Schur–Weyl–Howe-type duality and Ellipticity of symplectic twistor complexes

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Introduction

- Twistor operators in *classical spin geometry* for manifolds with $Spin(p, q)$-structure
- **[Penrose]** for signature $(1, 3)$
- They form elliptic complexes in definite signature, special cases of complexes on Hermitian manifolds whose Laplacians are so called Stein–Weiss operators, e.g. [Baston]
- *Elliptic complex* $=$ *complex of symbols is exact out of the zero section of the contangent bundle*

Manifold and Structure Groups

- (M^{2n}, ω) symplectic manifold
- $G = Sp(2n, \mathbb{R})$ symplectic group (non-compact) of $(\mathbb{R}^{2n}, \omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})$
  $\pi_1(Sp(2n, \mathbb{R})) = \pi_1(U(n)) = \mathbb{Z}$
- $\lambda : \widetilde{G} = Mp(2n, \mathbb{R}) \to G$ connected two-fold cover; metaplectic group

Segal–Shale–Weil/oscillator/metaplectic/symplectic spinor representation

- $\rho : \widetilde{G} \to U(H)$, $H = L^2(\mathbb{R}^n)$ Segal–Shale–Weil representation, $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ splitting in maximal isotropic vector subspaces

- $L^2(\mathbb{R}^n) \supseteq \mathcal{S}(\mathbb{R}^n) \supseteq \mathbb{C}[x^1, \ldots, x^n]e^{-|x|^2} = S^\bullet(\mathbb{C}^n)e^{-|x|^2}$ (symmetric algebra) $\times\ e^{-|x|^2}$

- $H = H_+ \oplus H_-$ where $H_\pm$ are the even and odd square integrable functions

- Complex OG-spinors $\mathbb{S} = \mathbb{S}_+ \oplus \mathbb{S}_- = \bigwedge^\bullet(\mathbb{C}^n)$ for $Spin(2n, \mathbb{R}) \to SO(2n, \mathbb{R})$ one of the similarities

Symplectic Spinors

- Symplectic spinor multiplication is the action $\cdot : \mathbb{R}^{2n} \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. For canonical basis $(e_i)_{i=1}^{2n}$, $f \in \mathcal{S}(\mathbb{R}^n)$ and $x = (x^1, \ldots, x^{2n})$

$$(e_i \cdot f)(x) = \imath x^i f(x), \; e_{i+n} \cdot f = \frac{\partial f}{\partial x^i}, \; i = 1, \ldots, n$$

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- Let M admit symplectic spinors ([ForgerHess]); construct the associated vector bundle; we denote it by $\mathcal{H}$ (Kostant's bundle of symplectic spinors)

- $\mathcal{H}^i = \bigwedge^i T^*M \otimes \mathcal{H}$ exterior forms twisted by SSW-repr.

- $\mathcal{T}^i \subseteq \mathcal{H}^i$, $\mathcal{T}^i = \mathrm{Ker}\,(Y)$, $Y(\alpha \otimes s) = \sum_{i,j=1}^{2n} \omega^{ij} i_{e_i}\alpha \otimes e_j \cdot s$, ith symplectic twistor bundle

- $p^i : \bigwedge^i T^*M \otimes \mathcal{H} \to \mathcal{T}^i$

Fedosov connection, Covariant derivatives, Symplectic twistor operators

- ∇ symplectic connection, i.e., $\nabla\omega = 0$. If $T^{\nabla} = 0$ as well, called Fedosov connection;
  affine space of symplectic connections isomorphic to $\Gamma(S^3 T^* M)$
- ∇^i ith exterior covariant derivative
  $\nabla^i : \Gamma(\bigwedge^i T^* M \otimes \mathcal{H}) \to \Gamma(\bigwedge^{i+1} T^* M \otimes \mathcal{H})$
- Symplectic twistor operators $T^i = p^{i+1}\nabla^i_{|\Gamma(\mathcal{T}^i)}$
- Theorem (Krysl, Monats. Math., 2010): (M, ω) symplectic manifold, ∇ Fedosov connection. If ∇ is Weyl-flat, $(\Gamma(\mathcal{T}^i), T_i)_{i=0,\ldots,n}$ and $(\Gamma(\mathcal{T}^i), T_i)_{i=n,\ldots,2n}$ are complexes.

Symbols of the symplectic twistor complexes

- Are these complexes elliptic, i.e., are complexes of operators' symbols fibre-wise exact for any $0 \neq \xi \in T^*_m M$ in the category of vector spaces?

- Symbols are $\sigma^i = \sigma(T^i, \xi)(\alpha \otimes s) = p^{i+1}((\xi \wedge \alpha) \otimes s)$

- $(\Gamma(\mathcal{T}^i), T^i)_{i=0,\dots,n}$ and $(\Gamma(\mathcal{T}^i), T^i)_{i=n,\dots,2n}$ form complexes $\implies \sigma^{i+1}\sigma^i = 0$

- $\mathrm{Ker}\,\sigma^{i+1} \subseteq \mathrm{Im}\,\sigma^i$? Compute p^i.

- Projections $p^i \in \mathrm{End}_{\widetilde{G}}(\bigwedge^i \mathbb{R}^{2n} \otimes H)$ since they are projection onto a $\widetilde{G}$-subrepresentation; $\mathrm{End}_{\tilde{G}}$ = so-called commutant algebra, space of $\widetilde{G}$-equivariant maps, space of $\widetilde{G}$-homomorphisms

Schur–Weyl–Howe-type duality

- Theorem (Krysl, J. Lie Thy. 2012, or arXiv 2008): There is a representation $\sigma : \mathfrak{osp}(1|2) \to \mathrm{End}_{\widetilde{G}}(\bigwedge^\bullet \mathbb{R}^{2n} \otimes H)$ such that $\langle \sigma(\mathfrak{osp}(1|2)), p_\pm \rangle = \mathrm{End}_{\widetilde{G}}(\bigwedge^\bullet \mathbb{R}^{2n} \otimes H)$, where $p_\pm : H \to H_\pm$.

- Schur: Image of the permutation representation for the symmetric group S_k generates $\mathrm{End}_{GL(V)}(V^{\otimes k})$.

- Weyl: Image of the permutation representation of an appropriate braid-group generates $\mathrm{End}_{SO(V)}(V^{\otimes k})$.

Schur–Weyl–Howe-type duality

- 1) R. Howe [Howe89] further generalizations and a systematization
  2) "Many" examples and extensions: Leites and Ščepočkina [LeitSca]

- Examples of application: 1) Decomposition of polynomials into sum of products of harmonic polynomials ($\Delta_{\mathbb{R}^n} p = 0$) $\times$ polynomials in variable $r^2 = \sum_{i=1}^n (x^i)^2$ ($SO(n)$ and $\mathfrak{sl}(2, \mathbb{C})$); 2) full set of invariants regarding similarity ($= g^{-1} - g$) of matrices on a f. dim. vector space; 3) $U(n)$ and $\mathfrak{sl}(2, \mathbb{C})$ regarding Kähler manifolds (Weil)

Symbols via $\mathfrak{osp}(1|2)$

$$F^+(\alpha\otimes s) := \sigma(f^+)(\alpha\otimes s) = \frac{\imath}{2}\epsilon^i\wedge\alpha\otimes e_i\cdot s$$
$$F^- := \sigma(f^-)(\alpha\otimes s) = F^-(\alpha\otimes s) = \frac{1}{2}\omega^{ij}i_{e_i}\alpha\otimes e_j\cdot s$$
$$E^\pm := \sigma(e^\pm) = \pm 2\{\sigma(f^\pm),\sigma(f^\pm)\}, \sigma(h) = 2\{\sigma(f^+),\sigma(f^-)\}.$$

For $i = 0,\ldots,n-1$

$$\sigma^i(\alpha\otimes s) = \xi\wedge\alpha\otimes s + \frac{2}{i-n}F^+(\alpha\otimes\xi^\sharp\cdot s) + \frac{\imath}{i-n}E^+(i_{\xi^\sharp}\alpha\otimes s)$$

(rather complicated for proving $\mathrm{Im} = \mathrm{Ker}$)

Assertion (Krysl, Arch. Math. (Brno), 2011): Let us suppse that a symmplectic manifold with a Fedosov connection is Weyl-flat and admits symplectic spinors. Then the symplectic spinor complexes $(\Gamma(\mathcal{T}^i), T^i)_{i=0,\ldots,n}$ and $(\Gamma(\mathcal{T}^i), T^i)_{i=n,\ldots,2n}$ are elliptic.

Scheme of the proof

Proof of $\operatorname{Ker}\sigma^i \subseteq \operatorname{Im}\sigma^{i+1}$: Apply E^- to reduce the form-degree by two $\Longrightarrow i_{\xi^\sharp}\alpha = 0$. Incorporate it.
Apply $i_{\xi^\sharp} \Longrightarrow \xi \wedge \alpha \otimes \xi^\sharp \cdot s = 0$
Use injectivity of $\xi^\sharp \cdot$ to deduce that $\xi \wedge \alpha = 0$.
Use Cartan lemma on exterior systems $\Longrightarrow \alpha \otimes s = \xi \wedge \beta \otimes s$ which is the pre-image.

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Supplement: Definition of Ricci and Weyl symplectic curvature tensors

- Ricci tensor $\sigma(X,Y) = \mathrm{Tr}(Z \mapsto R^\nabla(Z,X)Y)$, $\sigma_{ij} = R^k{}_{ijk} = +R^k{}_{ikj}$, coordinates with respect to a local symplectic frame $(e_i)_{i=1}^{2n}$, R^∇ classical curvature of affine connection ∇. Unlike the curvature of a Riemannian metric it maps anti-symmetric 2-tensor fields into *symmetric* 2-tensor fields (because of the duality between symmetric and anti-symmetric differential 2-forms defining the appropriate geometries; see Vaisman).
- Extended Ricci tensor: $\sigma_{ijkl} = \frac{1}{2n+2}(\omega_{il}\sigma_{jk} - \omega_{ik}\sigma_{jl} + \omega_{jl}\sigma_{ik} - \omega_{jk}\sigma_{il} + 2\sigma_{ij}\omega_{kl})$, $\widehat{\sigma} = \sigma_{ijkl}\epsilon^i \otimes \epsilon^j \otimes \epsilon^k \otimes \epsilon^l$ where $(\epsilon^i)_i$ is the dual basis (not ω-dual) (See [Vaisman])
- $W = R^\nabla - \hat{\sigma}$ symplectic Weyl curvature tensor
- **Definition:** A Fedosov connection is called symplectic *Weyl-flat* (or symplectic Ricci-type) if $W = 0$.