Conformal Killing trajectories on conformal Riemannian sphere

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We study flows of conformal Killing fields and we ask whether they are conformal circles or helices.

➪ *Conformal Riemannian structure* on a smooth manifold M of dimension n is a class of Riemannian metrics that differ by a multiple of an everywhere positive function.

➪ *Conformal Killing fields* are infinitesimal symmetries of the conformal structure.

➪ *Model conformal manifold* admitting the most conformal symmetries is the *sphere* S^n with the conformal class represented by the standard round metric.

➜ The Lie algebra of conformal Killing fields is isomorphic to

$$\mathfrak{g} = \mathfrak{so}(n+1,1)$$

and for general element we write

$$K = \begin{pmatrix} a & Z & 0 \\ X & A & -Z^t \\ 0 & -X^t & -a \end{pmatrix},$$

where $X, Z^t \in \mathbb{R}^n$, $a \in \mathbb{R}$ and $A \in \mathfrak{so}(n)$. In this notation, the grading of $\mathfrak{g}$ is according to the diagonal, where $\mathfrak{g}_{-1} \simeq \mathbb{R}^n$ is below the diagonal.

➜ As a homogeneous space, $S^n \cong G/P$, where we consider $G = O(n+1,1)$ as the principal group and $P \subset G$ its Poincaré subgroup.

➜ Each $K \in \mathfrak{so}(n+1,1)$ corresponds to a section of adjoint tractor bundle

$$\boldsymbol{K} = \mathrm{Ad}_{g^{-1}}(X), \quad g \in SO(n+1,1)$$

We focus on flows of conformal Killing fields.

➜ We consider key conformal invariants associated to a flow of a conformal Killing field as follows:

$$\alpha_1 = -\frac{\operatorname{proj} \boldsymbol{K}^4}{\operatorname{proj} \boldsymbol{K}^2}, \quad \alpha_2 = \frac{\operatorname{proj} \boldsymbol{K}^6}{\operatorname{proj} \boldsymbol{K}^2}, \quad \alpha_3 = -\frac{\operatorname{proj} \boldsymbol{K}^8}{\operatorname{proj} \boldsymbol{K}^2}, \quad \ldots.$$

Here proj denotes the projection to left-bottom position in the matrix (as K^{2k} can be viewed as an element of $S^2\mathbb{R}^{n+2}$).

⇛ Note that the first non-trivial quantity changes under a reparametrization of the curve.

⇛ Its vanishing determines a preferred family of parametrizations with freedom given by the projective group of the line (projective parameter).

Consider Hankel matrices

$$H_r^{odd} := \begin{pmatrix} 0 & -1 & \alpha_1 & & \\ -1 & \alpha_1 & -\alpha_2 & & \\ \alpha_1 & -\alpha_2 & \alpha_3 & & \\ & & & \ddots & \\ & & & & \alpha_{2r-3} \end{pmatrix},$$

$$H_r^{even} := \begin{pmatrix} 1 & -\alpha_1 & \alpha_2 & & \\ -\alpha_1 & \alpha_2 & -\alpha_3 & & \\ \alpha_2 & -\alpha_3 & \alpha_4 & & \\ & & & \ddots & \\ & & & & \alpha_{2r-2} \end{pmatrix}$$

and denote

$$h_r^{odd} := \det(H_r^{odd}), \quad h_r^{even} := \det(H_r^{even}).$$

➜ There is a sequence of natural relative conformal invariants of the curve given by

$$\Delta_j = \begin{cases} h_i^{odd} \cdot h_i^{even}, & \text{for } j = 2i, \\ h_{i+1}^{odd} \cdot h_i^{even}, & \text{for } j = 2i+1. \end{cases}$$

➜ They then provide the absolute conformal invariants, the *conformal curvatures* of the curve, by

$$\kappa_i = \begin{cases} -\frac{1}{2}(-\Delta_4)^{-\frac{5}{2}}\big(\alpha_1 \Delta_4{}^2 - \frac{1}{2}\Delta_4\Delta_4'' + \frac{9}{16}\Delta_4'^2\big), & \text{for } i = 1, \\ -(-\Delta_4)^{-\frac{1}{4}}\,(\Delta_{i+1}\Delta_{i+3})^{\frac{1}{2}}\,\Delta_{i+2}{}^{-1}, & \text{for } i = 2, \dots, n-1. \end{cases}$$

These invariants allow us to recognize nicely conformal circles and helices.

- ➪ *conformal circles* … all conformal curvatures vanish,
- ➪ *conformal helices* … the first conformal curvature κ_1 is constant and all others vanish.

➜ The behavior of invariants along all flows depends only on the Adjoint G–orbit of $\boldsymbol{K}$.

➜ These orbits can be classified in terms of normal forms, i.e. uniquely chosen representatives of non-zero adjoint orbits in $\mathfrak{g}$, according to [2, items 13–16 in table III].

➜ The number of orbits grows according to the dimension.

➜ We illustrate the principles in dimension 3 and 4.

Dimension 3

The normal representatives take form as follows, where where $a, b, c > 0$

$$K_1 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot \end{pmatrix}, \qquad K_2 = \begin{pmatrix} a & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -a \end{pmatrix},$$

$$K_3 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -b & \cdot & \cdot \\ \cdot & b & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \qquad K_4 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -b & \cdot \\ \cdot & \cdot & b & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot \end{pmatrix},$$

$$K_5 = \begin{pmatrix} a & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -b & \cdot & \cdot \\ \cdot & b & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -a \end{pmatrix}, \quad K_6 = \begin{pmatrix} \cdot & -c & \cdot & \cdot & \cdot \\ c & \cdot & \cdot & \cdot & c \\ \cdot & \cdot & \cdot & -b & \cdot \\ \cdot & \cdot & b & \cdot & \cdot \\ \cdot & -c & \cdot & \cdot & \cdot \end{pmatrix}.$$

In the following summary, invariants α start by first in order and Δ start by fourth in order (first non–trivial), and $*$ denotes that the first one is non-constant.

type	α's	Δ's	zeros	remark
K_1	0	0	1 point	translation
K_2	$-a^2, a^4, -a^6, \dots$	0	2 points	homothety
K_3	$b^2, b^4, b^6, \dots$	0	line	rotation
K_6	$c^2, c^4, c^6, \dots$	0	none	$b = \frac{c}{\sqrt{2}}$
K_4	$*$	$*$	1 point	
K_5	$*$	$*$	2 points	
K_6	$*$	$*$	none	$b \neq \frac{c}{\sqrt{2}}$

➜ Let us emphasize that there is no element for which $\Delta_4 \neq 0$ and $\Delta_5 = 0$ everywhere.

➜ Note that orbits in the first part have vanishing κ_1 and orbits in the second part have non-constant κ_1.

Dimension 4

➜ All orbits from above table appear, one only has to add a zero row and column to an appropriate position.

There is an additional orbit, where where $a, b, c > 0$

$$K_7 = \begin{pmatrix} a & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -b & \cdot & \cdot & \cdot \\ \cdot & b & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -c & \cdot \\ \cdot & \cdot & \cdot & c & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -a \end{pmatrix}$$

➜ Invariants α, Δ, κ for this orbit are non-constant for all possible a, b, c except the case $b = c$.

➜ In the case $b = c$, invariants take form

$$\alpha_1 = -(a-c)(a+c),$$
$$\alpha_2 = a^4 - a^2c^2 + c^4,$$
$$\alpha_3 = -(a-c)(a+c)(a^4 + c^4),$$
$$\dots$$
$$\Delta_4 = -a^2c^2,$$
$$\Delta_5 = 0,$$
$$\dots$$
$$\kappa_1 = \frac{(a-c)^2(a+c)^2}{4a^2c^2},$$
$$\kappa_2 = 0.$$

Theorem 1.

- ➡ *On the conformal sphere of dimension 3, all flow lines of Killing fields are conformal circles for K_1, K_2, K_3 and K_6 for $b = \frac{c}{\sqrt{2}}$ and there are no Killing fields with all flows conformal spirals.*
- ➡ *On the conformal sphere of dimension 4, all flow lines of Killing fields are conformal circles for K_1, K_2, K_3 and K_6 for $b = \frac{c}{\sqrt{2}}$ and flow lines of Killing fields are conformal spirals for K_7 for $b = c$.*

References

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