On non-geometric augmentations of Legendrian submanifolds

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Motivation

- Given a Legendrian submanifold of $(\mathbb{R}^{2n+1}, dz - ydx)$, Chekanov-Eliashberg algebra $\mathcal{A}(\Lambda)$ gives a powerful Legendrian invariant. Linearizations of this algebra (called augmentations) are important in order to define the computable Legendrian invariants called linearized Legendrian contact (co-)homology groups.

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- These linearizations are very often geometric, i.e. they are induced by Maslov 0 embedded exact Lagrangian fillings.
- We are interested in finding non-geometric augmentations in high dimensions.

Known obstructions

If Λ admits an exact Lagrangian filling L_Λ of Maslov number 0, then

- $tb(\Lambda) = (-1)^{\frac{(n-1)(n-2)}{2}+1}\chi(L)$,
- Obstruction A: there is an isomorphism of Seidel-Ekholm-Dimitroglou Rizell $LCH^i_\varepsilon(\Lambda; \mathbb{Z}_2) \simeq H_{n-i}(L; \mathbb{Z}_2)$,
- the result of Ekholm-Lekili comparing the higher structures holds,
- Obstruction B: the resut of Gao-Rutherford showing the existence of injective algebraic map $f_L^* : Aug(L, \mathbb{F}) \to Aug(\Lambda, \mathbb{F})$ holds.

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We will use Obstruction A and Obstruction B in order to find non-geometric augmentations.

Contact preliminaries

- A contact manifold $(M;\xi)$ is a $(2n+1)$-dimensional manifold M equipped with a smooth maximally nonintegrable hyperplane field $\xi \subset TM$, i.e., $\xi = \ker\alpha$, where α is a 1-form which satisfies $\alpha \wedge (d\alpha)^n \neq 0$. ξ is a contact structure and α is a contact 1-form which defines ξ.

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Example

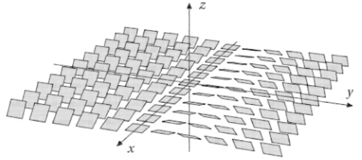

Figure: $(\mathbb{R}^{2n+1}, \xi_{st} = \ker(dz - \sum_i y_i dx_i))$, where $n = 1$.

Legendrian submanifolds and Reeb chords

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- A Reeb chord of Λ is a trajectory of ∂_z which starts and ends on Λ. The set of Reeb chords of Λ is denoted by $\mathcal{Q}(\Lambda)$.

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$$\partial(c^+) = \sum_{\dim \mathcal{M}^J(c^+;c_1^-,\dots,c_k^-)=1} \# \frac{M^J(c^+;c_1^-,\dots,c_k^-)}{\mathbb{R}} c_1^- \dots c_k^-,$$

where J is an almost complex structure on $(\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^s\alpha_{st}))$ such that $J : \xi \to \xi$, $d\alpha_{st}(v, Jv) > 0$ for $v \in \xi$, J is $\mathbb{R}$-invariant, $J(\partial_s) = \partial_z$ and $M^J(c^+; c_1^-, \dots, c_k^-)$ is a moduli space of punctured i-J holomorphic disks in $(\mathbb{R} \times \mathbb{R}^{2n+1}, J)$

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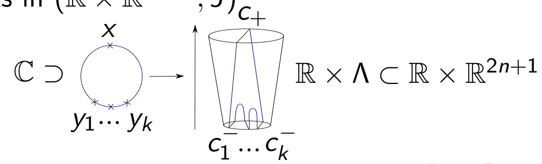

Exact Lagrangian cobordisms and fillings

We say that Λ^- is exact Lagrangian cobordant to Λ^+ if there is a smooth cobordism $(L; \Lambda^-, \Lambda^+)$ and an exact Lagrangian embedding $L \hookrightarrow S(\mathbb{R}^{2n+1}_{st})$, where $S(\mathbb{R}^{2n+1}_{st}) := (\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^t \alpha_{st}))$ and t is the coordinate on the first $\mathbb{R}$-factor, satisfying the following conditions:

- $L|_{(-\infty,-T)\times\mathbb{R}^{2n+1}_{st}} = (-\infty, -T) \times \Lambda^-$ and $L|_{(T,\infty)\times\mathbb{R}^{2n+1}_{st}} = (T, \infty) \times \Lambda^+$ for some $T \gg 0$,
- $L^c := L|_{[-T,T]\times\mathbb{R}^{2n+1}_{st}}$ is compact.
- There exists $f : L \to \mathbb{R}$ such that $e^t \alpha_{st}|_L = df$ and $f|_{(-\infty,-T)\times\Lambda^-}$, $f|_{(T,\infty)\times\Lambda^+}$ are constant functions.

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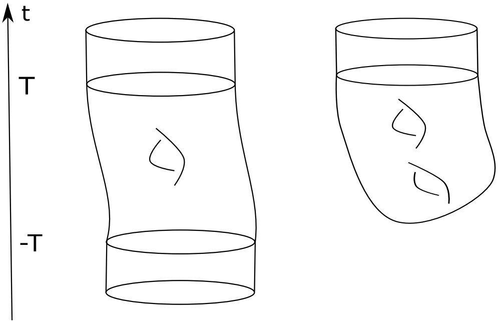
If L is an exact Lagrangian cobordism with empty negative end and whose positive end is equal to Λ, then we say that L is an exact Lagrangian filling of Λ.

Exact Lagrangian fillings and augmentations



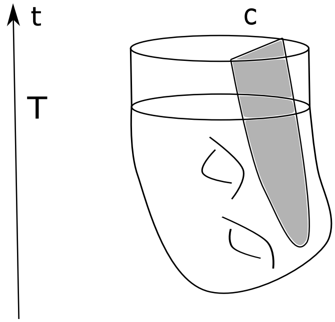
In general (by Ekholm-Honda-Kalman), having an exact Lagrangian cobordism L from Λ^- to Λ^+, there is a DGA homomorphism

$$\mathcal{A}(\Lambda^+) \to \mathcal{A}(\Lambda^-).$$

In particular, if $\Lambda^- = \emptyset$, then there is a DGA homomorphism that we call augmentation

$$\varepsilon_L : \mathcal{A}(\Lambda^+) \to (\mathbb{F}, 0).$$

Exact Lagrangian fillings and augmentations



For $c \in \mathcal{Q}(\Lambda)$, it is defined by

$$\varepsilon_L(c) = \#\mathcal{M}^J_L(c).$$

Isomorphism of Seidel–Ekholm–Dimitroglou Rizell, Obstruction A

Theorem (Seidel–Ekholm–Dimitroglou Rizell)

Let Λ be a Legendrian submanifold of Maslov number 0 of $\mathbb{R}^{2n+1}_{st}$, which admits an exact Lagrangian filling L of Maslov number 0. Then

$$LCH^i_{\varepsilon}(\Lambda;\mathbb{Z}_2) \simeq H_{n-i}(L_{\Lambda};\mathbb{Z}_2)$$

where ε is the augmentation induced by L.

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The homology and cohomology groups in the above result are defined over $\mathbb{Z}_2$, but following the modern techniques due to Karlsson and Ekholm–Lekili, one can generalize the isomorphism of Seidel–Ekholm–Dimitroglou Rizell to an arbitrary field and $\mathbb{Z}$-coefficients, i.e.

$$LCH^i_{\varepsilon}(\Lambda; \mathbb{Z}) \simeq H_{n-i}(L_{\Lambda}; \mathbb{Z}).$$

Class A

In this class we consider Legendrian knots Λ in $\mathbb{R}^3_{st}$ such that the Chekanov–Eliashberg algebra admits an augmentation

$$\varepsilon : \mathcal{A}(\Lambda) \to (\mathbb{Z}_2, 0)$$

satisfying that for some $i \in \mathbb{Z}$, $LCH^i_\varepsilon(\Lambda; \mathbb{Z}_2)$ is not isomorphic to $H_{1-i}(L_\Lambda; \mathbb{Z}_2)$ for all exact Lagrangian fillings L_Λ of Maslov number 0, i.e. $H_{1-i}(L_\Lambda; \mathbb{Z}_2)$ is different from 0 for $1 - i < 0$ or $1 - i > 1$.

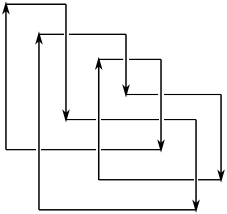

Figure: The grid diagram of the Legendrian representative of $m(8_{21})$.

Class A

We rely on the work of Chongchitmate–Ng. Legendrian representative of $m(8_{21})$ has a vanishing rotation number, and hence Maslov number 0, and two Poincaré polynomials, one of which is of the form

$$P_{m(8_{21})}(t) = t^{-1} + 4 + 2t,$$

The augmentation $\varepsilon_{m(8_{21})}$ which corresponds to this polynomial has the property that

$$LCH_{-1}^{\varepsilon_{m(8_{21})}}(\Lambda; \mathbb{Z}_2) \simeq LCH^{-1}_{\varepsilon_{m(8_{21})}}(\Lambda; \mathbb{Z}_2) \simeq \mathbb{Z}_2.$$

From Seidel's isomorphism it follows that $\varepsilon_{m(8_{21})}$ is not induced by a Maslov number 0 exact Lagrangian filling L, since otherwise $H_2(L; \mathbb{Z}_2) \simeq \mathbb{Z}_2$, which is impossible from the topological point of view.

Result of Gao-Rutherford, Obstruction B

Let L be an exact Lagrangian filling of a Legendrian submanifold Λ, then $H_1(L;\mathbb{Z}) \simeq \mathbb{Z}^k \oplus \mathbb{Z}/k_1 \oplus \cdots \oplus \mathbb{Z}/k_s$. We define

$$Aug(L;\mathbb{F}) \simeq (\mathbb{F}^*)^k \oplus C_{k_1}(\mathbb{F}) \oplus \cdots \oplus C_{k_r}(\mathbb{F}),$$

where $C_{k_i}(\mathbb{F})$ is the group of k_i-th roots of unity in $\mathbb{F}$.

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Theorem (Gao-Rutherford)

Let L be an exact Lagrangian filling of a Legendrian submanifold $\Lambda \subset (\mathbb{R}^{2n+1}, \alpha_{st})$ such that the Maslov number of L vanishes. If $\mathbb{F}$ has a characteristic differrent from 2, assume that L is equipped with a choice of spin structure. Then, the map $f_L^* : Aug(L,\mathbb{F}) \to Aug(\Lambda,\mathbb{F})$ is an injective, algebraic map, where $f_L : \mathcal{A}(\Lambda, \mathbb{F}[\pi_1(\Lambda, x_0))]) \to \mathbb{F}[\pi_1(L, x_0)]$.

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For Λ_n and a certain algebraically closed $\mathbb{F}$, one gets an injective algebraic map $\mathbb{F}^2 \to \{(a,b)\ ab \neq -1\}$ with $(0,0)$ in its image which is impossible (Gao-Rutherford).

Class B

Consider the following Legendrian representatives of twist knots Λ_n, n is odd and $n > 3$, investigated by Gao–Rutherford.

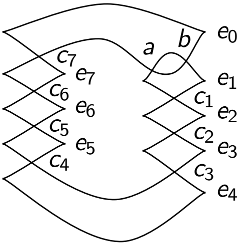


Figure: The front projection of Λ_7.

As proven by Gao–Rutherford, there is an augmentation ε_{Λ_n} to $\mathbb{Z}_2$ defined by $\varepsilon_{\Lambda_n}(a) = 0, \varepsilon_{\Lambda_n}(b) = 0, \varepsilon_{\Lambda_n}(c_i) = 1, \varepsilon_{\Lambda_n}(e_j) = 0$ that is not induced by any Maslov 0 exact Lagrangian filling.

Spherical spinning

Let Λ be a closed, orientable Legendrian submanifold of $\mathbb{R}^{2n+1}_{st}$ parameterized by $f_\Lambda : \Lambda \to \mathbb{R}^{2n+1}$ with

$$f_\Lambda(p) = (x_1(p), y_1(p), \dots, x_n(p), y_n(p), z(p))$$

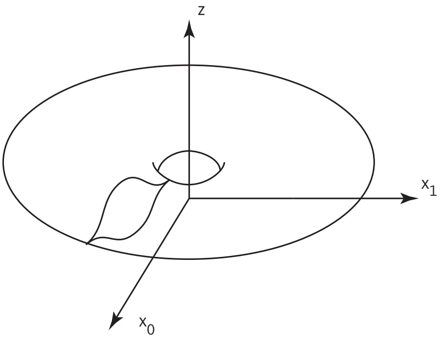
for $p \in \Lambda$. Without loss of generality assume that $x_1(p) > 0$ for all p. We define the S^m-spun $\Sigma_{S^m}\Lambda$ to be the Legendrian submanifold of $\mathbb{R}^{2(m+n)+1}$ whose xz-projection is parametrized by $\Phi : \Lambda \times S^m \to \mathbb{R}^{n+m+1}$ with

$$\Phi(p, \theta, \overline{\phi}) = (\tilde{x}_{-m+1}(p, \theta, \overline{\phi}), \dots, \dots, \tilde{x}_1(p, \theta, \overline{\phi}), x_2(p), \dots, z(p)),$$

where $\theta \in [0, 2\pi)$, $\overline{\phi} = (\phi_1, \dots, \phi_{m-1}) \in [0, \pi]^{m-1}$ and

$$\begin{cases} \tilde{x}_{-m+1}(p, \theta, \overline{\phi}) = x_1(p) \sin\theta \sin\phi_1 \dots \sin\phi_{m-1}, \\ \tilde{x}_{-m+2}(p, \theta, \overline{\phi}) = x_1(p) \cos\theta \sin\phi_1 \dots \sin\phi_{m-1}, \\ \dots \\ \tilde{x}_1(p, \theta, \overline{\phi}) = x_1(p) \cos\phi_{m-1}. \end{cases}$$

Spherical spinning



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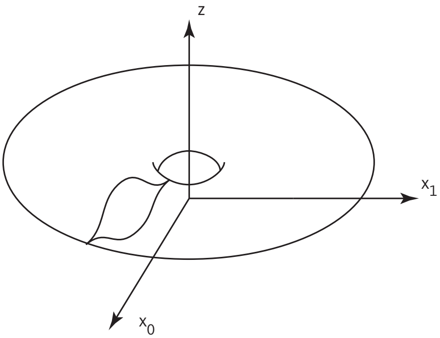

- From this definition it follows that $\Sigma_{S^m}\Lambda$ is diffeomorphic to $S^m \times \Lambda$.
- One can apply spherical spinning in a similar way to exact Lagrangian cobordisms and fillings. In particular, given an exact Lagrangian filling L of a Legendrian Λ, following the older result of [G], one gets an exact Lagrangian cobordism (filling) $\Sigma_{S^m}L$ of $\Sigma_{S^m}\Lambda$ such that $\Sigma_{S^m}L$ is diffeomorphic to $S^m \times L$.

Class A in high dimensions

Theorem (Dimitroglou Rizell-G)

There are dga maps $i : \mathcal{A}(\Lambda) \hookrightarrow \mathcal{A}(\Sigma_{S^m}\Lambda)$ *and its left inverse* $\pi : \mathcal{A}(\Sigma_{S^m}\Lambda) \twoheadrightarrow \mathcal{A}(\Lambda)$.

We take Λ from Class A with the corresponding ε and consider $\pi^*(\varepsilon)$. Using the fact that the potential topology of a filling contradicts SEDR isomorphism and the following Kunneth-type formula

Theorem (Dimitroglou Rizell-G)

$LCH_i^{\pi^*\varepsilon}(\Sigma_S^1\Lambda) \simeq LCH_i^{\varepsilon}(\Lambda) \oplus LCH_{i-1}^{\varepsilon}(\Lambda)$.

we get that $LCH_i^{\pi^*\varepsilon}(\Sigma_S^1\Lambda)$ contradicts the potential topology of a filling, and hence by SEDR $\pi^*\varepsilon$ is non-geometric. We can repeat the same procedure.

Class B in high dimensions

Here we take $\Sigma_{S^m}\Lambda_n$, where Λ_n is from Class B, and m is high enough.

Proposition (G)

$Aug(\Lambda, \mathbb{F}) \simeq Aug(\Sigma_{S^m}, \mathbb{F})$ for $m \geq 2$.

Proposition (G)

For a certain m large enough ("out of the grading window")
$LCH_i^{\pi^*\varepsilon}(\Sigma_S^m\Lambda, \mathbb{Z}) \simeq LCH_i^{\varepsilon}(\Lambda, \mathbb{Z}) \oplus LCH_{i-m}^{\varepsilon}(\Lambda, \mathbb{Z})$.

Then we need to redo the computations of $LCH_\varepsilon^0(\Lambda_n)$ for $\mathbb{Z}$-coefficients in order to show that $LCH_i^m(\Sigma_{S^m}\Lambda_n, \mathbb{Z}) \simeq \mathbb{Z}^2$. This using SEDR isomorphism implies that H_1(potential filling for $\pi^*\varepsilon) \simeq \mathbb{Z}^2$. Hence the map of Gao-Rutherford must be $\mathbb{F}^2 \to \{ab \neq -1\}$ with $(0,0)$ in the image, where $\mathbb{F}$ is algebraicly closed. Gao-Rutherford proved that it is impossible, and hence $\pi^*\varepsilon$ is non-geometric. Then we can do the same for multiple spuns.

Thank you for your attention!