Interpreting the standard cotractor connection associated to a (generalized) path geometry

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Definition

A *path geometry* on a manifold N is a collection of unparametrized smooth curves such that in each point $x \in N$ for each direction $\mathbb{R}X, 0 \neq X \in T_xN$, there is exactly one curve in the collection passing through the point x with a velocity tangent to the direction $\mathbb{R}X$.

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Example

Let $[\nabla]$ be a projective structure on N. Then each connection in the projective class gives rise to the same collection of unparametrized geodesics.

Path geometry $\longleftrightarrow$ generalized path geometry

A path geometry on N $\Leftrightarrow$ a rank 1 distribution $E \subseteq TPTN$ on

$$PTN := \bigcup_{x \in N} \{\text{1-dim'l subspaces of } T_x N\}$$

such that

$$\begin{aligned} &(\text{the tautological bundle } H \text{ of } PTN) \\ \cong\ & E \oplus (\text{the vertical bundle } V \text{ of } PTN \to N) \end{aligned}$$

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The decomposed distribution $E \oplus V \subseteq TPTN$ is a generalized path geometry on the manifold PTN.
Conversely, any generalized path geometry $E \oplus V \subseteq TM$ on a manifold M (with $dim(M) \neq 5$) is locally a path geometry on some manifold N restricted to an open subset of directions $\subseteq PTN$.

The corresponding parabolic geometries

A *parabolic geometry* is a Cartan geometry whose type (G, P) comes from a graded semisimple Lie algebra, in our case

$$\mathfrak{g} := \mathfrak{sl}(n+2,\mathbb{R}) = \left(\begin{array}{c|c|c} \mathfrak{g}_0 & \mathfrak{g}_1^E & \mathfrak{g}_2 \\ \hline \mathfrak{g}_{-1}^E & \mathfrak{g}_0 & \mathfrak{g}_1^V \\ \hline \mathfrak{g}_{-2} & \mathfrak{g}_{-1}^V & \mathfrak{g}_0 \end{array}\right) \text{ in blocks} (1,1,n)\times(1,1,n)$$

$\Rightarrow \mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

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A *Cartan geometry* of type (G, P) is

(i) a principal P-bundle $\mathcal{G} \to M$ together with

(ii) an equivariant (principal, Adjoint) global trivialisation $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})^P$ on $T\mathcal{G}$ which maps fundamental vector fields to their generators.

Then $TM \cong \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$ has subbundles $E := \mathcal{G} \times_P \mathfrak{g}_{-1}^E/\mathfrak{p}$, $V := \mathcal{G} \times_P \mathfrak{g}_{-1}^V/\mathfrak{p}$.

A path geometry on N^{n+1} $\rightsquigarrow$ a unique regular normal parabolic geometry of type (G, P) on $PTN =: M$

Via each Weyl structure (reduction to the frame bundle), the Cartan connection is encoded as the following:

- $Q \subseteq TM$: a complement of H
- ∇: a *Weyl connection* on all natural bundles (e.g. $TM, E, V, TM/H$ ...) on M
- $P \in \Gamma(\otimes^2 T^*M)$: *Rho tensor*

Which we use to describe the tractor calculus on the standard cotractor bundle.

Tractor calculus

The pair $(\mathcal{G}, \omega)$ can be equivalently encoded as a vector bundle endowed with a linear connection.

- $\mathcal{G} \hookrightarrow \mathcal{G} \times_P G$ to structure group G, the latter has a principal connection induced by ω.
- Any associated vector bundle of $\mathcal{G} \times_P G$ comes with a linear connection (tractor bundle, tractor connection).

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- Any associated vector bundle of $\mathcal{G} \times_P G$ comes with a linear connection (tractor bundle, tractor connection).
- Consider the standard cotractor bundle $\mathcal{T}^*$ w.r.t. the standard representation of $G = SL(n+2, \mathbb{R})$ on $(\mathbb{R}^{n+2})^*$.

(Remark: $\mathcal{T}^*$ is closely related to the projective standard cotractor bundle in the case of a projective structure on N, whose tractor calculus is well understood.)

- The canonical reduction of $\mathcal{G} \times_P G$ to structure group P induces a $(n \leq n+1 \leq n+2)$ filtration on $\mathcal{T}^*$.
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$$\mathcal{T}^* \cong \mathcal{E}(1,0) \oplus E^* \otimes \mathcal{E}(1,0) \oplus Q^* \otimes \mathcal{E}(1,0)$$

  Let $\pi : PTN \to N^{n+1}$. $\mathcal{E}(1,0) = \pi^* \mathcal{E}(1)$ where $\mathcal{E}(1)$ is a root of $\wedge^{n+1} T^* N$.

In these terms, an explicit formula for the tractor connection looks as follows:

General formula; the splitting operator

Let
$\Gamma(\mathcal{T}^*) \ni \sigma \cong (s, t, Y) \in \Gamma(\mathcal{E}(1,0) \oplus E^* \otimes \mathcal{E}(1,0) \oplus Q^* \otimes \mathcal{E}(1,0))$,
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$(\xi, \eta, \zeta) \in \Gamma(E \oplus V \oplus Q)$,

$$\nabla^{\mathcal{T}^*}_{\xi}(\sigma) \cong (\nabla_\xi s - t\xi, \nabla_\xi t - sP(\xi, \cdot|_E), \nabla_\xi Y - sP(\xi, \cdot|_Q) - tP(\xi, \cdot|_V))$$

$$\nabla^{\mathcal{T}^*}_{\eta}(\sigma) \cong (\nabla_\eta s, \nabla_\eta t - sP(\eta, \cdot|_E) - Y\eta, \nabla_\eta Y - sP(\eta, \cdot|_Q) - tP(\eta, \cdot|_V))$$

$$\nabla^{\mathcal{T}^*}_{\zeta}(\sigma) \cong (\nabla_\zeta s - Y\zeta, \nabla_\zeta t - sP(\zeta, \cdot|_E), \nabla_\zeta Y - sP(\zeta, \cdot|_Q) - tP(\zeta, \cdot|_V))$$

The splitting operator is $L : s \mapsto (s, \nabla^E s, \nabla^Q s - \frac{1}{2}\nabla^E \nabla^V s)$ in a non-standard form. In this case

$$\begin{aligned}
\nabla^{\mathcal{T}^*}_{E}(L(s)) &\cong (0, \quad \nabla^E\nabla^E s - P^{E^*\otimes E^*}s, \quad (*)) \\
\nabla^{\mathcal{T}^*}_{V}(L(s)) &\cong (\nabla^V s, \quad -\tfrac{1}{2}\nabla^E\nabla^V s, \quad (**)) \\
\nabla^{\mathcal{T}^*}_{Q}(L(s)) &\cong (0, \quad \nabla^Q\nabla^E s - P^{Q^*\otimes E^*}s, \quad (***))
\end{aligned}$$

Where we read off two invariant (BGG) operators.

Interpreting the two BGG operators

- $\nabla^V s = 0 \Leftrightarrow s = \pi^* \underline{s}$ for some $\underline{s} \in \Gamma(\mathcal{E}(1))$.
  - Moreover, $\underline{s} \mapsto L(\pi^* \underline{s})$ yields $\pi^* J^1(\mathcal{E}(1) \to N) \cong \mathcal{T}^*$

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  - Moreover, $\underline{s} \mapsto L(\pi^* \underline{s})$ yields $\pi^* J^1(\mathcal{E}(1) \to N) \cong \mathcal{T}^*$
- $\nabla^E \nabla^E s - P^{E^* \otimes E^*} s$: interpretation in terms of a 1-dim'l projective structure on the leaves of the foliation determined by E.

If $s = \pi^* \underline{s}$, then one can interpret it via an induced structure on the distinguished paths in N. In particular, this provides the right dimension bound on the joint kernel.

The remaining question is whether a joint solution of the BGG operators already leads to a parallel tractor. Steps towards this:

$$
\begin{array}{llll}
\nabla^{\mathcal{T}^*}_E(L(s)) \cong & (0, & \nabla^E\nabla^E s - P^{E^*\otimes E^*}s, & (*)) \\
\nabla^{\mathcal{T}^*}_V(L(s)) \cong & (\nabla^V s, & -\frac{1}{2}\nabla^E\nabla^V s, & (**)) \\
\nabla^{\mathcal{T}^*}_Q(L(s)) \cong & (0, & \nabla^Q\nabla^E s - P^{Q^*\otimes E^*}s, & (***))
\end{array}
$$

Denote by κ the curvature of the parabolic geometry.

- If $s = \pi^*\underline{s}$, i.e. $\nabla^V s = 0$, then $(**)$ equals the term $\kappa|_{V^*\wedge Q^*\otimes End_0} \bullet s$
- If these two terms are zero, then $(*)$ equals the term $\kappa|_{E^*\wedge Q^*\otimes End_0} \bullet s$
- $P^{E^*\otimes E^*}s$, $P^{Q^*\otimes E^*}s$ can be expressed with the standard torsion of the Weyl connections
- If $\nabla^{\mathcal{T}^*}_E(L(s)) = 0$ and $\nabla^{\mathcal{T}^*}_V(L(s))$, then $\nabla^{\mathcal{T}^*}_Q(L(s)) = 0$