An IDEAL characterization of pp-wave spacetimes

(to appear arXiv:230x.xxxxx w/ D.McNutt, L.Wylleman)

Igor Khavkine

Institute of Mathematics
Czech Academy of Sciences, Prague

20 Jan 2023
43nd Winter School on Geometry and Physics
15–20 Jan 2023, Srní, Czech Republic

Motivation

- The **fundamental symmetries** in General Relativity (GR) are **diffeomorphisms**.
- Two (Lorentzian) spacetime geometries (M, g) and (M, g') may appear to be very different but still be related by a diffeomorphism. The geometries are **isometric**.
- A lot of effort can go into deciding whether two geometries belong to the same (local) isometry class.

Definition (locally isometric)

(M, g) is **locally isometric** to (M_0, g_0) if $\forall x \in M\ \exists y \in M_0$ such that a neighborhood of x is isometric to a neighborhood of y. All such (M, g) constitute the **local isometry class** of (M_0, g_0).

IDEAL Characterization

- **Q:** Given a model geometry (M_0, g_0), is it possible to verify when (M, g) belongs to its **local isometry class** by checking a list of equations

$$T_\alpha[g] = 0 \quad (\alpha = 1, 2, \cdots, A),$$

  where each $T_\alpha[g]$ is a **tensor covariantly constructed** from g and its derivatives?
- If Yes, we call this an **IDEAL** (Intrinsic, Deductive, Explicit, ALgorithmic) characterization of the **local isometry class** of (M_0, g_0). Sometimes, also called **Rainich-like**.
- Generalizes to (M, g, Φ), including matter (tensor) fields, if we use covariant tensor equations of the form $T_\alpha[g, \Phi] = 0$.
- An **alternative** to the Cartan(-Karlhede) moving-frame-based characterization.
- Also, the **linearizations** $T_\alpha[g + \varepsilon p] = T_\alpha[g] + \varepsilon \dot{T}_{\alpha,g}[p] + O(\varepsilon^2)$ constitute a **complete list of local gauge invariant observables** $\dot{T}_{\alpha,g_0}[-]$ for linearized GR on (M_0, g_0).

Examples

- Relatively **few examples** of IDEAL characterizations are actually known. To my knowledge, they are either classical, or due to the work of Ferrando & Sáez (València), or myself + coauth.
- Examples:
  - **Constant curvature** (1800s): $R = R[g]$ — Riemann tensor,

$$R_{ijkh} = k(g_{ik}g_{jh} - g_{jk}g_{ih})$$

  - **Schwarzschild** of mass M in 4D (F&S 1998): $W = W[g]$ — Weyl tensor,

$$\begin{aligned} R_{ij} &= 0, & S_{ijlm}S^{lm}{}_{kh} + 3\rho S_{ijkh} &= 0, \\ P_{ab} &= 0, & \rho/\alpha^{3/2} - \mathrm{M} &= 0, \end{aligned}$$

$$\text{where} \quad \begin{aligned} \rho &= -(\tfrac{1}{12}\operatorname{tr} W^3)^{1/3}, & S_{ijkh} &= W_{ijkh} - \tfrac{1}{6}(g_{ik}g_{jh} - g_{jk}g_{ih}), \\ \alpha &= \tfrac{1}{9}(\nabla \ln \rho)^2 - 2\rho, & P_{ij} &= ({}^*W)_i{}^k{}_j{}^h \nabla_k\rho\nabla_h\rho. \end{aligned}$$

  - More F&S: Reissner-Nordström (2002), Kerr (2009), . . . (2010, 2017)
  - IK *et al.:* FLRW $+\phi$ (2018), Schwarzschild-Tangherlini (2019)

Current Ad-Hoc Strategy

- Fix a class of reference geometries $(M, g_0(\beta))$, with parameters β.
- Suppose there already exists a characterization of this class by the **existence** of tensor fields σ satisfying equations

$$S_\alpha[g, \sigma] = 0,$$

  covariantly constructed from σ, g_{ij}, R_{ijkl} and their covariant derivatives.
- Exploiting the geometry of $(M, g_0(\lambda))$, we try to **find formulas** for $\sigma = \Sigma[g_0]$ covariantly constructed from g_{ij}, R_{ijkl} and their covariant derivatives. If successful, we get an IDEAL characterization of **this class** by

$$T_\alpha[g] := S_\alpha[g, \Sigma[g]] = 0.$$

- If necessary, find **further covariant expressions** for the parameters $\beta = B[g_0]$, adding equations $B[g] - \beta = 0$ to the above list, until we can IDEALly characterize **individual isometry classes**.

IDEAL vs Cartan

Approaches to classification and equivalence of metrics.

- **Cartan** moving frame:
  - Supplements the metric with a progressively specialized frame.
  - Has a systematic foundation.
    Quite generally applicable.
- **IDEAL** characterization:
  - Relies only on the metric and covariant tensorial constructions from it.
  - Has only been worked out in *ad hoc* examples.
    Domain of applicability not well-understood.
  - More convenient in some applications (cf. linear observables).

- Try to push the IDEAL approach to its limits.
  $\rightsquigarrow$ **pp-waves** (maximally hard case?)

P(lane)P(arallel)-wave Spacetimes in 4d

Def: vacuum **pp-waves** take the form (with $\zeta = x + iy$, $\partial_{\bar\zeta} f = 0$)

$$\mathrm{d}s^2 = 2\mathrm{d}\zeta\mathrm{d}\bar\zeta - 2\mathrm{d}u\mathrm{d}v - 2\left(f(\zeta,u) + \bar f(\bar\zeta,u)\right)\mathrm{d}u^2,$$

$\iff$ Weyl-Petrov $\mathbf{C}^\dagger = \mathbf{W} - i{}^*\mathbf{W}$ *type N* and *Weyl recurrent* $\nabla\mathbf{C}^\dagger = \mathbf{K}\otimes\mathbf{C}^\dagger$.

- Sub-classified by isometry Lie algebra type (Ehlers & Kundt 1962).
- Further sub-classification into isometry classes possible. (**our work**)
- All curvature scalars vanish! Scalars cannot distinguish from flat space (maximally different from Riemannian signature).
- Cartan approach (McNutt 2013 PhD). Contains some of the most difficult cases for Cartan's method.

$$f(\zeta,u) = \begin{cases} 4\alpha u^{2i\kappa-2}\zeta^2 & G_{6a} \\ e^{2i\lambda u}\zeta^2 & G_{6b} \\ A(u)\zeta^2 & G_5 \\ 4\alpha u^{-2}\ln\zeta & G_{3a} \\ \ln\zeta & G_{3b} \\ e^{2\lambda\zeta} & G_{3c} \\ e^{i\gamma}\zeta^{2i\kappa} & G_{3d} \\ u^{-2}f(\zeta u^{i\kappa}) & G_{2a} \\ f(\zeta e^{i\lambda u}) & G_{2b} \\ A(u)\ln\zeta & G_{2c} \\ f(\zeta,u) & G_1 \end{cases}$$

Progress and Lessons Learned

Theorem (IDEAL identification of pp-waves)

*Vacuum 4d (M, g) is pp-wave iff **(a)** $C^{\dagger}_{ab}{}^{cd} C^{\dagger}_{cdef} = 0$, **(b)** $T_{abc[d} T_{e]fgh;i} = 0$, where $T_{abcd} = -C^{\dagger}{}_{e(ac|f|} \bar{C}^{\dagger}{}_{b}{}^{e}{}_{d)}{}^{f}$ is the Bel-Robinson tensor.*

Proof: (a) Weyl-Petrov type N $\rightsquigarrow$ $T_{abcd} = \beta \ell_a \ell_b \ell_c \ell_d$,
(b) $T_{abc[d} T_{e]fgh;i} = \beta \ell_a \ell_b \ell_c \ell_f \ell_g \ell_h (\ell_{[d} \ell_{e];i}) = 0$ $\rightsquigarrow$ recurrent $\nabla \mathbf{C}^{\dagger} = \mathbf{K} \otimes \mathbf{C}^{\dagger}$. □

problems going further to isometry classes

Lessons learned:

- **Recall:** No non-vanishing curvature scalars!
- Any covariant relation $F(\mathbf{T}_1, \ldots, \mathbf{T}_k) = 0$ between **non-scalar** invariants is at most **polynomial**.
- Any isometry classes with **non-polynomial relations** between invariants **cannot** be characterized IDEALly!

Highly Symmetric pp-waves

- **Q:** Is the problem of **non-polynomial relations** realized for pp-waves?
- **A:** Yes.
- At least the G_5 classes contain isometry classes characterized by arbitrary C^∞ functions $F(y)$.
- **Solution:** Introduce extra **conditional scalar invariants**.
- We have sub-classified **highly symmetric** pp-waves (dim.isom. ≥ 2, $G_{2\text{–}6}$) by isometry classes.
- Generic G_1 classes currently outside (our) reach.

Ex.: G_5, G_6 **isometry classes**

class		invariant parameters
$G_{5^\circ;\boldsymbol{F}}$:	$f(\zeta,u) = e^{B(u)}\zeta^2$	$\partial_u(\operatorname{Re}\dot{B}e^{-\frac{\operatorname{Re}B}{2}}) \neq 0$, $\begin{pmatrix}\operatorname{Re}\ddot{B}e^{-\operatorname{Re}B}\\ (\operatorname{Im}\dot{B})^2e^{-\operatorname{Re}B}\end{pmatrix} = \boldsymbol{F}(\operatorname{Re}\dot{B}e^{-\frac{\operatorname{Re}B}{2}})$, $\boldsymbol{F} = \begin{pmatrix}\boldsymbol{F}_{\mathrm{Re}}\\ \boldsymbol{F}_{\mathrm{Im}}\end{pmatrix} : U\subset\mathbb{R}\to\mathbb{R}^2$, $\boldsymbol{F}_{\mathrm{Im}}(y)\geq 0$, $\boldsymbol{F}_{\mathrm{Re}}(y)\neq \frac{1}{2}y^2$
$G_{5';\alpha,\boldsymbol{F}}$:	$f(\zeta,u) = \frac{4\alpha}{u^2}e^{i\operatorname{Im}B(u)}\zeta^2$	$\partial_u(\operatorname{Im}\dot{B}e^{-\frac{\operatorname{Re}B}{2}}) \neq 0$, $-\operatorname{Re}\dot{B}e^{-\frac{\operatorname{Re}B}{2}} = \alpha^{-1/2} \geq 0$, $\operatorname{Im}\ddot{B}e^{-\operatorname{Re}B} = \boldsymbol{F}(\operatorname{Im}\dot{B}e^{-\frac{\operatorname{Re}B}{2}})$, $\boldsymbol{F}\colon U\subset\mathbb{R}\to\mathbb{R}$, $\boldsymbol{F}(-y) = -\boldsymbol{F}(y)$, $\boldsymbol{F}(y)\neq -\frac{y}{2\sqrt{\alpha}}$
$G_{6a;\alpha,\kappa}$:	$f(\zeta,u) = \frac{4\alpha}{u^2}u^{2i\kappa}\zeta^2$	$\alpha > 0, \kappa \geq 0$
$G_{6b;\lambda}$:	$f(\zeta,u) = e^{2i\lambda u}\zeta^2$	$\lambda \geq 0$

ODE integration constants $\rightsquigarrow$ gauge parameters

Conditional Invariants

A **standard** tensor invariant $\mathbf{A} = \mathbf{A}[g]$ satisfies $\mathcal{L}_v \mathbf{A}[g] = \dot{\mathbf{A}}_g[\mathcal{L}_v g]$.

For any $g \in \mathcal{G}_0 \subset \Gamma(S^2 T^* M)$, suppose that

$$A_{c_1 \cdots c_n\, a_1 \cdots a_m} B_{b_1 \cdots b_m} - A_{c_1 \cdots c_n\, b_1 \cdots b_m} B_{a_1 \cdots a_m} = 0.$$

Lemma (Conditional Invariants)

(a) $\exists$ *a unique* $\mathbf{X}$ *such that* $\mathbf{A} = \mathbf{X} \otimes \mathbf{B}$. *(b) If* $\mathbf{A} = \mathbf{A}[g]$, $\mathbf{B} = \mathbf{B}[g]$ *are invariants, then also* $\mathcal{L}_v \mathbf{X}[g] = \dot{\mathbf{X}}_g[\mathcal{L}_v g]$ *for* $g \in \mathcal{G}_0$ *in a diff-stable family.*

Corollary

For covariant $\mathbf{F}[g] = F(\mathbf{A}, \mathbf{B}, \mathbf{X}, \ldots)$, *if* $\mathbf{F}[g] = 0$ *at* $g \in \mathcal{G}_0$, *then* $\dot{\mathbf{F}}_g[\mathcal{L}_v g] = 0$.

Linearization of an IDEAL characterization using **conditional invariants** still gives **linear invariants** (one of our motivations!).

Conditional Invariants for pp-waves

- Recurrence vector $\mathbf{K}$: $\mathbf{K} \otimes \mathbf{C}^\dagger := \nabla \mathbf{C}^\dagger$
- Weyl contraction: $D^\dagger_{ac} = \frac{1}{(\overline{\mathbf{K}}\cdot\mathbf{K})^2} \overline{K}^b \overline{K}^d C^\dagger_{abcd}$
- Conditional scalars: supposing $\overline{\mathbf{K}} \cdot \mathbf{K} = 0$

$$\left(I^{(2)}_{\mathrm{a}}\right)^4 \mathbf{T} := 16\mathbf{K}^{\otimes 4}, \qquad \left(I^{(2)}_{\mathrm{b}}\right)^2 \mathbf{T} := 16(\nabla\mathbf{K})^{\otimes 2}$$

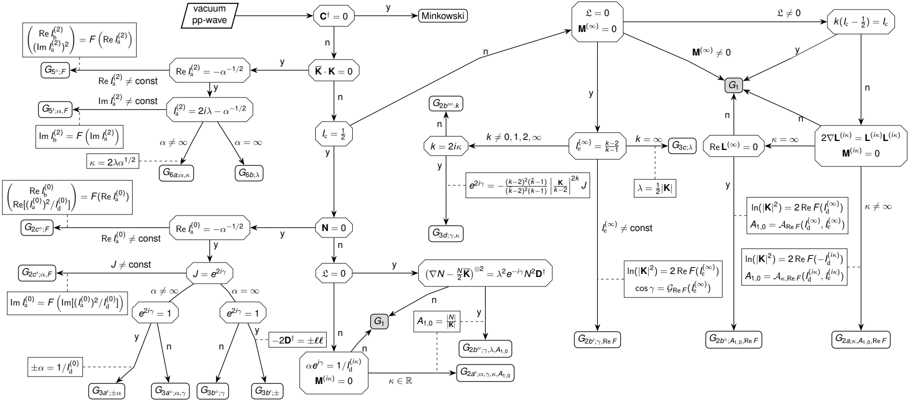
N.B.: The conditional invariant scalars $I^{(2)}_{\mathrm{a}}$, $I^{(2)}_{\mathrm{b}}$ can now participate in non-polynomial relations.

- **Example:** $G_{5^\circ;\boldsymbol{F}}$ isometry class, $f(\zeta, u) = e^{B(u)}\zeta^2$

$$\begin{pmatrix} \operatorname{Re} \ddot{B} e^{-\operatorname{Re} B} \\ (\operatorname{Im} \dot{B} e^{-\frac{1}{2}\operatorname{Re} B})^2 \end{pmatrix} = \boldsymbol{F}(\operatorname{Re} \dot{B} e^{-\frac{\operatorname{Re} B}{2}}) \quad \Longleftrightarrow \quad \begin{pmatrix} \operatorname{Re} I^{(2)}_{\mathrm{b}} \\ (\operatorname{Im} I^{(2)}_{\mathrm{a}})^2 \end{pmatrix} = \boldsymbol{F}\left(\operatorname{Re} I^{(2)}_{\mathrm{a}}\right)$$

with any smooth $\boldsymbol{F}(y)$

Flowchart: highly symmetric pp-waves



Using relative invariants, we have IDEALly characterized all (regular) highly symmetric vacuum pp-wave isometry classes.

Discussion

- An IDEAL characterization of the (local) isometry class of a physically interesting spacetime is a **natural problem** of geometric interest.
- **Q:** Examples where the IDEAL approach **fails**?
  A: In at least in some examples of **pp-waves** we **must** extend the IDEAL approach by **conditional tensor invariants**.
- **TODO:** Try to extend the IDEAL approach to **generic pp-waves** by systematic use of **differential invariants** (cf. Kruglikov, McNutt, Schneider).
- **TODO:** Continue to compare with Cartan method and find limitations of the IDEAL approach.

Discussion

- An IDEAL characterization of the (local) isometry class of a physically interesting spacetime is a **natural problem** of geometric interest.
- **Q:** Examples where the IDEAL approach **fails**?
  A: In at least in some examples of **pp-waves** we **must** extend the IDEAL approach by **conditional tensor invariants**.
- **TODO:** Try to extend the IDEAL approach to **generic pp-waves** by systematic use of **differential invariants** (cf. Kruglikov, McNutt, Schneider).
- **TODO:** Continue to compare with Cartan method and find limitations of the IDEAL approach.

Thank you for your attention!