On the relation between discrete series representations and BGG-complexes

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General setup

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Explicitly, $T^*(G/P)$ is a bundle of Lie algebras, which induces an invariant codifferential $\partial^* \colon \Lambda^k T^*(G/P) \to \Lambda^{k-1} T^*(G/P)$. Defining the G-bundles $\mathcal{H}_k := \ker(\partial^*)/\operatorname{im}(\partial^*)$ we can find invariant differential operators $D_k \colon \Gamma(\mathcal{H}_k) \to \Gamma(\mathcal{H}_{k+1})$ so that $(\Gamma(\mathcal{H}_k), D_k)$ is a complex.

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Question: Can we relate the BGG-complex to the geometry of G/K?

Intertwining operators

Using that $G/K \times G/P$ is isomorphic to G/M with $M := K \cap P$ we can easily construct G-equivariant integral operators

$$\Phi \colon \Omega^k(G/P) \to \Omega^\ell(G/K).$$

Their kernels are elements in $\Omega^*(G/M)^G$, which correspond to M-invariant elements in the underlying finite dimensional M-module.

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In general, the condition $\Delta \circ \Phi = 0$ is equivalent to Φ factoring to the BGG complex. This means that Φ induces a G-equivariant map

$$\underline{\Phi}\colon \Gamma(\mathcal{H}_k) \to \Omega^\ell(G/K)$$

so that $\underline{\Phi}(D_k[\alpha]) = \Phi(d\alpha)$ for all $\alpha \in \Gamma(\ker(\partial^*))$.

The case $G = SO_0(n+1,1)$

Let $G = SO_0(n+1,1)$ with maximal compact $K \cong SO(n+1)$ and minimal parabolic $P \cong CO(n) \ltimes \mathbb{R}^n$. Then G/K is the real hyperbolic space in dimension $n+1$, whereas G/P is the conformal n-sphere.

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Theorem (Gaillard '86, H. '16)

For all $0 \leq k \leq n$ there is a G-equivariant operator

$$\Phi \colon \Omega^k(G/P) \to \Omega^k(G/K)$$

whose image consists of coclosed and harmonic differential forms. Moreover,

$$d \circ \Phi_k = (n-2k)\Phi_{k+1} \circ d$$

Complex hyperbolic space

For $G = \mathrm{SU}(n+1,1)$ the symmetric space G/K is the complex hyperbolic space in (complex) dimension $n+1$. This is naturally a Kähler manifold, so we have a decomposition

$$\Omega^k(G/K, \mathbb{C}) = \bigoplus_{p+q=k} \Omega^{p,q}(G/K)$$

into (p,q)-types. Accodingly, we split the exterior derivative as $d = \partial + \overline{\partial}$, where the first and second operator maps (p,q)-forms to $(p+1,q)$-forms and $(p,q+1)$-forms, respectively.

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The adjoint of the wedge product with the Kähler form defines the Colefschetz map $L^*\colon \Omega^{p,q}(G/K) \to \Omega^{p-1,q-1}(G/K)$. Elements in the kernel of L^* are called primitive, which can only exist in degrees $\leq n+1$.

CR-sphere and Rumin complex

Let P be the minimal parabolic subgroup of $G = \mathrm{SU}(n+1,1)$, then $G/P \cong S^{2n+1}$ is a CR-sphere. Let $H \subset T(G/P)$ be the contact subbundle and put $Q := T(G/P)/H$. Denote by $\Lambda_0^k H^*$ the tracefree elements in $\Lambda^k H^*$ with respect to the Levi bracket.

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The bundle $\mathcal{H}_k$ is isomorphic to $\Lambda_0^k H^*$ for $k \leq n$ and a quotient of $\Lambda^{k-1} H^* \otimes Q^*$ for $k \geq n+1$. The resulting complex $(\Gamma(\mathcal{H}_k), D_k)$ is called the Rumin complex.

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The contact subbundle H carries an invariant complex structure, inducing a decomposition of $\Lambda^k H^* \otimes_{\mathbb{R}} \mathbb{C}$ into (p,q)-types. Thus, for $k \leq n$ the bundle $\mathcal{H}_k \otimes_{\mathbb{R}} \mathbb{C}$ decomposes into $\bigoplus_{p+q=k} \mathcal{H}_{p,q}$ with $\mathcal{H}_{p,q} \cong \Lambda_0^{p,q} H^*$. Each $\mathcal{H}_{p,q}$ is an irreducible G-invariant subbundle.

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Accodingly, the BGG-operators split into the sum $D = \mathcal{D} + \overline{\mathcal{D}}$, where

$$\mathcal{D}\colon \Gamma(\mathcal{H}_{p,q}) \to \Gamma(\mathcal{H}_{p+1,q}), \qquad \overline{\mathcal{D}}\colon \Gamma(\mathcal{H}_{p,q}) \to \Gamma(\mathcal{H}_{p,q+1}).$$

These are again G-invariant differentials.

Rumin compatible intertwining operators

Theorem (Čap, H., Julg, 2020)

Let $0 \leq p, q \leq n$ *with* $p + q \leq n$. *There exists a* G*-equivariant operator*

$$\Phi_{p,q} \colon \Gamma(\mathcal{H}_{p,q}) \to \Omega^{p,q}(G/K)$$

whose image consists of primitive, harmonic and coclosed differential forms. Moreover,

$$\partial \circ \Phi_{p-1,q} = c_{p-1,q} \Phi_{p,q} \circ \mathcal{D} \qquad \overline{\partial} \circ \Phi_{p,q-1} = c_{p,q-1} \Phi_{p,q} \circ \overline{\mathcal{D}}$$

for constants $c_{*,*} \neq 0$.

Discrete series representations

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2. If W_G and W_K are the Weyl groups of G and K, then there are exactly $|W_G|/|W_K|$ DSR with the same infinitesimal character.
3. DSR with trivial character are represented by L^2-harmonic forms on G/K. These can only exist in degree $\frac{1}{2}\dim_{\mathbb{R}}(G/K)$.

L^2-norm of intertwining operator

Theorem (Lott '00)

Let $G = SO_0(n+1,1)$ *and assume that* n *is odd. Then the image of* $d\Phi_{\frac{n-1}{2}}$ *consists of* L^2*-harmonic* $\frac{n+1}{2}$*-forms on* G/K.

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Let $G = SU(n+1,1)$. For all $p+q=n$ the image of $d\Phi_{p,q}$ consists of L^2-harmonic $n+1$-forms on G/K. Moreover, the image of $d\Phi_{p,q}$ is dense the L^2-harmonic forms of type $(p+1,q)$ as well as $(p,q+1)$ on G/K.

Therefore, the image of $d\Phi_{p,q}$ generates the discrete series representations of $SU(n+1,1)$ with trivial infinitesimal character.

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Therefore, the image of $d\Phi_{p,q}$ generates the discrete series representations of $\mathrm{SU}(n+1,1)$ with trivial infinitesimal character.

The proof of the above result is again in the realms of finite dimensional representation theory.

Sketch of proof I

Step 1: Consider the holonomy reduction of $G/P \cong S^{2n+1}$ to K/M. This induces a K-invariant pseudo-Hermitian structure $\alpha \in \Omega^1(K/M)$ with $\ker(\alpha) = H$. Let ζ be the Reeb vector field. The bundle metric $d\alpha(\ ,J\ )$ induces a partial Laplace Δ_H and a K-invariant L^2-norm $\|\ \|$ on $\Gamma(\Lambda^k H^*)$.

Aim: Relate $\|d\Phi_{p,q}(\sigma)\|$ to $\|\sigma\|$.

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Step 2: Let $\mathbb{W}$ be an irreducible M-representation and $\mathbb{V}$ irreducible K-module. Computing the possible M-weights of $\mathbb{V}$ it follows easily that $\mathrm{Hom}_M(\mathbb{V},\mathbb{W})$ is at most 1-dimensional. By Frobenius reciprocity this shows that

$$\dim(\mathrm{Hom}_K(\mathbb{V},\Gamma(K\times_M\mathbb{W}))) \leq 1.$$

We refer to this as the *multiplicity* 1 *property.*

Sketch of proof II

Let $\hat{K}$ be the set of all isomorphism classes of irreducible K-representations $\mathbb{V}$ so that $\mathrm{Hom}_K(\mathbb{V}, \Gamma(\mathcal{H}_{p,q})) \neq 0$. Since K is compact, Peter-Weyl implies that $\bigoplus_{\mathbb{V}\in\hat{K}} \mathbb{V}$ is dense in $\Gamma(\mathcal{H}_{p,q})$.

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Thus: Show Theorem for all $\sigma \in \mathrm{Hom}_K(\mathbb{V}, \Gamma(\mathcal{H}_{p,q}))$ and all $\mathbb{V} \in \hat{K}$.

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Step 3: If $\sigma \in \mathrm{im}(\mathcal{D}) \cap \mathrm{im}(\overline{\mathcal{D}})$, then the properties of $\Phi_{p,q}$ imply that $d\Phi_{p,q}(\sigma) = 0$.

Thus: Assume $\sigma \in \left(\mathrm{im}(\mathcal{D}) \cap \mathrm{im}(\overline{\mathcal{D}})\right)^{\perp}$

Let $\mathcal{D}^*$ and $\overline{\mathcal{D}}^*$ be the formal L^2-adjoints of $\mathcal{D}$ and $\overline{\mathcal{D}}$. By adjointness this is equivalent to $\sigma \in \ker(\mathcal{D}^*) + \ker(\overline{\mathcal{D}}^*)$.

Sketch of proof III

Step 4: Consider the Poincaré ball model of G/K, i.e. open unit ball $B \subset \mathbb{C}^{n+1}$ endowed with the Bergman metric

$$g(z)(\xi, \eta) = \frac{\langle \xi, \eta \rangle}{1 - |z|^2} + \frac{\langle \xi, z \rangle \langle z, \eta \rangle}{(1 - |z|^2)^2}.$$

Let

$$(r, \theta) \colon B \setminus \{0\} \to (0,1) \times S^{2n+1}, \qquad z \mapsto (|z|, |z|^{-1} z)$$

be the polar decomposition of $B \setminus \{0\}$. The K-orbits of $B \setminus \{0\}$ are the level sets S_r of θ, which are all isomorphic to S^{2n+1}.

Aim: Determine explicit formulae for $\Phi_{p,q}(\sigma)$ via restriction to S_r.

Sketch of proof IV

Step 5: Let $\tau = \sum_j z_j dz_j \in \Omega^{1,0}(B \setminus \{0\})$. Using $\sigma \in \ker(\mathcal{D}^*) + \ker(\overline{\mathcal{D}}^*)$, primitivity of $\Phi_{p,q}(\sigma)$ and the multiplicity 1 property it follows that

$$\Phi_{p,q}(\sigma) = f_1(r)\theta^*\sigma + f_2(r)\tau \wedge \theta^*\mathcal{D}^*\sigma + f_3(r)\overline{\tau} \wedge \theta^*\overline{\mathcal{D}}^*\sigma$$

for smooth functions $f_j \colon (0,1) \to \mathbb{C}$.

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Step 6: Distinguish the cases of σ contained in

$$\mathrm{im}(\mathcal{D}) \cap \ker(\overline{\mathcal{D}}^*), \qquad \ker(\mathcal{D}^*) \cap \mathrm{im}(\overline{\mathcal{D}}), \qquad \ker(\mathcal{D}^*) \cap \ker(\overline{\mathcal{D}}^*).$$

Use $\Delta\Phi_{p,q} = 0$, $\delta\Phi_{p,q} = 0$ and the differential calculi on G/K and G/P to obtain hypergeometric differential equations for f_j. These depend on the parameter $\lambda \in i\mathbb{R}$ and $\mu \in \mathbb{R}_{\geq 0}$ defined by $\mathcal{L}_\zeta\sigma = \lambda\sigma$ and $\Delta_H\sigma = \mu\sigma$.

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Step 7: Use the formula for $\Phi_{p,q}$ on B and Stokes' Theorem to relate $\|d\Phi_{p,q}(\sigma)\|^2_{L^2}$ to a multiple of $\|\sigma\|^2_{L^2}$.

Thank you for your attention!