BGG sequences and generalizations of the Korn inequality

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Introduction

The classical Korn inequality is an important tool in applied maths (elasticity theory). For appropriate domains $U \subset \mathbb{R}^n$ it states that for $f \in H^1(U, \mathbb{R}^n)$ one has $\|f\|^2_{H^1} \leq C(\|f\|^2_{L^2} + \|\mathrm{Sym}(Df)\|^2_{L^2})$.

Here $\mathrm{Sym}(Df)$ is the symmetrized derivative, which is the Killing operator for the flat metric in $\mathbb{R}^n$. There is a conformal analog for $\mathrm{tfp}(\mathrm{Sym}(Df))$ and both cases have been generalized to Riemannian manifolds.

In both cases one deals with a first BGG operator (in a Sobolev setting), and on $\mathbb{R}^n$ the BGG machinery extends to this setting. I will start by discussing an extension of this to Riemannian manifolds. There is a classical proof of the inequality, where the main step is proving a regularity statement, namely that for $f \in L^2$ with $Sym(Df) \in L^2$ one has $f \in H^1$. Via the BGG machinery, this leads to a vast generalization of the inequality.

Distributional sections of vector bundles

Let (M, g) be a compact Riemannian manifold. Recall that (regardless of orientation), there is a volume density on M and hence there is a well defined integral $\int_M f$ for $f \in C^\infty(M, \mathbb{R})$.

For a vector bundle $E \to M$ we get the dual bundle $E^* \to M$ and for sections $\sigma \in \Gamma(E)$, $\lambda \in \Gamma(E^*)$ we have the dual pairing $\langle \sigma, \lambda \rangle \in C^\infty(M, \mathbb{R})$. Defining $\mathcal{D}'(M, E)$ as the topological dual of $\Gamma(E^*)$, $\Gamma(E)$ injects into $\mathcal{D}'(M, E)$ via $\sigma \mapsto (\lambda \mapsto \int_M \langle \sigma, \lambda \rangle)$.

Via this inclusion, one extends operations to distributional sections. For example, for $f \in C^\infty(M, \mathbb{R})$ and $\alpha \in \mathcal{D}'(M, E)$, one defines $f\alpha(\lambda) := \alpha(f\lambda)$. Similarly, for a vector bundle homomorphism $\Phi : E \to F$, one gets $\Phi^* : F^* \to E^*$ and extends the operator on sections to $\Phi : \mathcal{D}'(M, E) \to \mathcal{D}'(M, F)$ via $\Phi(\alpha)(\lambda) := \alpha(\Phi^*(\lambda))$ for $\alpha \in \mathcal{D}'(M, E)$ and $\lambda \in \Gamma(F^*)$.

covariant (exterior) derivative

For a linear connection on E, we obtain the dual connection on E^* and we denote these and the Levi-Civita connection by ∇. For $\psi \in \Gamma(TM \otimes E^*)$ we define $\text{div}(\psi) = \mathcal{C}(\nabla\psi)$, where $\mathcal{C}$ contracts the first two indices. Using partial integration, one proves that $\nabla\alpha(\psi) := -\alpha(\text{div}(\psi))$ extends the covariant derivative to an operator $\mathcal{D}'(M, E) \to \mathcal{D}'(M, T^*M \otimes E)$.

This similarly works for the covariant exterior derivative $d^\nabla : \Omega^k(M, E) \to \Omega^{k+1}(M, E)$. Given $\psi \in \Gamma(\Lambda^{k+1}TM \otimes E^*)$ we again define $\text{div}(\psi)$ as the contraction of $\nabla\psi$ over the first two indices. Then we define $(d^\nabla\alpha)(\psi) := -\alpha(\text{div}(\psi))$ and using partial integration shows that this extends the definition on smooth forms.

One could now go ahead to define Sobolev norms and then Sobolev spaces as completions of $C^\infty(M, \mathbb{R})$ with respect to these norms. We prefer to initially take an alternative route via charts.

Sobolev sections

Let (U, u) be a chart for M, $f \in C^\infty(M, \mathbb{R})$ a function with $\mathrm{supp}(f) \subset U$ and $\{\lambda_i\}$ be a smooth local frame for E^* defined on U. For $h \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$, $f(h \circ u)\lambda_i$ extends by zero to a smooth section of E^*. Given $\alpha \in \mathcal{D}'(M, E)$, we can thus define $(f\alpha)^i \in \mathcal{D}'(\mathbb{R}^n)$ by $(f\alpha)^i(h) := \alpha(f(h \circ u)\lambda_i)$.

Definition

For $s \in \mathbb{R}$ we say that α lies in $H^s(M, E) \subset \mathcal{D}'(M, E)$ if and only if for any (U, u) and f and one (or equivalently any) local frame $\{\lambda_i\}$ as above, each of the distributions $(f\alpha)^i$ lies in $H^s(\mathbb{R}^n) \subset D'(\mathbb{R}^n)$.

If E carries a bundle metric, then for $s = k \in \mathbb{N}$, there is a (pre-Hilbert) H^k-norm $\| \ \|_{H^k}$ on $\Gamma(E)$ induced by the natural L^2-norms of $\sigma \in \Gamma(E)$ and its symmetrized iterated covariant derivatives of order up to k. One then proves that for $k \in \mathbb{N}$, $H^k(M, E)$ can be identified with the completion of $(C^\infty(M, \mathbb{R}), \| \ \|_{H^k})$, and hence carries a natural norm.

The Lions lemma for the covariant derivative

For $k \in \mathbb{N}$ it is more or less by definition true that if for an H^k-funktion f on $\mathbb{R}^n$ also the partial derivatives $\partial_i f$ lie in H^k, then f lies in H^{k+1}. This extends to arbitrary Sobolev indices (and suffiently nice domains in $\mathbb{R}^n$) and this extension is known as the *Lions lemma*. Via the chart interpretation, this generalizes further:

Proposition (Lions lemma for ∇)

Let (M, g) be a compact Riemannian manifold and $E \to M$ a vector bundle and take $\alpha \in \mathcal{D}'(M, E)$. Suppose that for some $s \in \mathbb{R}$ we have $\alpha \in H^s(M, E)$ and $\nabla\alpha \in H^s(M, T^*M \otimes E)$. Then $\alpha \in H^{s+1}(M, E)$.

In what follows, this will be mainly needed in the case that s is a negative integer.

Input from representation theory

We need a representation $\mathbb{V} = \oplus_{i=0}^{N} \mathbb{V}_i$ of $O(n)$ for some $N \in \mathbb{N}$ which is endowed with an $O(n)$-equivariant action of the Abelian Lie algebra $\mathbb{R}^n$ written as $(X, v) \mapsto X \bullet v$ such that $\mathbb{R}^n \bullet \mathbb{V}_i \subset \mathbb{V}_{i-1}$.

This induces an $O(n)$-equivariant Lie algebra cohomology differential $\partial : \Lambda^k \mathbb{R}^{n*} \otimes \mathbb{V} \to \Lambda^{k+1} \mathbb{R}^{n*} \otimes \mathbb{V}$ which sends $\mathbb{V}_i$-valued maps to $\mathbb{V}_{i-1}$-valued ones. Explicitly, $\partial\varphi(X_0, \dots, X_k) := \sum_i (-1)^i X_i \bullet \varphi(X_0, \dots, \widehat{X}_i, \dots, X_k)$.

Examples come from representations $\mathbb{V}$ of Lie groups G with $|1|$-graded Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that $\mathfrak{o}(n) \subset \mathfrak{g}_0$. The two main examples are $G = SL(n+1, \mathbb{R})$ with $G_0 = GL(n, \mathbb{R})$ and $G = O(n+1, 1)$ with $G_0 = CO(n)$. For these examples, there is Kostant's algebraic Hodge theory, which leads to additional structure and to an algorithm to compute the cohomology of ∂.

If one requires only $O(n)$ equivariancy, this can be formulated as existence of a differential $\partial^{\#} : \Lambda^{k+1}\mathbb{R}^{n*} \otimes \mathbb{V} \to \Lambda^{k}\mathbb{R}^{n*} \otimes \mathbb{V}$ such that $\partial^{\#}\partial\partial^{\#} = \partial^{\#}$ and $\partial\partial^{\#}\partial = \partial$. In addition, for each k there is an $O(n)$-invariant decomposition
$\Lambda^{k}\mathbb{R}^{n*} \otimes \mathbb{V} = \mathrm{Im}(\partial) \oplus (\ker(\partial) \cap \ker(\partial^{\#})) \oplus \mathrm{Im}(\partial^{\#})$.
Here the first two summands add up to $\ker(\partial)$, so the middle summand is isomorphic to the cohomology (of both ∂ and $\partial^{\#}$).

The $O(n)$-representations induce natural vector bundles $\mathcal{V}M = \oplus_i \mathcal{V}_i M$ on n-dimensional Riemannian manifolds and the $O(n)$-equivariant maps give rise to natural bundle maps. The corresponding operators on sections are denoted by $S : \Omega^k(M, \mathcal{V}M) \to \Omega^{k+1}(M, \mathcal{V}M)$ and T in the opposite direction.

By construction, we get $S^2 = 0$ and $T^2 = 0$. Moreover, for $\varphi \in \Omega^k(M, \mathcal{V}M)$ we get $\varphi = ST\varphi + (\varphi - ST\varphi - TS\varphi) + TS\varphi$. Here the middle summand lies in $\ker(S) \cap \ker(T)$ and hence occurs only in places where cohomology is present.

The twisted de Rham sequence

Now one defines the *twisted covariant exterior derivative* $d_V : \Omega^k(M, \mathcal{V}M) \to \Omega^{k+1}(M, \mathcal{V}M)$ as $d_V := d^\nabla - S$. Observe that d^∇ preserves the subspaces $\Omega^*(M, \mathcal{V}_i M)$ while S maps $\Omega^k(M, \mathcal{V}_i M) \to \Omega^{k+1}(M, \mathcal{V}_{i-1} M)$. Recall that bundle maps induced by $O(n)$-equivariant maps are automatically parallel for the Levi-Civita connection. Using this, one proves the following result.

Proposition

In any degree, we get $d_V \circ d_V = d^\nabla \circ d^\nabla$, so this is given by the tensorial action of the Riemann curvature R of ∇. In particular, for $\varphi \in \Omega^k(M, \mathcal{V}_i M)$, we get $d_V d_V \varphi \in \Omega^{k+2}(M, \mathcal{V}_i M)$.

The operator d_V extends without problems to distributional forms and by constructions it maps H^s-Sobolev forms to H^{s-1}-Sobolev forms. Also the description of the curvature extends, in particular this shows that for $\alpha \in H^s(M, \Lambda^k T^*M \otimes \mathcal{V}_i M)$ we get $d_V(d_V \alpha) \in H^s(M, \Lambda^{k+2} T^*M \otimes \mathcal{V}_i M)$.

The simplified BGG sequence

To define the BGG splitting operator $L:\Gamma(\mathcal{V}_0M)\to\Gamma(\mathcal{V}M)$, take $\sigma\in\Gamma(\mathcal{V}_0)$ and define the components $s_i\in\Gamma(\mathcal{V}_iM)$ of $L(\sigma)$ recursively by $s_0=\sigma$ and $s_i=T(\nabla s_{i-1})$ for $i>1$. Then the components b_i of $d_V(L(\sigma))$ are given by $b_i=\nabla s_i-ST(\nabla s_i)$, and hence $T(d_V(L(\sigma)))=0$. Recursively, one immediately concludes

Proposition

The properties that $L(\sigma)_0=\sigma$ and that $T(d_V(L(\sigma)))=0$ uniquely determine the operator L.

Next, one defines the first BGG operator D for $\sigma\in\Gamma(\mathcal{V}_0M)$ as $D(\sigma):=d_V(L(\sigma))-TS(d_VL(\sigma))$, so this lies in $\ker(T)\cap\ker(S)$. Similarly, splitting operators and BGG operators can be defined in higher degrees, but we'll continue the analysis in a different direction.

Kostant's theorem implies that in our setting and for irreducible $\mathbb{V}$, $\mathbb{H}_1 := \ker(\partial) \cap \ker(\partial^{\#}) \subset \mathbb{R}^{n*} \otimes \mathbb{V}$ is always irreducible as a representation of G_0. Moreover, if $\mathbb{W}$ is any irreducible representation of G_0, there is a unique representation $\mathbb{V}$ of G such that $\mathbb{V}_0 = \mathbb{W}$ und such that $\mathbb{H}_1 = \mathbb{R}^{n*} \circledcirc \mathbb{W} \subset \mathbb{R}^{n*} \otimes \mathbb{V}_0$, the ($G_0$–) Cartan product.

In the SL-case, $\mathbb{W} = S^k\mathbb{R}^{n*}$ leads to $\mathbb{H}_1 = S^{k+1}\mathbb{R}^{n*}$, but $\mathbb{V}$ is the irreducible component of highest weight in $S^k(\Lambda^2\mathbb{R}^{(n+1)*})$. Similarly, for $\mathbb{W} = \Lambda^k\mathbb{R}^{n*}$, $\mathbb{H}_1$ is the kernel of the complete alternation. For the O-case, $\mathbb{W} = S^k_0\mathbb{R}^n$ similarly leads to $\mathbb{H}_1 = S^{k+1}_0\mathbb{R}^n$ with $\mathbb{V}$ the irreducible component of highest weight in $S^k_0(\Lambda^2\mathbb{R}^{n+1,1})$.

In either case, the is a unique projection $\pi : \mathbb{R}^{n*} \otimes \mathbb{W} \to \mathbb{R}^{n*} \circledcirc \mathbb{W}$ and correspondingly we get a bundle map $\pi : T^*M \otimes \mathcal{W}M \to T^*M \circledcirc \mathcal{W}M$. The resulting first BGG operator is $D(\sigma) = \pi(\nabla\sigma)$ and can be viewed as giving the "main component" of $\nabla\sigma$.

The generalized Lions lemma

The BGG operator D extends to H^s sections without problems, and using this, we formulate:

Theorem

Let $\mathcal{W}M$ be the natural bundle induced by a G_0-irreducible representation $\mathbb{W}$ and let $D : \Gamma(\mathcal{W}M) \to \Gamma(T^*M \circledcirc \mathcal{W}M)$ the corresponding first BGG operator. Then for any $s \in \mathbb{R}$ if $\alpha \in H^s(M, \mathcal{W}M)$ has the property that $D(\alpha) \in H^s(M, T^*M \circledcirc \mathcal{W}M)$, then $\alpha \in H^{s+1}(M, \mathcal{W}M)$.

Sketch of proof: The recursive definition of $L(\alpha)$ applies to distributional forms, including the characterization. So the components s_i of $L(\alpha)$ are given by $s_0 = \alpha$ and $s_i = T(\nabla s_{i-1})$. For $\alpha \in H^s$, this inductively implies that $s_i \in H^{s-i}$ for each $i = 1, \dots, N$. Also for the components b_i of $d_V(L(\alpha))$, we see that $b_i = \nabla s_i - S(s_{i+1}) \in H^{s-i-1}$ for each $i = 0, \dots, N$.

proof (continued)

Now by definition $b_0 = \nabla\alpha - ST(\nabla\alpha) = \nabla\alpha - ST(s_1)$. The first expression shows that $b_0 = D(\alpha)$ so by assumption $b_0 \in H^s$. The second expression then shows that it suffices to prove that $s_1 \in H^s$, because then $\nabla\alpha \in H^s$ and the Lions lemma for the covariant derivative applies.

For $i > 0$, we know that $T(b_i) = 0$ and hence $b_i = TS(b_i)$ and $S(b_i) = d^{\nabla} b_{i-1} - (d_V d_V L(\alpha))_{i-1}$. But since s_{i-1} lies in H^{s-i+1}, so does the second summand. Hence if b_{i-1} lies in H^{s-i+1}, then $b_i \in H^{s-i}$, so by induction, this holds for all i.

For the last component, we have $b_N = \nabla s_N$. Knowing that this lies in H^{s-N} (and that $s_N \in H^{s-N}$), we conclude that $s_N \in H^{s-N+1}$. But then for $i < N$, we get $b_i = \nabla s_i - S(s_{i+1})$ and this lies in H^{s-i}. If we know that s_{i+1} also lies in H^{s-i} then we conclude that s_i lies in H^{s-i+1}. Hence by backwards induction this holds for all i, so $s_1 \in H^s$ and this completes the proof.

The generalized Korn inequality

Theorem

Let $\mathcal{W}M$ be the natural bundle induced by a G_0-irreducible representation $\mathbb{W}$ and let $D : \Gamma(\mathcal{W}M) \to \Gamma(T^*M \circledcirc \mathcal{W}M)$ the corresponding first BGG operator. Then there is a constant C such that for any $\alpha \in H^1(M, \mathcal{W}M)$ we get

$$\|\alpha\|_{H^1}^2 \leq C(\|\alpha\|_{L^2}^2 + \|D(\alpha)\|_{L^2}^2).$$

Proof: This is rather simple functional analysis. Consider the subspace $E := \{\alpha : D(\alpha) \in L^2(M, TM \circledcirc \mathcal{W}M)\} \subset L^2(M, \mathcal{W}M)$. The generalized Lions lemma applied to $s = 0$ shows that this is contained in $H^1(M, \mathcal{W}M)$ and hence equals $H^1(M, \mathcal{W}M)$.Now $\|\alpha\|^2 := \|\alpha\|_{L^2}^2 + \|D(\alpha)\|_{L^2}^2$ defines a (pre-Hilbert) norm on E and it is easy to see that E is complete for this norm. By the closed graph theorem, the identity from H^1 to E is a bounded operator, which implies the result.