Curvature in almost-complex and complex geometry

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Plan

1. **Motivating open problems** (Yau's Challenge and S^6 problem)
   - S^6 **progress:** (conformal) almost-hermitian version due to LeBrun, Salamon, and others
2. **Intrinsic torsions vs. "obstructures"**
3. **Question A:** What is the effect of constraining the (various) curvatures of a Riemannian metric in the almost-hermitian case?

1. Motivating open problems

- **Yau's Challenge (YC):** Determine if there are any (compact) almost-complex, but not complex manifolds of real dimension at least 6.
- S^6 **problem:** Is the 6-dimensional sphere S^6 a complex manifold?
  - S^6 is a YC solution candidate...

1. Motivating open problems, S^6 **progress**

Let g be the round metric on the unit 6-sphere S^6,

$$AC(S^6) := \{\text{ a-cx structures on } S^6\}, \text{ and}$$

$$AC(S^6)_g := \{J \in AC(S^6) \mid g(JX, JY) = g(X, Y), \forall X, Y \in \mathfrak{X}(S^6)\}$$
$$= \{g - \text{orthogonal a-cx structures on } S^6\}.$$

- **LeBrun (1987):** There is no integrable $A \in AC(S^6)_g$.
- **Salamon, and others ($\gtrapprox$ 1992):** For any conformal to g Riemannian metric g', $A \in AC(S^6)_{g'}$ cannot be integrable.

1. Motivating open problems, S^6 **progress**

The proofs of both results are extrinsic.

- **LeBrun (1987):** For any $A \in AC(S^6)$, there is a smooth embedding $\mathcal{R}_A : S^6 \hookrightarrow \mathrm{Gr}_3^{\mathbb{C}}(\mathbb{C}^7)$ that is holomorphic when $A \in AC(S^6)_g$ is integrable. So if S^6 had a complex structure $A \in AC(S^6)_g$, $\mathcal{R}_A$ would embed S^6 as a complex manifold into the Kähler manifold $\mathrm{Gr}_3^{\mathbb{C}}(\mathbb{C}^7)$. Then, S^6 would be Kähler too, which is impossible because $H^2(S^6, \mathbb{R}) = 0$.

1. Motivating open problems, S^6 **progress**

The proofs of both results are extrinsic.

- **Salamon, and others ($\gtrapprox$ 1992):** Suppose that S^6 carried an integrable $A \in AC(S^6)_{g'}$, where g' is conformal to g. As in LeBrun's proof, there exists a holomorphic embedding $\mathcal{S}_A$ of S^6 into a hermitian symmetric space (H, J_H) (twistor bundle of S^6). But H is Kähler, forcing $S^6 \simeq \mathcal{S}_A(S^6)$ to be Kähler too, which again, is false.

2. Intrinsic torsions vs. "obstructures"

- Intrinsic torsions are a cohomological measurement of the non-integrability of a G-structure (they are given by Spencer cohomology classes). As their name suggest, they allow for an intrinsic study of (non-)integrability.
  - Indeed, almost-complex structures are $GL_n(\mathbb{C})$-structures, and almost-hermitian structures (i.e. pairs (g, A), where A is a g-orthogonal a-cx structure) are $U(n)$-structures.
- The 1st intrinsic torsion of a linear connection ∇^J that is compatible with an a-cx structure J ($\nabla^J J = 0$) coincides with the Nijenhuis tensor.

2. Intrinsic torsions vs. "obstructures"

- It's not clear how to use the classical obstruction theory of G-structures to study the interfacing between the almost-complex and (bounded curvature) Riemannian geometries of a manifold. Almost-hermitian structures for metrics of some prescribed curvature aren't a usual kind of G-structure. This is why the standard theory cannot be readily applied. Anyhow,
  - if ∇ is a linear connection compatible with a G-structure P, then the 1st 2 intrinsic torsions have the following geometric interpretation:
    - $T^1_{intr}(P) = [T^\nabla]$, and
    - $T^2_{intr}(P) = [R^\nabla]$,

    where T^∇ and R^∇ are the torsion, respectively the curvature of the connection.

2. Intrinsic torsions vs. "obstructures"

- We will obtain curvature obstruction equations to the integrability of almost-complex structures that allow us to probe the almost-complex geometry with Riemannian metrics of prescribed curvature. We will covariantly differentiate the Nijenhuis tensor. Observe that $D^k N_J = 0$ – the vanishing of any order $k \geq 1$ covariant derivative of the Nijenhuis tensor of an a-cx structure J – can be viewed as an obstruction equation.
- Let's call the left-hand-side of such an obstruction equation an "obstructure;" e.g. if $J \in AC(M)$, the differential form $D^k N_J \in \Omega^{k+2}(M, T_M)$ is an obstructure.

2. Intrinsic torsions vs. "obstructures"

Lemma

Let ∇ be any torsion-free connection on T_M, and d^∇ be its associated covariant exterior derivative. Then, $A \in AC(M)$ is integrable iff $d^\nabla A$ is A-invariant; i.e.

$$I_A^\nabla(\zeta,\eta) := d^\nabla A(A(\zeta),A(\eta)) - d^\nabla A(\zeta,\eta) = 0.$$

2. Intrinsic torsions vs. "obstructures"

- The spaces $\Omega^\bullet(M, \mathrm{End}_{\mathbb{R}}(T_M)) = \bigoplus_{k\geq 0} \Omega^k(M, \mathrm{End}_{\mathbb{R}}(T_M))$, and $\Omega^\bullet(M, \bigwedge^\bullet T_M) = \bigoplus_{k\geq 0} \bigoplus_{p+q=k} \Omega^p(M, \bigwedge^q T_M)$ are graded algebras, and

$$\Omega^\bullet(M, T_M) = \bigoplus_{k\geq 0} \Omega^k(M, T_M)$$

can be viewed as both a left $\Omega^\bullet(M, \mathrm{End}_{\mathbb{R}}(T_M))$-module and a right $\Omega^\bullet(M, \bigwedge^\bullet T_M)$-module. One can unambiguously denote all of the products and actions involved by "$\wedge$."

2. Intrinsic torsions vs. "obstructures"

- The correct notion of covariant differentiation of the integrability form $I_A^\nabla \in \Omega^2(M, T_M)$ is supplied by d^∇. Since I_A^∇ is a function of A and $d^\nabla A$, its first covariant exterior derivative will be a function of A, $d^\nabla A$, and $(d^\nabla)^2 A = R^\nabla \wedge A$. The same thinking reveals that the k-th covariant exterior derivative of I_A^∇ will depend on $A, d^\nabla A, (d^\nabla)^2 A = R^\nabla \wedge A, \dots,$

$$(d^\nabla)^{k+1} A = \begin{cases} (R^\nabla)^{\frac{k+1}{2}} \wedge A & \text{if } k \text{ is odd} \\ (R^\nabla)^{\frac{k}{2}} \wedge d^\nabla A & \text{if } k \text{ is even.} \end{cases}$$

2. Intrinsic torsions vs. "obstructures"

The integrability form can be re-expressed as

Lemma

$$I_A^\nabla = d^\nabla A \wedge (A \wedge A) - d^\nabla A$$

- Once we develop an appropriate calculus of T_M-valued forms, this Lemma makes it easier to compute the kth covariant exterior derivative of I_A^∇.

2. Intrinsic torsions vs. "obstructures"

For example, the 1st 2 obstructures are:

Lemma

$$d^\nabla I_A^\nabla = (R^\nabla \wedge A) \wedge (A \wedge A) + 2d^\nabla A \wedge (d^\nabla A \wedge A) - R^\nabla \wedge A$$

$$(d^\nabla)^2 I_A^\nabla = (R^\nabla \wedge d^\nabla A) \wedge (A \wedge A) + 4(R^\nabla \wedge A) \wedge (d^\nabla A \wedge A) +$$
$$2d^\nabla A \wedge ((R^\nabla \wedge A) \wedge A) - R^\nabla \wedge d^\nabla A$$

3. Question A

- **Question A**: What is the effect of constraining the (various) curvatures of a Riemannian metric in the almost-hermitian case?
  - **Program to study Question A:**
    - Recover LeBrun's result via an obstructure refinement to constant curvature $c = 1$
    - Generalize Lebrun's result via perturbed obstructures corresponding to perturbations of the round metric on S^6 with controlled curvature

3. Question A

- Without using obstructures, we can already see that constant curvature rules out certain special complex structures.

Theorem

Let (M, g) be a Riemannian manifold of real dimension at least 4. If g has non-zero constant sectional curvature, then M does not admit a complex structure $A \in AC(M)$ satisfying $d^{\nabla} A = 0$, where ∇ is the Levi-Civita connection. So in particular, M does not admit a Kähler complex structure.

Thank you.

- G. Clemente, *A curvature obstruction to integrability*, arXiv:2108.03376, to appear in Math. Commun. (2023).
- C. LeBrun, *Orthogonal complex structures on* S^6, Proc. Amer. Math. Soc., 101(1)(1987), 136 – 138.
- S. Salamon, *Orthogonal complex structures*, in: Differential geometry and applications, Proceedings of the 6th international conference, Brno, Czech Republic, August 28 – September 1, 1995, Brno: Masaryk University (1996), 103 – 117.