Geometry of spaces of split skew-torsion

Ana Cristina Ferreira

(Work in progress with I. Agricola and S. Vasilev)

43rd Winter School in Geometry and Physics – 2023
Srní, Czech Republic

Split torsion

$\left(M = {}^{G}\!/\!_{H}, g\right)$ Riemannian homogeneous space of dim n

$\rightarrow$ reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ such that

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$$

is a splitting of $\mathfrak{m}$ into 3 (not necessarily irreducible) $\mathfrak{h}$-modules.

$\rightarrow$ ∇ invariant metric connection on M with totally skew symmetric torsion tensor T.

Definition (split torsion)
T is said to be of split type if $T(X_i, X_j) = 0$, with $X_i \in \mathfrak{m}_i, X_j \in \mathfrak{m}_j$, $i \neq j$.

T is said to be non-degenerate if for every $X_1 \in \mathfrak{m}_1$ the map $\Gamma_{X_1} : \mathfrak{m}_2 \longrightarrow \mathfrak{m}_3$ s.t. $\Gamma_{X_1}(X_2) = T(X_1, X_2)$ is a linear isomorphism.

Naturally reductive homogeneous spaces

$\left(M = {}^{G}\!/_{H}, g\right)$ Riemannian homogeneous space of dim n

$\rightarrow$ reductive decomposition: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$.

g induces a scalar product on $\mathfrak{m}$ which is ad($\mathfrak{h}$)-invariant, i.e. for $Z \in \mathfrak{h}$

$$\langle [Z, X]_{\mathfrak{m}}, Y\rangle + \langle [Z, Y]_{\mathfrak{m}}, X\rangle = 0.$$

If, moreover, the scalar product is ad($\mathfrak{g}$)-invariant, i.e. or $Z \in \mathfrak{g}$

$$\langle [Z, X]_{\mathfrak{m}}, Y\rangle + \langle [Z, Y]_{\mathfrak{m}}, X\rangle = 0.$$

$(M = G/H, g)$ is said to be a naturally reductive homogeneous space.

Naturally reductive homogeneous spaces

- $T(X,Y) = -[X,Y]_{\mathfrak{m}}$ is totally skew symmetric

→ Levi-Civita connection: $\nabla^g_X Y = \frac{1}{2}[X,Y]_{\mathfrak{m}}$

→ One-parameter family of connections with skew torsion

$$\nabla^s := \nabla^g + 2s\,T,\ s \in \mathbb{R}.$$

→ $\nabla^{1/4} := \nabla^c$ canonical connection satisfies

$$\nabla^c T^c = \nabla^c R^c = 0.$$

Classification results

→ dim 3: Tricerri & Vanhecke (LNM, 1983)

→ dims 4 and 5: Kowalski & Vanhecke (1983, 1985)

→ dim 6: Agricola, F., & Friedrich (2015)

3-locally-symmmetric spaces

Definition

A reductive homogeneous space $M = {}^{G}/_{H}$, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, with three isotropy summands $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$, satisfying the condition

$$[\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{h}$$

is called a 3-locally-symmetric space or a generalized Wallach space.

→ Classification ($\mathfrak{m}_i$ irreducible $\mathfrak{h}$-modules) by Y. Nikonorov (2016).

Proposition

A naturally reductive generalized Wallach space is a space of split torsion.

Idea of proof: The relations $[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}_k$ are satisfied.

Wallach spaces revisited

Theorem (Wallach, 1972)

Compact, simply connected, homogeneous manifold of even dimension with metric of positive sectional curvature:

- Compact, rank one, symmetric spaces (CROSS)
- Three exceptional examples:
  - $\rightarrow W^6 = {}^{\mathrm{U}(3)}\!\big/_{\mathrm{U}(1)^3}$
  - $\rightarrow W^{12} = {}^{\mathrm{Sp}(3)}\!\big/_{\mathrm{Sp}(1)^3}$
  - $\rightarrow W^{24} = {}^{\mathrm{F}_4}\!\big/_{\mathrm{Spin}(8)}$

Classification in odd dimensions: B. Bergery (1976)

Modern account with some corrections: B. Wilking and W. Ziller (2015)

Wallach spaces revisited

Theorem (Wallach, 1972)

Compact, simply connected, homogeneous manifold of even dimension with metric of positive sectional curvature:

- Compact, rank one, symmetric spaces (CROSS)
- Three exceptional examples:
  - $\rightarrow W^6 = {}^{\mathrm{U}(3)}\!/\!_{\mathrm{U}(1)^3}$ dim = 9 - 3 = 6
  - $\rightarrow W^{12} = {}^{\mathrm{Sp}(3)}\!/\!_{\mathrm{Sp}(1)^3}$ dim = 21 - 9 = 12
  - $\rightarrow W^{24} = {}^{\mathrm{F}_4}\!/\!_{\mathrm{Spin}(8)}$ dim = 52 - 28 = 24

Classification in odd dimensions: B. Bérgery (1976)

Modern account with some corrections: B. Wilking and W. Ziller (2015)

6-dimensional Wallach space

$W^6 = {}^{\mathrm{U}(3)}\!/\!{}_{\mathrm{U}(1)^3}$

→ flag manifold, twistor space of $\mathbb{C}\mathrm{P}^2$

→ carries two almost Hermitian structures

- one is Kähler-Einstein
- one is nearly-Kähler (therefore Einstein)

$$\mathfrak{g} = \mathfrak{u}(3) = \{A \in M_3(\mathbb{C}) \colon A + \overline{A}^t = 0\}$$
$$\mathfrak{h} = \mathfrak{u}(1)^3 = \{A \in M_3(\mathbb{C}) \colon A = \mathrm{diag}(h_1, h_2, h_3),\ h_i \in \mathrm{Im}(\mathbb{C})\}$$

$$\mathfrak{m} = \mathfrak{h}^\perp = \left\{ \begin{pmatrix} 0 & a & b \\ -\bar{a} & 0 & c \\ -\bar{b} & -\bar{c} & 0 \end{pmatrix} : a, b, c \in \mathbb{C} \right\}$$

→ $\mathfrak{m}$ splits as $\mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ with $[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}_k$.

12-dimensional Wallach space

$W^{12} = {}^{\mathrm{Sp}(3)}\!\big/_{\mathrm{Sp}(1)^3}$

$\mathfrak{g} = \mathfrak{sp}(3) = \{A \in M_3(\mathbb{H})\colon A + \overline{A}^t = 0\}$
$\mathfrak{h} = \mathfrak{sp}(1)^3 = \{A \in M_3(\mathbb{H})\colon A = \mathrm{diag}(h_1, h_2, h_3),\ h_i \in \mathrm{Im}(\mathbb{H})\}$

$\mathfrak{m} = \mathfrak{h}^\perp$ splits as $\mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ with $[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}_k$.

- Geometric structures on W^{12} ?

→ no almost complex structure

→ no quaternionic Kähler structure

→ no quaternionic Kähler with torsion structure

→ W^{12} carries an $\mathrm{Sp}(1)^3$-structure

24-dimensional Wallach space

$W^{24} = {}^{\mathrm{F}_4}\!/\!_{\mathrm{Spin}(8)}$

$\rightarrow M_3(\mathbb{O})$...

$\rightarrow \mathfrak{f}_4 = \mathfrak{so}(8) \oplus \mathbb{O} \oplus \mathbb{O} \oplus \mathbb{O}$ (Baez, 2001; Kollross 2020)

Baez, 2001:
'This formula emphasizes the close relation between $\mathfrak{f}_4$ and triality: the Lie bracket in $\mathfrak{f}_4$ is completely built out of maps involving $\mathfrak{so}(8)$ and its three 8-dimensional irreducible representations!'

Kollross, 2020
Explicit construction of Lie bracket structure on $\mathfrak{f}_4 = \mathfrak{so}(8) \oplus \mathbb{O} \oplus \mathbb{O} \oplus \mathbb{O}$.

A unified approach

Wallach spaces via Jordan algebras

(Massey, 1974; Ishikawa, 1999; Yokota, 2009)

- $\mathrm{H}_3(\mathbb{K}) = \{A \in M_3(\mathbb{K}) \colon A^* = A\}$.
- Jordan product $A \circ B = \frac{1}{2}(AB + BA)$

  → $(\mathrm{H}_3(\mathbb{K}), \circ)$ is an algebra with unit Id_3.

  → $\langle A, B \rangle = \mathrm{tr}(A \circ B)$ is a positive definite inner product

- Automorphism group of $\mathrm{H}_3(\mathbb{K})$:

  → $G_{\mathbb{R}} = \mathrm{O}(3)$

  → $G_{\mathbb{C}} = \mathrm{U}(3)$

  → $G_{\mathbb{H}} = \mathrm{Sp}(3)$

  → $G_{\mathbb{O}} = \mathrm{F}_4$

- $D_1 = \mathrm{diag}(1, 0, 0), \quad D_2 = \mathrm{diag}(0, 1, 0), \quad \mathrm{diag}(0, 0, 1)$

  Then $\mathrm{H}_{\mathbb{K}} = \cap_{i=3} \mathrm{Stab}(D_i)$ is equal to

  → $H_{\mathbb{R}} = \mathbb{Z}_2^3$

  → $H_{\mathbb{C}} = \mathrm{U}(1)^3$

  → $H_{\mathbb{H}} = \mathrm{Sp}(1)^3$

  → $G_{\mathbb{O}} = \mathrm{Spin}(8)$

Dirac operators with skew torsion

M spin manifold and $\mathcal{S}$ spin bundle on M

One-parameter family of metric connections with skew torsion

$$\nabla^s_X Y = \nabla^g_X Y + 2sT(X, Y, -)$$

Lifts to spin bundle

$$\nabla^s_X \psi = \nabla^g_X \psi + s(X \lrcorner T)\psi$$

Dirac operator $\mathcal{D}^s$ on $\mathcal{S}$:

$$\mathcal{D}^s\colon \Gamma(\mathcal{S}) \xrightarrow{\nabla^s} \Gamma(T^*M \otimes \mathcal{S}) \longrightarrow \Gamma(TM \otimes \mathcal{S}) \xrightarrow{\bullet} \Gamma(\mathcal{S})$$

$\bullet$ denotes Clifford multiplication

If $e_1, \cdots, e_n$ is on ON frame then $\mathcal{D}^s\psi = \sum_{i=1}^n e_i \nabla_{e_i}\psi$

Remark: $\mathcal{D}^{s/3} = \mathcal{D}^g + sT$

Dirac operators with parallel skew torsion

Recall: estimate for the first eigenvalue of $(\mathcal{D}^g)^2$

$\lambda \geq \frac{n}{4(n-1)} \mathrm{scal}^g_{min}, \quad n = \dim M$ (Friedrich, 1980)

Several estimates for Dirac operators with parallel skew torsion:

If $\nabla^c T = 0$ then for $\mathcal{D} := \mathcal{D}^{1/3}$ we have $\mathcal{D}^2 \circ T = T \circ \mathcal{D}^2$.

$\rightarrow \mathcal{S} = \bigoplus_{\mu_i} \mathcal{S}_{\mu_i}$ eigenbundles for the action of T: $\mu_1, \cdots, \mu_k$

(Agricola, Becker-Bender, Kim, 2013; Agricola, Kim, 2014)

β_{univ}: $\lambda \geq \frac{1}{4}\mathrm{scal}^g_{min} + \frac{1}{8}\|T\|^2 - \frac{1}{4}\max(\mu_1^2, \cdots, \mu_k^2)$

β_{twist}: $\lambda \geq \frac{n}{4(n-1)}\mathrm{scal}^g_{min} + \frac{n(n-5)}{8(n-3)^2}\|T\|^2 - \frac{n(n-4)}{4(n-3)^2}\max(\mu_1^2, \cdots, \mu_k^2)$

β_{split}: $\lambda \geq \frac{n_p}{4(n_p-1)}\mathrm{scal}^g_{min} + \frac{n_p}{8(n_p-1)}\|T\|^2 - \frac{1+n_p}{4(n_p-1)}\max(\mu_1^2, \cdots, \mu_k^2)$

$n_1 \leq \cdots \leq n_p$ dimensions of the modules $\mathfrak{m}_i$, $i = 1, \cdots, p$.

Examples

- W^6 (8)

  $\mathrm{scal}^g = 30,\ \|T\| = 2,\ \mu = 4,\ n = 6,\ n_p = 2$

  $\beta_{univ} = \beta_{twist} = \beta_{split} = 4$

- W^{12} (64)

  $n = 12,\ n_p = 4, \mathrm{scal}^g = 168,\ \|T\| = 4,\ \mu = 8$

  $\beta_{univ} = 28, \quad \beta_{twist} = \frac{2864}{99} \simeq 28.9, \quad \beta_{split} = 32$

- W^{24} (4096)

  $n = 24,\ n_p = 8, \mathrm{scal}^g = 768,\ \|T\| = 8,\ \mu = 20$

  $\beta_{univ} = 100, \quad \beta_{twist} = \frac{103616}{1127} \simeq 91.94, \quad \beta_{split} = 100$

[Thanks to M. Panizzut and T. Beuer (Aachen) for the GAP code which computed the minimum polinomial for the action of T on W^{24}]

Many thanks for the attention!