Quantitative Perspective on Legendrians and non-Legendrians

and applications to C^0-contact geometry

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Plan of the Talk

This is based on joint work with Michael Sullivan:

Mostly:

- arXiv:2212.09190 [math.SG] [DRS22b]

But also:

- arXiv:2201.04579 [math.SG] [DRS22a]
- arXiv:2111.11975 [math.SG] [DRS21]

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Plan of the Talk

1. Basics of Contact Geometry
2. The Main Application
3. Contact Isotopies
4. Quantitative Flexibility of non-Legendrians

1. Basics of Contact Geometry

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DEFINITION 1.1

A *contact manifold* is an $2n+1$-dimensional smooth manifold Y equipped with a maximally non-integrable distribution of tangent hyperplanes $\xi \subset TY$.

We will assume that ξ is cooriented so that $\xi = \ker \alpha$ for some auxiliary $\alpha \in \Omega^1(Y)$ such that $\alpha \wedge d\alpha^{\wedge n}$ is a volume form.

The **contactomorphism group** consists of

$$\mathrm{Cont}(Y, \xi) := \{\Phi \in \mathrm{Diff}(Y);\ D\Phi(\xi) = \xi\}$$

i.e. $\Phi^*\alpha = e^f \alpha$ for some $f\colon Y \to \mathbb{R}$.

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1. Basics of Contact Geometry

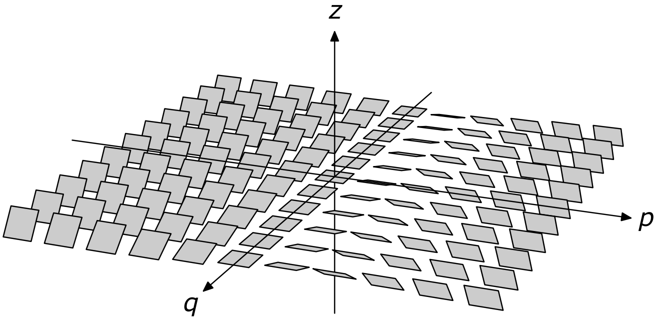

Figure 1: The contact planes on $\mathbb{R}^3_{q,p,z}$ for $\alpha = dz - p\,dq$. (Source: Wikipedia.)

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Definition 1.2

Let $M^k \subset Y^{2n+1}$ be a smooth k-dimensional submanifold.

- M is *Legendrian* if $k = n$ and $TM \subset \xi$;
- M is *non-Legendrian* if either
  - $k < n$; or
  - $k = n$, M is connected, and $TM \not\subset \xi$.

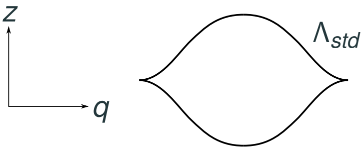

Figure 2: Front projection of the standard Legendrian unknot $p = \partial_q z$.

Examples

1. $J^1 M = T^*M \times \mathbb{R}_z$ is a contact manifold with $\alpha = dz - p\,dq$;
2. The section $j^1 f \subset (J^1 M, \ker \alpha)$ is Legendrian for any smooth $f\colon M \to \mathbb{R}$;
3. Any Legendrian Λ has a neighbourhood contactomorphic to $J^1\Lambda$, in which Λ becomes $j^1 0$;
4. Germs of non-Legendrians of dimension n can be modeled by non-vanishing sections of $T^*M \times \{0\} \to M$.

PROOF.

Infinitesimal classification of germs: Consider an n-dimensional submanifold $M \subset (Y^{2n+1}, \xi = \ker \alpha)$. The goal is to extend the contact form preserving embedding

$$M \hookrightarrow T^*M \times \{0\} \subset J^1M$$

given by the graph of $-\alpha|_{TM}$ to a smooth embedding of a neighborhood that preserves the contact form on M (not just along TM).

Main point: This is possible since, by dimensional reasons, any vector field V along M which is transverse to ξ can be perturbed to become normal to M (while remaining transverse to ξ). □

2. The Main Application

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2. The Main Application

Theorem 2.1 ([DRS22b])

Let $\Lambda \subset (Y, \xi = \ker \alpha)$ be a properly embedded Legendrian submanifold, and $\Psi_i \in \mathrm{Cont}(Y, \xi)$ a sequence of contactomorphisms, all supported in some fixed compact subset, such that

- $\Psi_i \to_{C^0} \Psi_\infty$ *where* Ψ_∞ *is a homeomorphism of* Y*;*
- $\Psi_\infty(\Lambda)$ *is a smooth submanifold;*

Then $\Psi_\infty(\Lambda)$ *is Legendrian as well.*

Remark 2.2 (The symplectic case)

The analogous problem is well-studied:

- Laudenbach–Sikorav established it for Lagrangians [LS94];
- Opshtein established it for certain coisotropics [Ops09]; and
- Humilière–Leclercq–Seyfaddini [HLS15] have established the general coisotropic case.

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REMARK 2.3

This result implies Eliashberg's Theorem [Eli87]; Namely, if the limit Ψ_∞ is smooth, then $\Psi_\infty \in \mathrm{Cont}(Y, \xi)$.

PROOF.

For any Lagrangian plane $L_{\mathrm{pt}} \subset (\xi_{\mathrm{pt}}, d\alpha)$, there exists a closed Legendrian that is tangent to L_{pt}. □

Previous results:

- [Ush20] Usher proved it under assumption on the behaviour of the conformal factors f_i, where $\Psi_i^*\alpha = e^{f_i}\alpha$.
- [Nak20] Nakamura proved it under assumptions on the length of Reeb chords on $\Psi_i(\Lambda)$.
- [DRS22a] We proved it when $\dim Y = 3$ using the Thurston–Bennequin inequality.
- [Sto22] Stokić excluded the existence of an “almost Reeb invariant” neighbourhood of $\Psi_\infty(\Lambda)$.

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PROOF (1/2).

We modify Stokić's argument by showing that $\Psi_\infty(\Lambda)$ cannot admit arbitrarily small positive contact loops;

i.e. a non-trivial loop induced by Φ_t satisfying

- $\Phi_1 = \mathrm{Id}$ in some small neighbourhood of $\Psi_\infty(\Lambda)$ and
- $\alpha(\dot{\Phi}_t) \geq 0$;

Indeed, if this was the case, then we could produce such a positive loop of $\Psi_N(\Lambda)$ for $N \gg 0$ in an arbitrarily small standard jet-neighbourhood.

Note that, since Ψ_∞ is a homeomorphism, a one-jet neighborhood U of Λ has an image $\Psi_N(U)$ that contains $\Psi_\infty(\Lambda)$ for all $N \gg 0$.

2. The Main Application

PROOF (2/2).

This contradicts Chernov–Nemirovski's result from [CN10] (Colin–Ferrand–Pushkar [CFP17]) in the closed case; $j^1 0 \subset J^1 M$ does not admit a non-trivial non-negative (positive) loop.

What now remains is to show that $\Psi_\infty(\Lambda)$ admits a small non-trivial negative loop whenever it is non-Legendrian! □

REMARK 2.4

Some Legendrian submanifolds (e.g. the standard unknot) admit positive loops. However, by Chernov–Nemirovski, such an isotopy must leave the standard neighbourhood of the original Legendrian.

3. Contact Isotopies

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The **contact isotopies** are the identity component of $\mathtt{Cont}(Y,\xi)$, i.e.

$$\mathtt{Cont}_0(Y,\xi) := \mathtt{Cont}(Y,\xi) \cap \mathrm{Diff}_0(Y).$$

Let $V_t \in \Gamma(TY)$ be the infinitesimal generator of a contact isotopy Φ_t, where $(\Phi_t)^*\alpha = e^{f_t}\alpha$.

CARTAN'S FORMULA:

$$\dot{f}_t e^{f_t}\alpha = \frac{d}{dt}(\Phi_t)^*\alpha = \Phi_t^*(d(\iota_V\alpha) + \iota_V d\alpha).$$

$H_t := \iota_V\alpha = \alpha(\dot{\Phi}_t \circ \Phi_t^{-1})\colon Y \to \mathbb{R}$ is called the **Contact Hamiltonian** (depends on the choice of α!).

The contact Hamiltonian vanishes precisely where the infinitesimal generator is tangent to the contact distribution ξ; it is positive (resp. negative) where it is positively (resp. negatively) transverse to ξ.

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Conversely, any smooth $H_t\colon Y \to \mathbb{R}$ gives rise to a contact isotopy Φ_t by solving

$$\begin{cases} \iota_{V_t}(\alpha) = H_t, \\ \iota_{V_t} d\alpha|_\xi = -dH_t|_\xi. \end{cases}$$

Facts:

1. Any path of embeddings $\phi_t\colon M \hookrightarrow Y$ such that $\phi_t^*\alpha = e^{f_t}\phi_0^*\alpha$ is generated by a global contact isotopy;
2. If $\phi_0^*\alpha \equiv 0$, i.e. ϕ_0 is a Legendrian embedding, then the contactomorphisms vanishing along M are precisely those which are induced by reparametrisation.

PROOF OF FACTS:

1. Extend the function $H_M := \iota_{V_0}\alpha \colon M \to \mathbb{R}$ to a smooth function $H \colon Y \to \mathbb{R}$ that satisfies
$$dH = \dot{f}_0\alpha - \iota_V d\alpha$$
along the normal bundle of M.
2. If $M \subset (Y, \alpha)$ is Legendrian, there is a bijection between vector fields V that are tangent to M and one-forms $\iota_V d\alpha$ that vanish along M. The latter one-form can be extended to the differential of a function $H \colon Y \to \mathbb{R}$ that vanishes on M.

□

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4. Quantitative Flexibility of non-Legendrians

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4. Quantitative Flexibility of non-Legendrians

Theorem 4.1

If $M \subset (Y, \xi)$ is a properly embedded, connected, non-Legendrian, and Φ_t is a compactly supported contact isotopy, then there exists a compactly supported contact isotopy $\tilde{\Phi}_t$ that satisfies:

- $\tilde{\Phi}_1|_M = \Phi_1|_M$ *(can be extended to hold in a nbhd.)*
- $\tilde{\Phi}_t(M)$ *is contained in an ϵ-nbhd. of $\Phi_t(M)$;*
- $H_t \circ \tilde{\Phi}_t = \alpha\left(\dot{\tilde{\Phi}}_t\right)$ *vanishes along M;*

See Usher's work [Ush14] for the symplectic case (also previous relevant work [LS94] by Laudenbach–Sikorav).

The result we need is an easy consequence:

COROLLARY 4.2

Any properly embedded, connected, non-Legendrian submanifold $M \subset (Y, \xi)$ *admits a compactly supported non-negative loop* Ψ_t *such that:*

- $\Phi_1|_M = \mathrm{Id}_M$ *(can be extended to hold in a nbhd.)*
- $\Phi_t(M)$ *is contained in an* ϵ*-nbhd. of* M*;*
- $H_t \circ \Phi_t = \alpha\left(\dot{\Phi}_t\right) \geq 0$ *and is positive for some* t *and* $\mathrm{pt} \in M$*.*

COROLLARY 4.3

The parametrised Chekanov–Hofer–Shelukhin pseudo-metric

$$\delta^\alpha(\phi, \Phi_1 \circ \phi) = \inf_{\left\{H_t; \tilde{\Phi}_1^{H_t} \circ \phi = \Phi_1 \circ \phi\right\}} \int_0^1 \max_Y |H_t| dt$$

vanishes completely on any parametrised contact isotopy class of a non-Legendrian $\phi\colon M \hookrightarrow Y$.

Rosen–Zhang have proved that the unparametrised Chekanov–Hofer–Shelukhin pseudo-metric is completely vanishing on the non-Legendrians [RZ20].

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PROOF OF THEOREM 4.1 (1/4).

The main step consists of constructing a contact displacement Φ_t of $\mathrm{pt} \in M$ from M with a vanishing contact Hamiltonian.

More precisely: We want a contact isotopy Φ_t such that $\Phi_1(\mathrm{pt}) \cap M = \emptyset$ and with $H_t \circ \Phi_t = \alpha(\dot{\Phi}_t)$ vanishing on M. □

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PROOF OF THEOREM 4.1 (2/4).

We consider the **Legendrian locus** of M:

$$\mathcal{L}(M) := \{\mathrm{pt} \in M;\ T_{\mathrm{pt}}M \cap \xi \subset (\xi_{\mathrm{pt}}, d\alpha) \text{ is Lagrangian}\}$$

(a closed proper subset of M).

We consider these three separate cases of $\mathrm{pt} \in M$:

1. $\mathrm{pt} \in M \setminus \mathcal{L}(M)$;
2. $\mathrm{pt} \in \operatorname{int} \mathcal{L}(M)$; and
3. $\mathrm{pt} \in \operatorname{bd}(M \setminus \mathcal{L}(M)) \subset \mathcal{L}(M)$.

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PROOF OF THEOREM 4.1 (3/4).

1. The case $\mathrm{pt} \in M \setminus \mathcal{L}(M)$*:*

I.e. $T_{\mathrm{pt}}M \cap \xi_{\mathrm{pt}}$ is either of dimension $< n$, or $T_{\mathrm{pt}}M \subset \xi_{\mathrm{pt}}$ is non-Lagrangian. Hence, we can find a vector

$$0 \neq V_{\mathrm{pt}} \in (T_{\mathrm{pt}}M \cap \xi_{\mathrm{pt}})^{d\alpha} \setminus T_{\mathrm{pt}}M \subset \xi_{\mathrm{pt}}$$

normal to M, and thus a function $H\colon Y \to \mathbb{R}$ satisfying

- $H|_M \equiv 0$
- $dH_{\mathrm{pt}} = -\iota_{V_{\mathrm{pt}}} d\alpha$

PROOF OF THEOREM 4.1 (4/4).

2. The case $\mathrm{pt} \in \operatorname{int} \mathcal{L}(M)$*:*

Move pt close to $\mathrm{bd}(M \setminus \mathcal{L}(M))$ by a reparametrisation of $\operatorname{int} \mathcal{L}(M)$.

3. The case $\mathrm{pt} \in \mathrm{bd}(M \setminus \mathcal{L}(M)) \subset \mathcal{L}(M)$*:*

Show that there are contact isotopies that vanish on M, but which do not induce local reparametrisations of M near pt. □

On the rigid side, we have

THEOREM 4.4 ([DRS21])

The unparametrised Chekanov–Hofer–Shelukhin pseudo-metric

$$\delta^{\alpha}(\Lambda, \Phi_1(\Lambda)) = \inf_{\left\{H_t; \tilde{\Phi}_1^{H_t}(\Lambda) = \Phi_1(\Lambda)\right\}} \int_0^1 \max_Y |H_t| dt$$

is non-degenerate on any Legendrian isotopy class of a closed Legendrian $\Lambda \subset (Y, \xi)$ *in a closed contact manifold.*

PROOF.

Continuous dependence of the spectral invariants / barcode of the Rabinowitz Floer homology of a Legendrian and its push-off; this complex is well-defined in a small action window. Then we use the dichotomy proven in [RZ20]; this pseduo-metric is either non-degenerate or vanishes completely. □

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