal{E}$ has no compatibility conditions, then for every $\xi\in\mathcal{E}^{(l)}$ there exists a local solution $u\in\mathrm{Sol}(\mathcal{E})$ satisfying $j^{k+l}_xu=\xi$.*

## Algorithmic implementation

The above criterion allows for an effective implementation of evaluation of $\dim K_d$ for a given metric $g$ using computer algebra systems. The Killing PDE $\mathcal{E}$ as well as its prolongations $\mathcal{E}^{(k)}$ are linear in $(k+1)$-jets of the dependent variables. We convert this linear system of equations to a matrix-valued function $M_k(x)$ on the spacetime. For our class of metrics $g$ the entries are polynomial with rational coefficients. Hence to make use of computer algebra software, we insert a *rational* point $x_0\in M$ to obtain a matrix with *rational* coefficients (in this case computer calculations are exact!).

**Algorithm. (Cartan's Prolongation Method for Geodesic Flow)**.
(**Input**: A nonnegative integer $d$, a point $x_0$.)

- Step 1.) Compute the Poisson bracket $\{H,I_d\}$ of a polynomial in momenta $p$ function $I_d$ with the Hamiltonian.
- Step 2.) Collect the coefficients of $\{H,I_d\}$ with respect to the momentum variables. Define the first order linear PDE $\mathcal{E}:=\{F=0:F\in\mathrm{coeffs}_p(\{H,I_d\})\}$.
- Step 3.) Set $k:=0$.
  - Convert the linear system of equations $\mathcal{E}^{(k)}$ w.r.t. the variables $\mathcal{V}_{k+1,d}:=\{a^{i_1\cdots i_d}_\alpha:|\alpha|\leq k+1\}$ into a matrix $M_k(x)$ that depends on the $x$-coordinates.
  - Substitute $x_0$ to obtain a matrix $M_k:=M_k(x_0)$, the $k$'th *prolongation matrix*.
  - Set $\delta_k:=\mathrm{columns}(M_k)-\mathrm{rank}(M_k)$.

  If $(k\leq d)$ or $(k>d$ and $\delta_k\neq\delta_{k-1})$, increase $k$ by $1$ and repeat Step 3.
- Step 4.) Return $(\delta_k,k)$.

(**Output**: The dimension of the space of Killing $d$-tensors is $\dim K_d=\delta_k$. The integer $k$ indicates the number of prolongations necessary to find all compatibility conditions of $\mathcal{E}$.)

*Proof.* (Sketch correctness algorithm). Rows of the matrix $M_k$ represent equations defining $\mathcal{E}_d^{(k)}$, so they consist of the original Killing PDE, their differential corollaries and compatibility conditions. Consequently, $\delta_k$ is number of free jets (coordinates on fibers of the equation $\mathcal{E}_d\to N$). In view of the Frobenius theorem, each free variable corresponds to a $(k+1)$ jet-solution of the Killing PDE. The first condition $(k\leq d)$ addresses whether the prolongation has achieved Frobenius type, see Theorem 3. The second part $(k>d$ and $\delta_k\neq\delta_{k-1})$ checks whether all compatibility conditions have been computed. □

### Improving computability of the algorithm.

- (**Exploiting Sparsity**.) The prolongation matrices $M_k$ that we encounter here are sparse. We generate a matrix with only zeroes and then substitute the nonzero values.
- (**LinBox**). The LinBox package in Sage allows for fast rank computations of large sparse integer matrices.
- (**Combinatorial Description of Prolongations**.) We have that $\{H,I_d\}$ is of degree $d+1$ in momenta. Given a multi-index $\tau$ of length $d+1$, we compute the $p^\tau$-coefficient in terms of the coefficients $a^{i_1\cdots i_d}(x)$ of $I_d$. Using the multi-index Leibniz rule, we can then determine the general expression for the derivative $\partial^\alpha(\mathrm{coeff}_\tau(\{H,I_d\}))$, where $\alpha$ is a multi-index. We obtain the equations of the prolongation as a function of the multi-indices $\tau$ and $\alpha$. This combinatorial description significantly reduces generation time in Maple.

## Conformally Flat pp-Waves

These are given by the following formula:

$$g=2dx^3dx^4+\big(f(x^3)((x^1)^2+(x^2)^2)\big)(dx^3)^2-(dx^1)^2-(dx^2)^2 \tag{0.5}$$

Sippel and Goenner classified pp-waves in terms of their isometry groups. For conformally flat pp-waves there are three classes: $f(x^3)=c$, $f(x^3)=c(x^3)^{-2}$ and the generic case with $\dim K_1=6$. We apply our prolongation-projection algorithm to the following four metrics (rescaling of $f$ does not play a role for the first three metrics):

$$\text{(i) } f(x^3)=1,\quad \text{(ii) } f(x^3)=x^3,\quad \text{(iii) } f(x^3)=(x^3)^2,\quad \text{(iv) } f(x^3)=2(x^3)^{-2}.$$

If two subsequent values $\delta_k$, $\delta_{k+1}$ are equal (with $k\geq d$), the sequence of $\delta$-values stabilizes and we can read off the number of Killing $d$-tensors. In the table this is shown by circling this $\delta$-value.

| Linear | $\mathcal{E}$ | $\mathcal{E}^{(1)}$ | $\mathcal{E}^{(2)}$ | $\mathcal{E}^{(3)}$ | $\mathcal{E}^{(3)}$ |
|---|---|---|---|---|---|
| $\delta$ | 10 | 10 | 7 | 7 | … |
| Quadratic | $\mathcal{E}$ | … | $\mathcal{E}^{(4)}$ | $\mathcal{E}^{(5)}$ | $\mathcal{E}^{(6)}$ |
| $\delta$ | 30 | … | 29 | 28 | 28 |
| Cubic | $\mathcal{E}$ | … | $\mathcal{E}^{(6)}$ | $\mathcal{E}^{(7)}$ | $\mathcal{E}^{(8)}$ |
| $\delta$ | 65 | … | 87 | 84 | 84 |
| Quartic | $\mathcal{E}$ | … | $\mathcal{E}^{(10)}$ | $\mathcal{E}^{(11)}$ | $\mathcal{E}^{(12)}$ |
| $\delta$ | 119 | … | 211 | 210 | 210 |

| Linear | $\mathcal{E}$ | $\mathcal{E}^{(1)}$ | $\mathcal{E}^{(2)}$ | $\mathcal{E}^{(3)}$ | $\mathcal{E}^{(3)}$ |
|---|---|---|---|---|---|
| $\delta$ | 10 | 10 | 7 | 6 | 6 |
| Quadratic | $\mathcal{E}$ | … | $\mathcal{E}^{(5)}$ | $\mathcal{E}^{(6)}$ | $\mathcal{E}^{(7)}$ |
| $\delta$ | 30 | … | 24 | 22 | 22 |
| Cubic | $\mathcal{E}$ | … | $\mathcal{E}^{(11)}$ | $\mathcal{E}^{(12)}$ | $\mathcal{E}^{(13)}$ |
| $\delta$ | 65 | … | 63 | 62 | 62 |
| Quartic | $\mathcal{E}$ | … | $\mathcal{E}^{(17)}$ | $\mathcal{E}^{(18)}$ | $\mathcal{E}^{(19)}$ |
| $\delta$ | 119 | … | 150 | 148 | 148 |

Table 1: Metrics 2,3. Metrics 1,4.

The number of Killing $d$-tensors include those which are reducible, i.e., can be written as a linear combination of products of lower order Killing tensors. A variation on Cartan's algorithm computes the relations among Killing tensors. Counting these allows us to compute the number of irreducible Killing tensors. For conformally flat pp-waves, we can obtain:

**Theorem 6.** *For a generic conformally flat pp-wave* (0.5) *all* 3- *and* 4- *Killing tensors are combinations of Killing vectors, the Hamiltonian and an irreducible Killing 2-tensor.*

## Absence of Killing Tensors for the Wils Metric

The Wils metric is given by

$$g=2x^1dx^3dx^4-2x^4dx^1dx^3+\big(f(x^3)x^1((x^1)^2+(x^2)^2)-(x^4)^2\big)(dx^3)^2-(dx^1)^2-(dx^2)^2. \tag{0.6}$$

Application of Cartan's algorithm yields:

**Theorem 7.** *The Wils metric* (0.6) *for $f(u)=u^m$, $m=0,1,2$, admits no Killing tensors up to degree 6 except for powers of the Hamiltonian.*

This also implies that for generic values of the functional parameter $f$ there are no lower degree Killing tensors. Now we want to be more specific on those exceptional parameters.

**Theorem 8.** *The Wils metric generically admits no Killing fields and Killing 2-tensors. It admits a Killing field if and only if $f$ is of the form*

$$f(x^3)=(c_0+c_1x^3+c_2(x^3))^{-2}. \tag{0.7}$$

*for some constants $c_1\neq 0, c_2, c_3$. The only Killing 2-tensor is given by the square of the Killing vector.*


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### References


[1] A. Koutras and C. McIntosh "A metric with no symmetries or invariants".


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