Sam Blitz

(A Step) Toward a Classification of Conformal Hypersurface Invariants

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Srní, 43rd Winter School, January 2023

Motivation

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- Ph.D. work on extrinsic Paneitz operator, tractor holography, conformal geometry

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Ubiquitous appearances:

- Asymptotically Poincaré–Einstein structures
- Anomalies of renormalized volume
- Willmore invariants
- Dirichlet–Neumann maps

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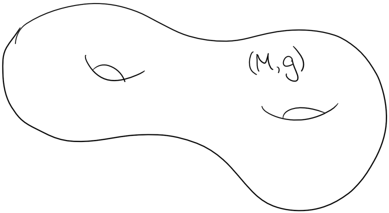
Q: Do conformal fundamental forms locally characterize extrinsic conformal hypersurface data?

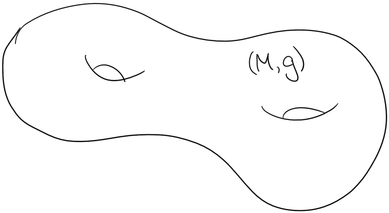
(M, g) smooth



Setup

Riemannian manifolds

(M, g) smooth



Weyl's classical invariant theory
$\Rightarrow$ "natural" invariants built from $\{g, g^{-1}, \nabla, R\}$

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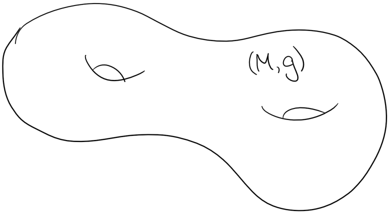

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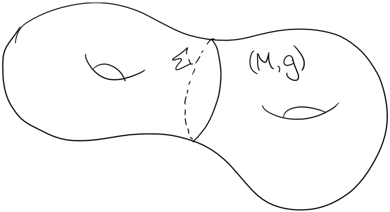
$\Rightarrow$ "natural" invariants built from $\{g, g^{-1}, \nabla, R\}$
(Broader notions of natural not used here, but maybe later!)

$\Sigma \hookrightarrow (M, g)$ smooth



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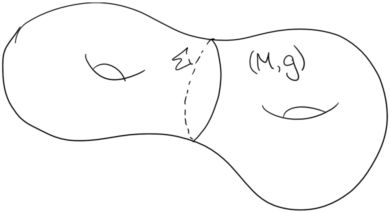

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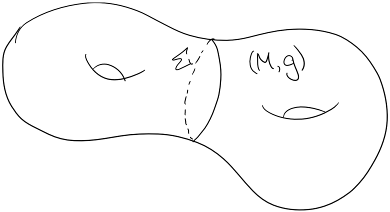

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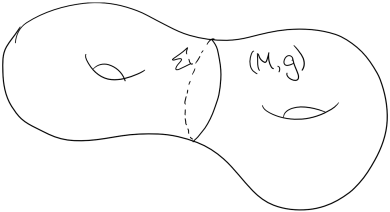

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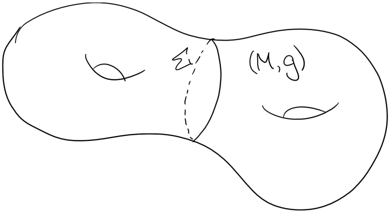

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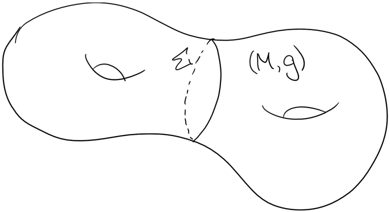

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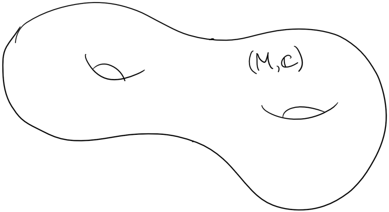
$$\Rightarrow \{\bar{g}, \bar{g}^{-1}, \hat{n}, (\bar{\nabla}^\ell \mathrm{I\!I}), (\nabla^m R)|_\Sigma\} \text{ (Gover–Waldron)}$$

$(M, \boldsymbol{c})$ smooth: $g, g' \in \boldsymbol{c} \quad \Leftrightarrow \quad \exists \Omega \in C^{\infty}_{+} M$ s.t. $g' = \Omega^2 g$



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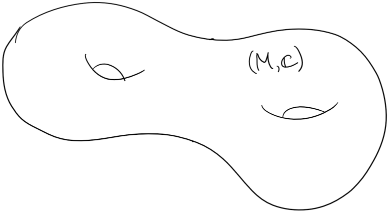

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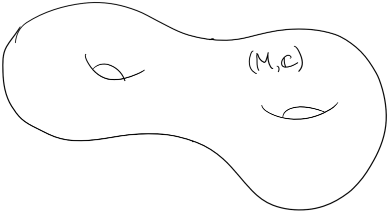

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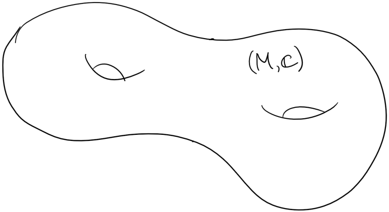

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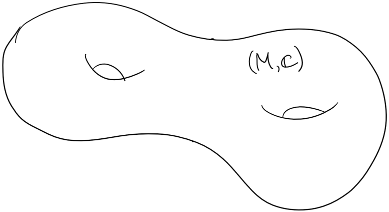

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$\Rightarrow$ **tractors**, **BGGs**

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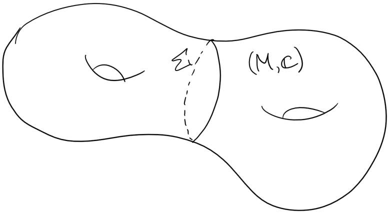

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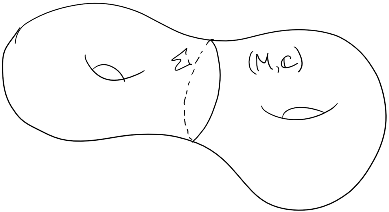

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$\{\bar{g}, \bar{g}^{-1}, \hat{n}, (\bar{\nabla}^{\ell} \mathrm{II}), (\nabla^m R)|_{\Sigma}\}$

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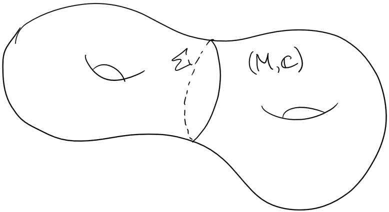

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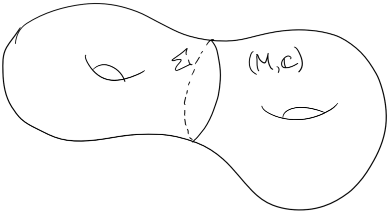

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$\{\bar{g}, \bar{g}^{-1}, \bar{\nabla}, \bar{R}, \hat{n}, \mathring{\mathrm{II}}, \dots, \overset{\circ}{\overline{\underline{\mathrm{d}-1}}}\} \leftarrow$ often works

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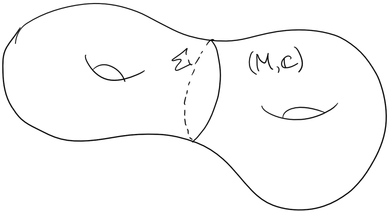

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$\underline{\mathring{\mathrm{m}}} = [g; \underline{\mathring{\mathrm{m}}}] = [\Omega^2 g; \Omega^{3-m}\underline{\mathring{\mathrm{m}}}] =$ m^{th} conf. fundamental form

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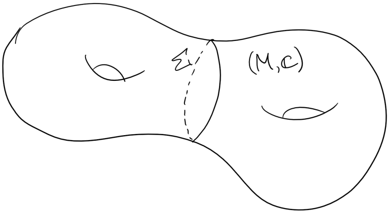

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$\underline{\mathring{\mathrm{m}}} :=$ **"higher order"** trace-free $\mathrm{I\!I}$

Definition

Let s be any defining function for $\Sigma \hookrightarrow (M, g)$. Let $I[g, s]$ be the restriction to Σ of a (partial) contraction polynomial in the set $\{s, |ds|_g^{-1}, g, g^{-1}, \nabla, R\}$. Then $I[g, s]$ is a *natural hypersurface invariant* (NHI) when $I[g, s] = I[g, \tilde{s}]$ for any defining functions $s, \tilde{s}$ for Σ.

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Contraction polynomial in $\{g_{ab}, g^{ab}, X_a, Y_{ab}, Z_{abc}\}$:
e.g. $\quad g^{ab} X_a X_b + g^{aa'} X_{a'} g^{bb'} g^{cc'} Y_{b'c'} Z_{abc}$

Definitions

Natural hypersurface invariant

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Partial contraction polynomial:
e.g. $\quad X_a X_b + Y_{ab} + Z_{abc} g^{cc'} X_{c'}$

Definition

Let $\Sigma \hookrightarrow (M, \boldsymbol{c})$ be a conformal hypersurface embedding, and for some $g \in \boldsymbol{c}$, let $I[g]$ be an NHI for $\Sigma \hookrightarrow (M, g)$. Then $I[\boldsymbol{c}, \sigma]$ is a *natural conformal hypersurface invariant (NCHI) of weight* w when, for any $\Omega \in C^\infty_+ M$,

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Examples:

- $w(\hat{n}_a) = 1$
- $w(\mathring{\mathrm{II}}_{ab}) = 1 \leftarrow$ trace-free part of II
- $w(W_{abcd}|_\Sigma) = 2$

Definition

Let $I[g]$ be an NHI for $\Sigma \hookrightarrow (M, g)$ with g generic and s a defining function, and suppose that

$$I[g + s^k h] \neq I[g] = I[g + s^{k+1} h']$$

for some h and any h' such that $g + s^k h$ and $g + s^{k+1} h'$ are metrics on M. Then $I[g]$ has *transverse order* k, and write $\mathrm{TO}(I[g]) = k$.

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Important: TO is compatible with NCHIs

Definitions

Transverse order equivalence

Definition

Let $I[g], L[g]$ be NHIs for $\Sigma \hookrightarrow (M, g)$ with g generic of the same tensor type. Then for $k \in \mathbb{N}$ we say that

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Allows us to work "modulo lower order"
Example: *Theorema egregium*

$$Sc - Ric_{\hat{n}\hat{n}} \overset{\Sigma}{=} \bar{S}c + \mathrm{II}^2 - (\operatorname{tr} \mathrm{II})^2$$

$$\Rightarrow Sc \overset{2}{\sim} Ric_{\hat{n}\hat{n}}$$

Definition

Let $\Sigma \hookrightarrow (M, \boldsymbol{c})$, let $2 \leq m \in \mathbb{N}$, and let $g \in \boldsymbol{c}$. For an NCHI $I[\boldsymbol{c}]$ represented by $I[g] \in \Gamma(\odot_\circ^2 T^*\Sigma)$, if

$$\mathrm{TO}(I[\boldsymbol{c}]) = m - 1 \quad \text{and} \quad w(I[\boldsymbol{c}]) = 3 - m\,,$$

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(Caveat: canonical construction only exists for M even dimensional and for $m \leq d - 1$.)

Main Result

Statement

Can we specify NCHIs entirely with tensors on Σ?

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Yes (to a certain order):

Theorem

Let $\Sigma \hookrightarrow (M^d, \boldsymbol{c})$ with d even and let $I[\boldsymbol{c}]$ be an NCHI with $\mathrm{TO}(I[\boldsymbol{c}]) = k$ for $k \in \{0, 1, \ldots, d-2\}$. Then, for any $g \in \boldsymbol{c}$, $I[g]$ can be expressed as a (partial) contraction polynomial in elements of the set

$$\{\bar{g}, \bar{g}^{-1}, \bar{\nabla}, \bar{R}, \hat{n}, \mathring{\mathrm{II}}, \ldots, \overset{\circ}{\underline{\overline{\mathrm{k}+1}}}\}\,.$$

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Proof idea: Relies on decomposition of tensors in Riemannian setting to projections and then eliminates all but the remaining tensors from the list of possible terms.

Main Result

Hypersurface Tensor Projection

Theorema egregium:

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Fialkow–Gauss equation:

$$W_{\hat{n}ab\hat{n}} + (d-3)P^{\top}_{ab} \stackrel{\Sigma}{=} \mathring{\mathrm{II}}^2_{ab} - \frac{1}{2(d-2)}\mathring{\mathrm{II}}^2 \bar{g}_{ab} + (d-3)\left(\bar{P}_{ab} - H\,\mathring{\mathrm{II}}_{ab} - \frac{1}{2}H^2 g_{ab}\right)$$

Expected:

$$\mathrm{TO}(R_{\hat{n}ab\hat{n}}) = \mathrm{TO}(Ric^{\top}_{ab}) = \mathrm{TO}(Ric_{\hat{n}\hat{n}}) = 2$$

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"Unexpected" (lower):

$$\mathrm{TO}(R^{\top}_{abcd}) = \mathrm{TO}(R^{\top}_{abc\hat{n}}) = \mathrm{TO}(Ric^{\top}_{a\hat{n}}) = 1$$

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$$\mathrm{TO}(R_{\hat{n}ab\hat{n}}) = \mathrm{TO}(Ric^{\top}_{ab}) = \mathrm{TO}(Ric_{\hat{n}\hat{n}}) = 2$$

"Unexpected" (lower):

$$\mathrm{TO}(R^{\top}_{abcd}) = \mathrm{TO}(R^{\top}_{abc\hat{n}}) = \mathrm{TO}(Ric^{\top}_{a\hat{n}}) = 1$$

Generalize to higher derivatives?

Compute modulo lower order:

$$(:\nabla_{\hat{n}}^{m}: R_{abcd})^{\top} \overset{m+2}{\sim} 0$$

$$(\hat{n}^{d}:\nabla_{\hat{n}}^{m}: R_{dabc})^{\top} \overset{m+2}{\sim} 0$$

$$(\hat{n}^{c}\hat{n}^{d}:\nabla_{\hat{n}}^{m}: R_{cabd})^{\top} \overset{m+2}{\sim} (d-2)(:\nabla_{\hat{n}}^{m}: P_{ab})^{\top}_{\circ} + \bar{g}_{ab}:\nabla_{\hat{n}}^{m}: J|_{\Sigma}$$

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$$:\nabla_{\hat{n}}^{m}: Sc|_{\Sigma} \overset{m+2}{\sim} 2(d-1):\nabla_{\hat{n}}^{m}: J|_{\Sigma}\,.$$

$$(:\nabla_{\hat{n}}^{m}: \;= \hat{n}^{a_1}\cdots\hat{n}^{a_m}\nabla_{a_1}\cdots\nabla_{a_m})$$

Compute modulo lower order:

$$
\begin{aligned}
(:\nabla_{\hat n}^m : R_{abcd})^\top &\overset{m+2}{\sim} 0\\
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$(:\nabla_{\hat n}^m : \; = \hat n^{a_1} \cdots \hat n^{a_m} \nabla_{a_1} \cdots \nabla_{a_m})$

(Slightly) better basis!

Canonical conf. FFs [B. '21]:

For d even, $2 \le m \le d-2$

$$(:\nabla_{\hat{n}}^{m-2}:P_{ab})^{\top}_{\circ} \overset{m}{\sim} \alpha \, \overset{\circ}{\underline{\overline{\mathrm{m}+1}}}_{ab}$$

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***T*-curvatures [Gover–Peterson '21]:** For $2 \leq m \leq d-2$:

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Important feature:

$$T_m^{\Omega^2 g} = \Omega^{-m}(T_m^g + \delta_m \log \Omega)$$

where $\delta_m = \gamma :\nabla_{\hat{n}}^m: +$ lower order

Can write NHIs (TO $\leq d-2$):

$$\{\bar{g}, \bar{g}^{-1}, \bar{\nabla}, \bar{R}, \mathring{\mathrm{II}}, \ldots, \overline{\underline{\mathrm{d}-1}}^{\circ}, T_1^g, \ldots, T_{d-2}^g\}$$

Main Result

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Proof done!

(Still in d even.)

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Future Work

Tractor classification

(Still in d even.)

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Conjecture: Besides the known exceptions, all NCHIs with TO $\leq d-2$ can be formed from a tractor basis

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Experimental evidence: *yes?*

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Higher transverse order

Main result: $\mathrm{TO} \leq d - 2$

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Thank you

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