Losik classes and Reeb foliations

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(joint work with Ya.Bazaikin and P. Gumenyk)

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References

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Ya.V. Bazaikin, A.S. Galaev, Losik classes for codimension one foliations. Journal of the Institute of Mathematics of Jussieu 21 (2022), 1391—1419.

Mark Losik (1935–2013)

M. V. Losik, On some generalization of a manifold and its characteristic classes. Functional Anal. Appl. 24 (1990), 26–32.

M. V. Losik, Categorical differential geometry. Cahiers de topol. et geom. diff. cat. 35 (1994), no. 4, 274–290.

M. V. Losik, Orbit spaces and leaf spaces of foliations as generalized manifolds. arXiv:1501.04993

Godbillon-Vey class (1971). $(M^3, \mathcal{F}^2)$, normal bundle is oriented; $\mathcal{F}$ is the kernel of a 1-form ω.

$d\omega = \eta \wedge \omega$

$GV := [\eta \wedge d\eta] \in H^3(M)$.

The geometric meaning is still unclear (Thurston, Sullivan, Duminy, Hurder,.....)

S. Hurder, Dynamics and the Godbillon-Vey classes: a history and survay. In Foliations: Geometry and Dynamics (Warsaw, 2000)

Berstein, Rozenfeld; Bott, Haefliger: GV class comes from the Gel'fand-Fuchs cohomology

$$\begin{array}{c} H^3(W_1, \mathrm{O}_1) \cong \mathbb{R} \\ \Big\downarrow \\ H^3(M) \end{array}$$

$$
\begin{array}{c}
H^3(W_1, \mathrm{O}_1) \cong \mathbb{R} \\
\downarrow \\
H^3(S(M/\mathcal{F})/\mathrm{O}_1 \times \mathbb{Z}) \\
\downarrow \\
H^3(S(M/\mathcal{F})/\mathrm{O}_1) \\
\downarrow \\
\check{H}^3(M/\mathcal{F}) = H^3(BG_T) \\
\downarrow \\
H^3(M)
\end{array}
$$

$$\begin{array}{ccc}
H^3(W_1, \mathrm{O}_1) \cong \mathbb{R} & & H^2(W_1, \mathrm{GL}_1) \cong \mathbb{R} \\
\downarrow & & \downarrow \\
H^3(S(M/\mathcal{F})/\mathrm{O}_1 \times \mathbb{Z}) & \dashleftarrow \dashrightarrow & H^2(S(M/\mathcal{F})/\mathrm{GL}_1) \\
\downarrow & & \\
H^3(S(M/\mathcal{F})/\mathrm{O}_1) & & \\
\downarrow & & \\
H^3(M) & &
\end{array}$$

Foliation. $(M^m, \mathcal{F})$, $\mathrm{codim}\mathcal{F} = n$

M is a union of submanifolds (leaves) of dimension $m - n$ such that locally

$$\phi = (g, f) : V \to W \times U \subset \mathbb{R}^{n-m} \times \mathbb{R}^n$$

and for any leaf L the connected components of $L \cap V$ are given by $f = \mathrm{const}$.

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Foliation atlas: $\varphi_i : V_i \to W_i \times U_i \subset \mathbb{R}^{n-m} \times \mathbb{R}^n$

Coordinate change: $\varphi_{ij}(x, y) = (g_{ij}(x, y), f_{ij}(y))$

Holonomy transformations: $f_{ij} : U_i \to U_j$

Complete transversal: $N = \cup U_i$

Holonomy pseudogroup: (N, G), $G = \langle f_{ij} \rangle$

Leaf space:

$$M/\mathcal{F} = N/G$$

Leaf holonomy.

$L \subset M$ is a leaf, $x \in L$, $\gamma : [0,1] \to L$ is a closed curve at x

$\mathrm{hol}(\gamma) \in \mathrm{Diff}_x(D)$

$$\mathrm{hol} : \pi_1(L, x) \to \mathrm{Diff}_x(D)$$

The image is the **holonomy group** of the leaf L

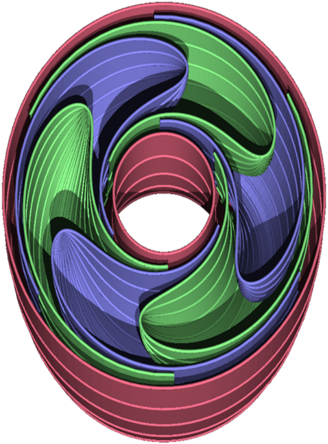
A Reeb foliation on $T = S^1 \times D^2$ is defined by a function

$f : (-1, 1) \to \mathbb{R}$, $f(-x) = f(x)$,

$f^{(k)}(x) \to +\infty$, $(1/f')^{(k)}(x) \to 0$ as $x \to \pm 1$

$$\varphi(t) = f^{-1}(f(t) + 1)$$

$$\varphi(1) = 1, \quad \varphi'(1) = 1, \quad \varphi''(1) = \cdots = 0$$



https://commons.wikimedia.org/wiki/File:
Reeb_foliation_half-torus_POV-Ray.png

$\mathcal{R}_f$, $\mathcal{R}_g$ two such foliations $\mathcal{R}_{f,g}$ is a Reeb foliation on

$$S^3 = T \cup_{S^1 \times S^1} T$$

Leaf holonomy

$Hol(L) = \mathbb{Z} \oplus \mathbb{Z},\ L = S^1 \times S^1$

$Hol(L) = 0,\ L \neq S^1 \times S^1$

$$\varphi(x) = f^{-1}(f(x) + 1),\ \ x < 1, \quad \varphi(x) = x,\ \ x \geq 1$$

$$\varphi(1) = 1, \quad \varphi'(1) = 1, \quad \varphi''(1) = \cdots = 0$$

Theorem (Morita, Tsuboi) $(M, \mathcal{F})$ is without holonomy $=>$ $\mathrm{GV}(\mathcal{F}) = 0$.

Theorem (Mizutani, Morita, Tsuboi) $(M, \mathcal{F})$ is **almost** without holonomy $=>$ $\mathrm{GV}(\mathcal{F}) = 0$.

Cor. $\mathrm{GV}(\mathcal{R}_{f,g}) = 0$

L is a **resilient leaf**: $\exists\ x \in L$, U through x, $y \in L \cap U$, f such that $f^n(y) \to x$ as $n \to +\infty$

Theorem (Duminy, Sergiescu) $(M, \mathcal{F})$ has no resilient leaf $=>$ $\mathrm{GV}(\mathcal{F}) = 0$.

$U \subset \mathbb{R}$ a transversal to $\mathcal{F}$

$S_2(U)/\mathrm{O}_1 = U \times \mathbb{R} \times \mathbb{R}$ with coordinates x_0, x_1, x_2

$m : U \to V$ holonomy transformation

$\tilde{m} : S_2(U)/\mathrm{O}_1 \to S_2(V)/\mathrm{O}_1$

$\alpha_0 = \tilde{m}(x_0)$,

$\alpha_1 = x_1 + \ln |m'(x_0)|$,

$\alpha_2 = \frac{x_2}{m'(x_0)} + \frac{m''(x_0)}{(m'(x_0))^2}$

Important: $\tilde{m}^*(d\alpha_0 \wedge d\alpha_1 \wedge d\alpha_2) = dx_0 \wedge dx_1 \wedge dx_2$

Leaf space $M/\mathcal{F}$

$\cup U_\alpha$ a complete transversal, $U_\alpha \subset \mathbb{R}$

$M/\mathcal{F} = (\cup U_\alpha)/\{\text{holonomy transformations}\}$

$S_2(M/\mathcal{F})/\mathrm{O}_1 = (\cup S_2(U_\alpha)/\mathrm{O}_1)/\{\text{holonomy transformations}\}$

$\omega \in \Omega^k(S_2(M/\mathcal{F})/\mathrm{O}_1) \longleftrightarrow \forall\alpha\ \omega_\alpha \in \Omega^k(S_2(U_\alpha)/\mathrm{O}_1)$,
$\tilde{m}^*\omega_\beta = \omega_\alpha$

De Rham cohomology $H^k(S_2(M/\mathcal{F})/\mathrm{O}_1)$

Godbillon-Vey-Losik class

$$\mathrm{GVL}(M/\mathcal{F}) = [-dx_0 \wedge dx_1 \wedge dx_2] \in H^3(S_2(M/\mathcal{F})/\mathrm{O}_1)$$

$$H^3(S_2(M/\mathcal{F})) \to H^3(M)$$

$$\mathrm{GVL}(M/\mathcal{F}) \mapsto \mathrm{GV}(\mathcal{F})$$

Remark

$$\mathrm{pt}_1 = \mathbb{R}/\{\text{all local diffeomorphisms}\}$$

$$H^*(S_2(\mathrm{pt}_1)/\mathrm{O}_1) \cong H^*(W_1, O_1)$$

The projection

$$M/\mathcal{F} \to \mathrm{pt}_1$$

induces

$$\mathbb{R} = H^3(S_2(\mathrm{pt}_1)/\mathrm{O}_1) \to H^3(S_2(M/\mathcal{F})/\mathrm{O}_1)$$

$S'_2(U) = S_2(U)/\mathrm{GL}_1$ with coordinates x_0, x_2

the first Chern-Losik class

$$\mathrm{CL}_1(M/\mathcal{F}) = [dx_2 \wedge dx_0] \in H^2(S_2(M/\mathcal{F})/\mathrm{GL}_1)$$

$$H^2(S_2(M/\mathcal{F})/\mathrm{GL}_1) \to H^2(M)$$
$$\mathrm{CL}_1(M/\mathcal{F}) \mapsto 0$$

Theorem. $\mathrm{CL}_1(M/\mathcal{F}) = 0 \Rightarrow \mathrm{GVL}(M/\mathcal{F}) = 0$

Foliations without holonomy. $(M, \mathcal{F})$

$\exists$ $(E, \tilde{\mathcal{F}})$ such that $E/\tilde{\mathcal{F}} = \mathbb{R}/\mathrm{Im}(q)$,

$$q : \pi_1(M) \to \mathrm{Diff}_+(\mathbb{R})$$

$\exists$ $\sigma : M \to E$ inducing $M/\mathcal{F} \to E/\tilde{\mathcal{F}}$

$\mathrm{Im}(q) = \langle \varphi_1, \dots \varphi_p \rangle$ is commutative and its elements have no fixed points in $\mathbb{R}$

Theorem If $\rho(\varphi_i, \varphi_j)$ is Diophantine for some $i \neq j$ then $\mathrm{Im}(q)$ is conjugated to a group of shifts, and $\mathrm{CL}_1(M/\mathcal{F}) = 0$.

Reeb foliation

$M/\mathcal{F} = \mathbb{R}/ < \varphi >$

$\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(0) = 0$, $\varphi'(0) = 1$, $\varphi''(0) = \varphi'''(0) = \cdots = 0$

$\mathrm{GVL}(M/\mathcal{F}) = 0 \Longleftrightarrow \exists\, \omega \in \Omega^2(\mathbb{R}^3)$

$\tilde{\varphi}^*\omega = \omega, \quad d\omega = -dx_0 \wedge dx_1 \wedge dx_2$

Theorem. For all Reeb foliations $\mathrm{CL}_1(M/\mathcal{F}) \neq 0$.

CL_1 detects the compact leaf with non-trivial holonomy!

Theorem. (generalization) If a holonomy diffeomorphism of a foliation has a non-hyperbolic fixed point, then $\mathrm{CL}_1(M/\mathcal{F}) \neq 0$.

Let a Reeb foliation be given by a local diffeomorphism φ

$\exists\ \varphi_t,\ \varphi_1 = \varphi$

φ_t defines V called the Szekeres vector field

$\mathrm{GVL}(M/\mathcal{F}) = 0 \iff \exists\ \omega \in \Omega^2(\mathbb{R}^3)$

$L_{\tilde{V}}\omega = \omega, \quad d\omega = -dx_0 \wedge dx_1 \wedge dx_2$

$$\tilde{V} = (V, V', -x_2 V' + V'')$$

$$V_\alpha(x) = \begin{cases} e^{-\frac{1}{|x|^\alpha}}, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$$

Theorem. $\alpha \notin \mathbb{N} \Rightarrow \mathrm{GVL}(M/\mathcal{R}_\alpha) \neq 0$

Theorem. $\alpha \in \mathbb{N} \Rightarrow \mathrm{GVL}(M/\mathcal{R}_\alpha) = 0$

Corollary. If $\alpha \notin \mathbb{N}$ is and $\beta \in \mathbb{N}$, then the foliations $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ are not diffeomorphic.