Algebraic Surgery over simplicial complexes and ball complexes

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Motivation

Question: Is X homotopy equivalent to a closed manifold?

One of the tools to help answer this question is **algebraic surgery**.

Idea: Try to abstract and generify algebraic properties of closed manifolds, for instance - Poincare duality, and test X against these properties.

Result: For high-dimensional ($>$4) geometric Poincare complexes there exists an algebraic invariant - **total surgery obstruction**, which strictly determines the answer to the question.

Algebraic Surgery Theory

Definition: Category with chain duality

A chain duality on additive category $\mathbb{A}$ is a pair (T, e), where:

- T is a contravariant functor $T : \mathbb{A} \longrightarrow \mathbb{B}(\mathbb{A})$;
- e is a natural transformation $e : T^2 \Longrightarrow \mathrm{id}$

such that:

- $e_M : T^2(M) \longrightarrow \mathrm{id}(M)$ is a chain equivalence in $\mathbb{B}(\mathbb{A})$;
- $e_{T(M)} \circ T(e_M) = \mathrm{id}$

Functor T is uniquely extended to a functor $T : \mathbb{B}(\mathbb{A}) \longrightarrow \mathbb{B}(\mathbb{A})$ on a category of bounded chain complexes $\mathbb{B}(\mathbb{A})$ over $\mathbb{A}$.

Example

A chain duality (T, e) on a category of finitely generated R-modules is given by $T(M) := \mathrm{Hom}_R(M, R)$ and $e : T^2 \cong id$.

Algebraic Surgery Theory

Definition:

An **algebraic bordism category** $\Lambda = (\mathbb{A}, \mathbb{B}, \mathbb{C})$ consists of:

- $\mathbb{A}$ an additive category with chain duality (T, e);
- $\mathbb{B}$ a full subcategory of $\mathbb{B}(\mathbb{A})$;
- $\mathbb{C}$ a full subcategory of $\mathbb{B}$ closed under taking cones;

Algebraic Surgery Theory

Given an algebraic bordism category $\Lambda = (\mathbb{A}, \mathbb{B}, \mathbb{C})$ we have

Structured complexes in Λ

An n-dim. $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$ algebraic complex in Λ is a pair $\begin{cases} (C, \varphi) \\ (C, \psi) \end{cases}$ where $C, C^{-*} \in \mathbb{B}$ and $\begin{cases} \varphi \in (W^{\%}(C))_n \\ \psi \in (W_{\%}(C))_n \end{cases}$ an n-cycle such that the boundary complex $\partial C = \begin{cases} \Sigma^{-1}\mathrm{cone}(\varphi_0) \\ \Sigma^{-1}\mathrm{cone}((\mathrm{id} + T)(\psi_0)) \end{cases}$ is in $\mathbb{C}$

We say that complexes in Λ are $\mathbb{B}$-contractible and $\mathbb{C}$-Poincare.

Algebraic Surgery Theory

Structured pairs in Λ

An (n+1)-dim. $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ algebraic pair in Λ is a chain map $f : C \longrightarrow D$ in $\mathbb{B}$ together with an (n+1)-cycle $\begin{cases}(\varphi, \delta\varphi) \in \mathrm{cone}(f^{\%})_n\\ (\psi, \delta\psi) \in \mathrm{cone}(f_{\%})_n\end{cases}$ such that $\begin{cases}\mathrm{cone}(\varphi, \delta\varphi)\\ \mathrm{cone}(\psi, \delta\psi)\end{cases}$ is in $\mathbb{C}$

We say that n-dim. symmetric (quadratic) complexes (C, φ) and (C', φ') are **cobordant** if there exists an (n+1)-dim. symmetric (quadratic) pair

$$(f \oplus f' : C \oplus C' \to D, (\varphi \oplus -\varphi', \delta\varphi))$$

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Definition: L-groups of Λ

The **symmetric L-groups of** Λ are defined as

$$L^n(\Lambda) := \{\text{n-dimensional SACs in } \Lambda\} \,/\, \text{cobordisms}$$

The **quadratic L-groups of** Λ are defined as

$$L_n(\Lambda) := \{\text{n-dimensional QACs in } \Lambda\} \,/\, \text{cobordisms}$$

The group operation is the direct sum. Inverse of (C, φ) is $(C, -\varphi)$.

Note:

Standard textbook L-groups of a ring $L_n(R)$ are the special case of these for $\Lambda(R) = (R\mathrm{Mod}, \mathrm{CH}_b(R), \mathrm{CH}_{cb}(R))$ a category of R-modules, bounded complexes and contractible bounded complexes.

Algebraic Surgery Theory

Given an additive category $\mathbb{A}$ and locally finite simplicial complex K we define two additive categories $\mathbb{A}^*(K)$ and $\mathbb{A}_*(K)$ as follows:

- Objects of both categories are $\left\{\bigoplus_{\sigma\in K} M_\sigma \mid M_\sigma \in \mathbb{A}\right\}$
- Morphisms of $\mathbb{A}^*(K)$ are:

$$\left\{\{f_{\sigma,\tau}\}_{\sigma\geq\tau} : \bigoplus_{\sigma\in K} M_\sigma \longrightarrow \bigoplus_{\tau\in K} N_\tau \mid (f_{\sigma,\tau} : M_\sigma \longrightarrow N_\tau) \in \mathbb{A}\right\}$$

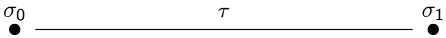
- Morphisms of $\mathbb{A}_*(K)$ are:

$$\left\{\{f_{\sigma,\tau}\}_{\sigma\leq\tau} : \bigoplus_{\sigma\in K} M_\sigma \longrightarrow \bigoplus_{\tau\in K} N_\tau \mid (f_{\sigma,\tau} : M_\sigma \longrightarrow N_\tau) \in \mathbb{A}\right\}$$

where ordering $\sigma \geq \tau$ is understood as an ordering on a dimension.

Algebraic Surgery Theory

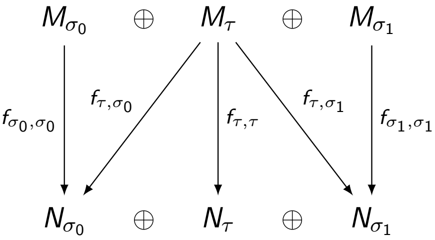
Example: For $\mathbb{A}$ some additive category and $K = \Delta^1 = \{\sigma_0, \tau, \sigma_1\}$ a simplicial complex of an interval:

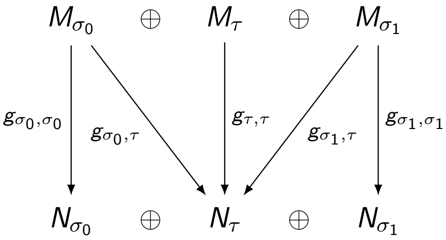


Objects are all triples of objects of $\mathbb{A}$, labeled by Δ^1:

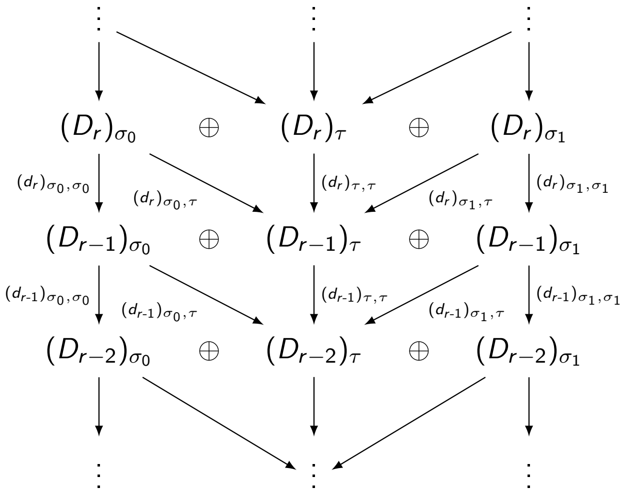
$$\{M_{\sigma_0} \oplus M_\tau \oplus M_{\sigma_1} \mid M_i \in \mathbb{A}\}$$

Morphisms $f \in \mathbb{A}^*(\Delta^1)$ and $g \in \mathbb{A}_*(\Delta^1)$ consist of the following collection of maps:





Chain complex in $\mathbb{A}_*(\Delta^1)$ is then depicted in the diagram:



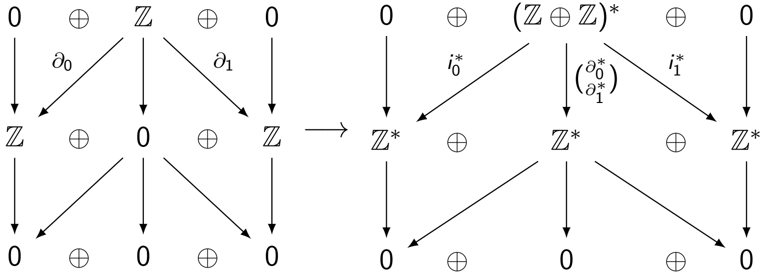
The categories $\mathbb{A}^*(K)$ and $\mathbb{A}_*(K)$ have induced chain dualities:

$$(T^*(\bigoplus_{\sigma\in K} M_\sigma)_\bullet)_\tau := (T(\bigoplus_{\tau\geq\bar{\tau}} M_{\bar{\tau}}))_{\bullet-|\tau|}$$

and

$$(T_*(\bigoplus_{\sigma\in K} M_\sigma)_\bullet)_\tau := (T(\bigoplus_{\tau\leq\bar{\tau}} M_{\bar{\tau}}))_{\bullet+|\tau|}$$

Example of a chain dual complex over $\mathbb{Z}\mathrm{Mod}^*(\Delta^1)$:



Algebraic Surgery Theory

Example: $\mathbb{Z}$-modules over K

Categories of finitely generated $\mathbb{Z}$-modules over K are denoted as:

$$\mathbb{Z}_*(K) := \mathbb{Z}\mathrm{Mod}_*(K) \qquad \mathbb{Z}^*(K) := \mathbb{Z}\mathrm{Mod}^*(K)$$

Note that, for $K = \{*\}$, we have just $\mathbb{Z}_*(\{*\}) = \mathbb{Z}^*(\{*\}) := \mathbb{Z}\mathrm{Mod}_{fg}$

Assembly

There is a well-defined **assembly functor**:

$$A : \mathbb{Z}_*(K) \longrightarrow \mathbb{Z}[\pi]\mathrm{Mod}$$
$$M \longmapsto \bigoplus_{\tilde{\sigma} \in \tilde{K}} M_{p(\tilde{\sigma})}$$

which induces a functor: $A : \Lambda(\mathbb{Z})_*(K) \longrightarrow \Lambda(\mathbb{Z}[\pi])$

The additive categories with chain duality $A^*(K)$ and $A^*(K)$ can be made into algebraic bordism categories in various ways yielding chain complexes with various types of Poincare duality:

Global and local Poincare dualities

For X a finite simplicial complex and $\mathbb{A} = \mathbb{Z}_*(X)$ we define categories:

- $\mathbb{B} = \mathbb{B}(\mathbb{Z}_*(X)) = \mathbb{Z}\mathrm{Ch}_{b*}$;
- $\mathbb{C} = \{C \in \mathbb{B} \,|\, A(C) \simeq *\}$ (global);
- $\mathbb{D} = \{C \in \mathbb{B} \,|\, C(\sigma) \simeq *,\ \forall \sigma \in X\}$ (local);

We then have the following algebraic bordism categories

$$\begin{aligned} \Lambda(X) &:= (\mathbb{A}, \mathbb{B}, \mathbb{C}) \equiv \Lambda(\mathbb{Z})(X) \\ \Lambda_*(X) &:= (\mathbb{A}, \mathbb{B}, \mathbb{D}) \equiv \Lambda(\mathbb{Z})_*(X) \\ \Lambda_*^c(X) &:= (\mathbb{A}, \mathbb{C}, \mathbb{D}) \end{aligned}$$

Algebraic Surgery Theory

For be a finite simplicial complex X we then have the following long exact sequence

$$
\begin{array}{ccccccccc}
 & & \text{globally contractible, locally Poincare} & & \text{locally Poincare} & & \text{globally Poincare} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \xrightarrow{\bar{\partial}} & L_n(\Lambda_*^c(X)) & \longrightarrow & L_n(\Lambda_*(X)) & \longrightarrow & L_n(\Lambda(X)) & \xrightarrow{\bar{\partial}} & \cdots \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
\cdots & \xrightarrow{\bar{\partial}} & \mathbb{S}_{n+1}(X) & \longrightarrow & H_n(X, \mathbb{L}) & \longrightarrow & L_n(\mathbb{Z}[\pi(X)]) & \xrightarrow{\bar{\partial}} & \cdots
\end{array}
$$

which is called an **algebraic surgery exact sequence for** X.

Symmetric construction: Given an identiy map on simplicial complex $f : X \longrightarrow X$ one can construct SAC complex over $\mathbb{Z}_*(X)$ via the following (naive) reasoning:

- We start by dissecting X into it's dual cells;
- SAC $(C(X[\sigma]), \varphi_\sigma(\mu_\sigma))$ constructed on each such small piece using Alexander-Whitney diagonal approximation;
- Finally we gather all such SACs into one big complex in $\mathbb{Z}_*(X)$ (indexed by simplexes of X) with appropriate compatibility properties;

Algebraic Surgery Theory

Visible symmetric signature

Given a finite n-dim. Poincare simplicial complex X we define **the visible symmetric signature of** X **over** X as the algebraic cobordism class of symmetric constructions on X

$$\mathrm{vssign}_X(X) := [(C(X), \varphi([X]))]_{\text{cobordism}} \in L^n(\Lambda\langle 1/2\rangle(X))$$

Note:

Originally, visible symmetric signature, lives inside a so-called **1/2-visible symmetric L-group** $VL^n(X)$, but there is a convenient isomorphism $VL^n(X) \cong L^n(\Lambda\langle 1/2\rangle(X))$.

Algebraic Surgery Theory

Total surgery obstruction

Then, **the total surgery obstruction** of n-dim. finite Poincare simplicial complex X is an element $s(X) \in \mathbb{S}_n(X)$ defined by

$$s(X) := \partial(\mathrm{vssign}_X(X)) \in \mathbb{S}_n(X) \cong L_{n-1}(\Lambda^c_*(X))$$

where the map ∂ is the boundary homomorphism.

Theorem: Ranicki

For X an $n \geq 5$-dimensional finite geometric Poincare complex there exists homotopy equivalence of X to manifold M iff $s(X_\Delta) = 0$ (here X_Δ is an n-dimensional finite geometric Poincare **simplicial** complex, such that $X \simeq X_\Delta$).

Algebraic Surgery Theory

Theorem: Product formula

For X, Y two finite Poincare complexes of dimensions n and m we have:

1. $X \times Y$ is an (n+m)-dimensional finite Poincare simplicial complex;
2. With visible signature over K given as

$$\mathrm{vssign}_{X \times Y}(X \times Y) = \mathrm{vssign}_X(X) \otimes_{(X,Y)} \mathrm{vssign}_Y(Y)$$

Problem: At the moment there is no general proof to this statement. Ranicki has proved this for the special case of $K, L = \{*\}$, but not in general.

Proposed solution: Ball complexes

Intuitive analogy for understanding the idea:

simplicial complex $\Longleftrightarrow$ **space split into triangles**

ball complex $\Longleftrightarrow$ **space split into general polyhedra**

Definition: Ball complex

A ball complex K is a finite set of PL balls in $\mathbb{R}^n$, for given $n \in \mathbb{N}$, such that:

1. all balls in K have disjoint interiors;
2. the boundary of a ball in K is a union of balls of K;

The product of two ball complexes is canonically a ball complex:

$$K \times L := \bigcup_{\sigma \in K, \tau \in L} \sigma \times \tau$$

The current roadmap, for K a ball complex:

1. Define categories $\mathbb{A}^*(K)$, $\mathbb{A}_*(K)$ (Macko, Spiros)
2. Define chain dualities (T, e) on those (Davis, Rovi)
3. Construct L-homology theory (Macko, Spiros)
4. Prove (simply connected) assembly (WIP)
5. Prove π-π theorem (TODO, hard)
6. Prove product formula (?)

Thank You for your attention!

Chain duality (T, e) allows us to define tensor products over $\mathbb{A}$:

$$C \otimes_{\mathbb{A}} D := \mathrm{Hom}_{\mathbb{B}(\mathbb{A})}(T(C), D)$$

Hence we can define W-complexes:

Definition: W-complexes

For bounded chain complex $C \in \mathbb{B}(\mathbb{A})$ we have:

$$W_{\%}(C) := W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes_{\mathbb{A}} C) \qquad W^{\%}(C) := \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_{\mathbb{A}} C)$$

where W is the canonical free $\mathbb{Z}[\mathbb{Z}_2]$-resolution.

Connectivity

For $\Lambda = (\mathbb{A}, \mathbb{B}, \mathbb{C})$ and an integer $q \in \mathbb{Z}$ we define an algebraic bordism categories

$$\Lambda\langle q\rangle := (\mathbb{A}, \mathbb{B}\langle q\rangle, \mathbb{C}\langle q\rangle) \qquad \Lambda\langle 1/2\rangle := (\mathbb{A}, \mathbb{B}\langle 0\rangle, \mathbb{C}\langle 1\rangle)$$

where:

$$\mathbb{B}\langle q\rangle := \{C \in \mathbb{B} \mid C \simeq \text{q-connected chain complex}\}$$
$$\mathbb{C}\langle q\rangle := \mathbb{B}\langle q\rangle \cap \mathbb{C}$$