Variational characterization of circles in low dimensions

Josef Šilhan (joint with M. Dafinger)

Masaryk University, Brno, Czech Republic

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Plan

— Variational problem for circles in $\mathbb{R}^n$

— Inverse problem and its solution in dim 3

Variational problem for circles in $\mathbb{R}^n$

Inverse problem and its solution in dim 3

— Let $X(t) : I \to \mathbb{R}^n$ be a parametrized curve on some interval I. Its velocity vector, acceleration vector etc. are denoted by

$$U = X' = \frac{d}{dt}X, \quad U' = X'' = \frac{d^2}{dt^2}X, \quad U'' = \frac{d^3}{dt^3}X$$

where $(\ )' = \frac{d}{dt} = U^r \nabla_r(\ )$. Here $\nabla_a = \frac{\partial}{\partial x^a}$ is the Levi-Civita connection of the Euclidean metric.

— Variational approach: given a Lagrangian

$$\mathcal{L} = \mathcal{L}(t, X, U, U', \ldots, U^{(k)}) : I \to \mathbb{R},$$

find extremal curves – or just critical curves – of the integral

$$\int_{t_1}^{t_2} \mathcal{L}\, dt, \quad I = [t_1, t_2]$$

among all curves with fixed endpoints $X(t_i)$ and derived vectors $U^{(j)}(t_i)$ at endpoints, $0 \le j \le k$.

— If the integral $\int_{t_1}^{t_2} \mathcal{L}\, dt$ is independent on reparametrization, then any reparametrization of a critical curve is again critical.

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— If the integral $\int_{t_1}^{t_2} \mathcal{L}\, dt$ is independent on reparametrization, then any reparametrization of a critical curve is again critical.

— Which families of curves are *variational* in the sense that they form the family of critical curves of suitable $\mathcal{L}$? And what is the lowest order for such $\mathcal{L}$?

- This is generally a difficult question
- Classical example – straight lines are variational:
  - $\mathcal{L} = \langle U, U \rangle \rightsquigarrow$ energy functional (not parametrization invariant)
  - $\mathcal{L} = \langle U, U \rangle^{1/2} \rightsquigarrow$ length functional (parametrization invariant)

— Our problem: is the family of all circles – or of all conformal circles – variational? If so, what is the lowest order Lagrangian?

- find the required Lagrangian (optimally parametrization invariant)
- or show its nonexistence

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A suprising answer in dimension 3

— The usual torsion τ of curves (more or less) solves the problem on $\mathbb{R}^3$ with parametrizarion independent Lagrangian,

$$\mathcal{L} = \tau(U, U', U'')\langle U, U\rangle^{1/2} = \frac{\epsilon(U, U' U'')}{G(U, U')}\langle U, U\rangle^{1/2}$$

where $G(U, U')$ is the Gramm matrix of U and U':

- critical curves are exactly all circles in $\mathbb{R}^3$ (but not straight lines)
- a geometrical explanation is unclear
- N. Thamwattana, J.A. McCoy, J.M. Hill. *Energy density functions for protein structures*, Q. J. Mech. Appl. Math. 61(3):431451 (2008).

— But is this the lowest order Lagrangian for circles in $\mathbb{R}^3$?

- No.

Differential equations for circles

— Circles are characterized by constant curvatuture κ and zero torsion τ. Including also straight lines, we have two ODEs

$$\kappa = \left(\frac{G(U,U')}{\langle U,U\rangle^3}\right)^{\frac{1}{2}} \in \mathbb{R} \quad \text{and} \quad U'' \in \text{ span}\{U, U'\}.$$

Put $u = \sqrt{\langle U, U\rangle}$. Expanding the latter condition, we obtain the system of n ODEs

$$\underbrace{U'' + u^{-2}\left[-\langle U, U''\rangle + 3u^{-2}\langle U, U'\rangle^2\right] U - 3u^{-2}\langle U, U'\rangle U'}_{E(U,U',U'')} = 0.$$

— There are many other systems of ODEs $\overline{E}_a = 0$ characterizing **(conformal) parameter-independent circles**. BUT the system $E_a = 0$ is in a suitable sense *nondegeretare* $\rightsquigarrow$ $\overline{E}_a = 0$ has the form

$$\overline{E}_a = M^r{}_a E_r \quad \text{for a } \underline{\text{multiplier}} \quad M^b{}_a = M^b{}_a(t, X, U, U', U'').$$

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Inverse problem and its solution in dim 3

Variational equations for circles

— Variationality of the system $\overline{E}_a = 0$ means it is the Euler-Lagrange equation of a Lagrangian

$$\mathcal{L} = \mathcal{L}(t, X, U, U', U''),$$

i.e. it has the form

$$\overline{E}_a = \frac{\partial \mathcal{L}}{\partial X_a} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial U_a} + \frac{d^2}{dt^2}\frac{\partial \mathcal{L}}{\partial U'_a} - \frac{d^3}{dt^3}\frac{\partial \mathcal{L}}{\partial U''_a} = 0.$$

— Equations of this form are characterized by Helmholtz conditions

$$\frac{\partial \overline{E}_a}{\partial X^b} - \frac{\partial \overline{E}_b}{\partial X^a} - \frac{1}{2}\frac{d}{dt}\Big(\frac{\partial \overline{E}_a}{\partial U^b} - \frac{\partial \overline{E}_b}{\partial U^a}\Big) + \frac{1}{4}\frac{d^3}{dt^3}\Big(\frac{\partial \overline{E}_a}{\partial U''^b} - \frac{\partial \overline{E}_b}{\partial U''^a}\Big) = 0,$$

$$\frac{\partial \overline{E}_a}{\partial U^b} + \frac{\partial \overline{E}_b}{\partial U^a} - \frac{d}{dt}\Big(\frac{\partial \overline{E}_a}{\partial U'^b} + \frac{\partial \overline{E}_b}{\partial U'^a}\Big) = 0,$$

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Helmholtz conditions in low dimensions

— We need to find (or to prove nonexistence) of a multiplier $M^b{}_a$ such that $\overline{E}_a = M^r{}_a E_r$ satisfies Helmholtz conditions. This is generally extremely difficult.

— Our setup: assume $M^b{}_a = M^b{}_a(t, X, U, U')$. Then the simplest Helmholtz condition simplifies the problem significantly in low dimensions:

- Dimension $n = 2$: such multpilier does not exist $\rightsquigarrow$ (conformal) circles in the plane are not variational as unparametrized curves.
- Dimension $n = 3$: the multiplier has the form

$$M^b{}_a = \varphi U^r \epsilon_r{}^b{}_a, \quad \varphi = \varphi(t, X, U, U')$$

  where ϵ_{abc} is the Euclidean volume form on $\mathbb{R}^3$. It remains to determine the function φ.
- Dimension $n \geq 4$: unclear.

— Remaining Helmholtz cond's for $n = 3 \rightsquigarrow \varphi$ is a constant.

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Lagrangian for (conformal) circles in $\mathbb{R}^3$ I.

— We found the variational system of ODEs

$$\overline{E}_a = \epsilon(U, U'', \_)_a - 3u^2 \langle U, U' \rangle \epsilon(U, U', \_)_a = 0$$

for (conformal) unparametrized circles in $\mathbb{R}^3$.

— What is the corresponding Lagrangian $\mathcal{L} = \mathcal{L}(t, X, U, U')$? The answer starts with the Vainberg-Tonti formula

$$\text{Lagrangian} := \int_0^1 \overline{E}(sU + V, sU', sU'')_a X^a ds$$

for any $V \in \mathbb{R}^3$.

- the case $V = 0$ is the usual Vainberg-Tonti formula but this integral does not exist (it is infinite).
- we can modify the Lagrangian by total derivative term
- then the integration is managable by hand
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Lagrangian for (conformal) circles in $\mathbb{R}^3$ II.

— Assume $||V|| = 1$. After integration and further manipulation, we obtain

$$\mathcal{L} := \frac{\epsilon(V,U,U')}{||U||\big(||U|| + \langle U,V\rangle\big)}.$$

One can directly verify that solutions of the EL-equations are exactly conformal circles.

— We found two Lagrangians for (conformal) unparametrized circles in $\mathbb{R}^3$ but both have sigularities:

- torsion τ excludes straight lines
- the Lagrangian $\mathcal{L}$ excludes tangent directions $-V$

— Final comments/questions:

- is there a globally defined Lagrangian for conformal circles?
- do we have something similar for non-flat metrics?
- what about higher dimensions?

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