The homological part of the total surgery obstruction

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Outline

Based on work of A. Ranicki

1. Surgery theory
2. The total surgery obstruction
3. Cobordisms of quadratic chain complexes
4. Ideas in the proof

Surgery theory

Let X be a finite n-dim geometric Poincaré complex.

A manifold structure on X is $f : M \xrightarrow{\simeq} X$ with M an n-mfd.

Define

$$(f_0 : M_0 \xrightarrow{\simeq} X) \sim (f_1 : M_1 \xrightarrow{\simeq} X)$$

if there exists $h : M_0 \xrightarrow{\cong} M_1$ such that

$$f_1 \circ h \simeq f_0.$$

Definition:

The structure set of X is

$$\mathcal{S}^{\mathrm{TOP}}(X) := \{f : M \xrightarrow{\simeq} X\}/\sim$$

Surgery theory questions

Uniqueness question

$$\mathcal{S}^{\mathrm{TOP}}(X) \quad \cong \quad ?$$

Existence question

$$\mathcal{S}^{\mathrm{TOP}}(X) \quad \neq \quad \emptyset \quad ?$$

Alternative question

What is the homotopy type of $\widetilde{\mathcal{S}}^{\mathrm{TOP}}(X)$?

$$\pi_k \widetilde{\mathcal{S}}^{\mathrm{TOP}}(X) = \mathcal{S}_{\partial}^{\mathrm{TOP}}(X \times D^k).$$

Surgery theory answers

Classical surgery theory (Browder-Novikov-Sullivan-Wall) gives

$$\mathcal{S}^{\mathrm{TOP}}(X) \quad \text{for} \quad X = S^n, S^k \times S^l, \mathbb{C}P^n, \mathbb{R}P^n, L_N^{2d-1}, T^n, \ldots$$

The Borel conjecture (BC)

$$\mathcal{S}^{\mathrm{TOP}}(BG) = \{\mathrm{id}_{BG}\}?$$

Existence answer $\mathcal{S}^{\mathrm{TOP}}(X) = \emptyset$ for X

- $X^5 = (S^2 \vee S^3) \cup_{\eta^2 + [\iota_2, \iota_3]} e^5$ (Gitler-Stasheff, Madsen-Milgram)
- $X^4 = e^0 \cup e^1 \cup 10e^2 \cup e^3 \cup e^4, \quad \pi_1 X = \mathbb{Z}/p$ (Wall)

Surgery theory method

The surgery exact sequence (Browder-Novikov-Sullivan-Wall)

For an n-manifold X with $n \geq 5$ nad $\pi = \pi_1(X)$ we have

$$\cdots \to \mathcal{N}_\partial^{\mathrm{TOP}}(X \times I) \xrightarrow{\theta} L_{n+1}(\mathbb{Z}\pi) \xrightarrow{\partial} \mathcal{S}^{\mathrm{TOP}}(X) \xrightarrow{\eta} \mathcal{N}^{\mathrm{TOP}}(X) \xrightarrow{\theta} L_n(\mathbb{Z}\pi).$$

Explanation

- $\mathcal{N}^{\mathrm{TOP}}(X)$ - normal cobordism - gen. cohomology theory
- $L_n(\mathbb{Z}\pi)$ - Witt group of quadratic forms

Existence versus uniqueness

$(X, \partial X)$, $(Y, \partial Y)$ mfds with ∂, $h \colon \partial X \xrightarrow{\cong} \partial Y \rightsquigarrow Z := X \cup_h Y$ is GPC.

The main theorem

Let X be a finite n-dimensional simplicial Poincaré complex.

Let $\Lambda^c_*(X)$ be the category of chain complexes of free $\mathbb{Z}$-modules which are

- quadratic
- X-based
- locally Poincaré
- globally contractible

Let $\mathbf{S}_n(X) := \mathbf{L}_{n-1}(\Lambda^c_*(X))$ be the $(n-1)$-st L-theory space, $n \geq 5$.

Theorem (Ranicki)

There exists a point $s(X) \in \mathbf{S}_n(X)$, the total surgery obstruction, and

$$\mathbf{qsign}_X : \widetilde{\mathcal{S}}^{\mathrm{TOP}}(X) \xrightarrow{\simeq} \mathrm{Path}_0^{s(X)} \mathbf{S}_n(X).$$

Applications

Algebraic surgery exact sequence (infinite in both directions)

Denote $\mathbb{S}_{n+k}(X) = \pi_k \mathbf{S}_n(X)$. The we have

$$\cdots L_{n+k}(\mathbb{Z}\pi) \longrightarrow \mathbb{S}_{n+k}(X) \longrightarrow H_{n+k-1}(X; \mathbf{L}_\bullet\langle 1 \rangle) \xrightarrow{\text{asmb}} L_{n+k-1}(\mathbb{Z}\pi) \cdots$$

The Farrell-Jones conjecture (torsion free version):

For G torsion free the map $H_m(BG; \mathbf{L}_\bullet) \xrightarrow{\text{asmb}} L_m(\mathbb{Z}G)$ is an iso for all m .

FJC implies BC

Ranicki's theorem + FJC for G say that if $s(BG) \sim 0$ then

$$\widetilde{\mathcal{S}}^{\text{TOP}}(BG) \simeq * \quad \Rightarrow \quad \mathcal{S}^{\text{TOP}}(BG) \cong \{\text{id}_{BG}\}.$$

Surgery obstructions

Let $(f, \overline{f}) \colon M \to X$ be a degree one normal map.

Surgery question

Can we change $(f, \overline{f})$ by normal cobordism to a homotopy equivalence?

Surgery answer

Yes if and only if $0 = \mathbf{qsign}_\pi(f, \overline{f}) \in L_n(\mathbb{Z}\pi)$.

Here $\pi = \pi_1(X)$ and $n \geq 5$.

Algebraic surgery

Idea

Use chain complexes.

Question

- How to define a symmetric bilinear form on a chain complex?
- How to define a quadratic form on a chain complex?

On modules

- A symmetric bilinear form on a module is a fixed point.
- A quadratic form on a module is an orbit.

Forms on modules

A bilinear form is $\varphi \in \mathrm{Hom}_R(P, P^*) \cong (P \otimes_R P)^* \ni \lambda$.

An involution on forms

$$T : \mathrm{Hom}_R(P, P^*) \to \mathrm{Hom}_R(P, P^*) \quad T : (P \otimes_R P)^* \to (P \otimes_R P)^*$$

$$T(\varphi) = \varphi^* \circ \mathrm{ev} \quad T(\lambda)(x, y) = \overline{\lambda(y, x)}$$

An ε-symmetric bilinear form for $\varepsilon = \pm 1$ is

$$\varphi \in \ker(1 - \varepsilon T) = \mathrm{Hom}_R(P, P^*)^{\mathbb{Z}_2} = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\mathbb{Z}, \mathrm{Hom}_R(P, P^*)).$$

An ε-quadratic form for $\varepsilon = \pm 1$ is

$$\psi \in \mathrm{coker}(1 - \varepsilon T) = \mathrm{Hom}_R(P, P^*)_{\mathbb{Z}_2} = \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}_R(P, P^*).$$

The symmetrization map is $1 + \varepsilon T : \mathrm{Hom}_R(P, P^*)_{\mathbb{Z}_2} \to \mathrm{Hom}_R(P, P^*)^{\mathbb{Z}_2}$.

Structured chain complexes

A "form" on a chain complex C is $\omega \in (C \otimes_R C) \cong \mathrm{Hom}_R(C^{-*}, C)$.

An involution on forms

$$\begin{aligned} C \otimes_R C &\to C \otimes_R C \\ x \otimes y &\mapsto (-1)^{|x| \cdot |y|} y \otimes x. \end{aligned}$$

But we need a homotopy invariant notion!

Homotopy invariant structures

fixed points $\rightsquigarrow$ homotopy fixed points
orbits $\rightsquigarrow$ homotopy orbits

Structured chain complexes II

The standard $\mathbb{Z}[\mathbb{Z}_2]$-resolution of $\mathbb{Z}$:

$$W := \cdots \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \longrightarrow 0$$

Notation

$$W^{\%}(C) := \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(W, C \otimes_R C) = (C \otimes_R C)^{h\mathbb{Z}/2}$$

$$W_{\%}(C) := W \otimes_{\mathbb{Z}[\mathbb{Z}/2]} (C \otimes_R C) = (C \otimes_R C)_{h\mathbb{Z}/2}$$

Definition

An n-dimensional symmetric structure on C is a cycle $\varphi \in W^{\%}(C)_n$.
An n-dimensional quadratic structure on C is a cycle $\psi \in W_{\%}(C)_n$.

The symmetric construction I

The Alexander-Whitney diagonal + higher homotopies give a chain map

$$\Delta_X \colon W \otimes C(X) \to C(X) \otimes C(X)$$
$$1_s \otimes x \mapsto \Delta_s(x)$$

The symmetric construction map

$$\varphi_X \colon C(X) \to W^{\%}(C(X))$$

is defined to be the adjoint of Δ_X. For a cycle $c \in C_n(X)$ we have

$$\varphi_X(c)_0 = - \cap c \colon C^{n-*}(X) \to C(X).$$

It is natural in X and there is an equivariant version.

The quadratic construction

The quadratic construction for $(f, \overline{f}) \colon M \to X$ a deg 1 normal map with $F = S(\overline{f}) \colon \Sigma^p_+ X \to \Sigma^p_+ M$ and $f^! \colon C(X) \to C(M)$ is

$$\psi_F \colon C(X) \to W_\%(C(M)) \quad \text{s.t.} \quad (1+T)\psi_F = \varphi_M f^!_* - (f^!)^\% \varphi_X$$

and using $e \colon C(M) \to \mathcal{C}(f^!)$ we get

$$(C, \psi) = (\mathcal{C}(f^!), e_\% \psi_F[X]).$$

Definition

An n-dim sym complex (C, φ) is called Poincaré if $\varphi_0 \colon C^{n-*} \xrightarrow{\simeq} C$.

An n-dim quad complex (C, ψ) is called Poincaré if $(1+T)\psi_0 \colon C^{n-*} \xrightarrow{\simeq} C$.

L-groups

Definition

$L^n(R)$ is the cobordism group of n-dim sym alg Poincaré cplxs.
$L_n(R)$ is the cobordism group of n-dim quad alg Poincaré cplxs.

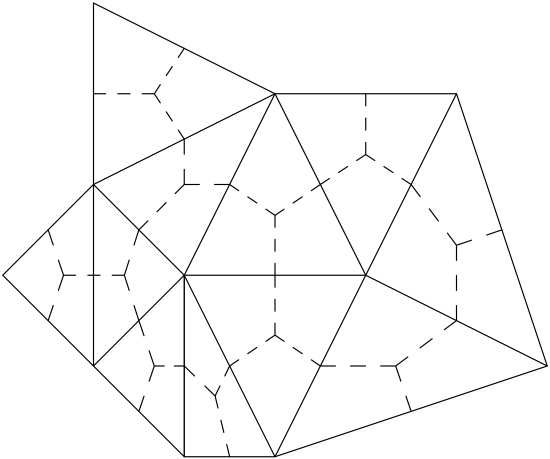
Theorem (Signatures)

There are symmetric and quadratic signatures maps:

$$\mathbf{ssign}_\pi : \Omega_n^{\mathrm{TOP}}(X) \to L^n(\mathbb{Z}\pi)$$
$$\mathbf{qsign}_\pi : \mathcal{N}^{\mathrm{TOP}}(X) \to L_n(\mathbb{Z}\pi)$$

such that $0 = \mathbf{qsign}_\pi(f, \overline{f})$ *iff* $(f, \overline{f}) \in L_n(\mathbb{Z}\pi)$ *is normally cobordant to a homotopy equivalence.*

Local Poincaré duality



Local Poincaré duality II

Additive category with chain duality $(\mathbb{A}, (T, e))$
$\rightsquigarrow$ $L^n(\mathbb{A})$ and $L_n(\mathbb{A})$

$\mathbb{A} = \mathbb{Z}_*(X)$ modules $M = \sum_{\sigma \in X} M(\sigma)$ and "lower triangular matrices".

$\rightsquigarrow$ X-based chain complexes

Algebraic bordism category $\Lambda = (\mathbb{A}, (T, e), \mathbb{B}, \mathbb{C})$ with $\mathbb{C} \subseteq \mathbb{B} \subseteq \mathbb{B}(\mathbb{A})$
$\rightsquigarrow$ $L^n(\Lambda)$ and $L_n(\Lambda)$

$\Lambda(X)$ globally Poincaré complexes in $\mathbb{Z}_*(X)$

$\Lambda_*(X)$ locally Poincaré complexes in $\mathbb{Z}_*(X)$

$\Lambda_*^c(X)$ locally Poincaré globally contractible complexes in $\mathbb{Z}_*(X)$

The total surgery obstruction

Recall

$$s(X) = (C, \psi) \in \mathbf{S}_n(X) = \mathbf{L}_{n-1}(\Lambda^c_*(X))$$

Need $(n-1)$-dim locally Poincaré globally contractible quadratic complex!
Start with $(C(X), \varphi_X([X])$ which is n-dim symmetric complex in $\mathbb{Z}_*(X)$

Boundary construction of Ranicki produces $(n-1)$-dim locally Poincaré globally contractible symmetric complex

Local normal structure (Quinn, Ranicki) or homological algebra (Weiss) produces quadratic refinement.

Locally

We have

$$C(\sigma) = \Sigma^{-1}\mathcal{C}(C^{n-|\sigma|-*}(D(\sigma, K)) \to C_*(D(\sigma, K), \partial D(\sigma, K)))$$

The homological part of the total surgery obstruction

Algebraic surgery exact sequence (infinite in both directions)

$$\begin{array}{ccccccc} L_n(\Lambda(X)) & \longrightarrow & L_{n-1}(\Lambda^c_*(X)) & \longrightarrow & L_{n-1}(\Lambda_*(X)) & \longrightarrow & L_{n-1}(\Lambda(X)) \\ \downarrow \cong & & \cong \downarrow & & \cong \downarrow & & \downarrow \cong \\ L_n(\mathbb{Z}\pi) & \longrightarrow & \mathbb{S}_n(X) & \longrightarrow & H_{n-1}(X; \mathbf{L}_\bullet\langle 1 \rangle) & \xrightarrow{\text{asmb}} & L_{n-1}(\mathbb{Z}\pi) \end{array}$$

Definition (Homological part of TSO)

Define $t(X)$ to be the image of $s(X)$ under the map

$$\mathbf{S}_n(X) \to \mathbf{H}_{n-1}(X; \mathbf{L}_\bullet\langle 1 \rangle).$$

Theorem (Ranicki)

$$\mathbf{qsign}_X : \widetilde{\mathcal{N}}^{\mathrm{TOP}}(X) \xrightarrow{\simeq} \mathrm{Path}_0^{t(X)} \mathbf{H}_{n-1}(X; \mathbf{L}_\bullet\langle 1 \rangle).$$

Main idea of the new proof 1

Use a Mayer-Vietoris principle thanks to the following proposition.

Proposition

Let X be an n-GPC. Then there exists an $N > 0$ such that the $(n+N)$-GPP

$$(X \times D^N, X \times S^{N-1})$$

is homotopy equivalent to an $(n+N)$-GPP $(Z, \partial Z)$ with Z the total space of some K-dimensional disk fibration ξ over some smooth manifold with boundary $(Y, \partial Y)$ and with $\partial Z = S(\xi) \cup E(\xi|_{\partial Y})$:

$$D^k \to Z = E(\xi) \to Y.$$

Main idea of the new proof 2

Theorem (M. and Adams-Florou (2018))

The suspension homotopy equivalence

$$\mathbf{H}_{n-1}(X; \mathbf{L}_\bullet\langle 1 \rangle) \to \mathbf{H}_{n+N-1}(X \times D^N, X \times S^{N-1}; \mathbf{L}_\bullet\langle 1 \rangle)$$

sends

$$t(X) \mapsto t(X \times D^N).$$

Proof.

Use ball complexes instead of simplicial complexes. □

Main idea of the new proof 3

Use a Mayer-Vietoris principle via a homotopy handlebody decomposition.

The homotopy handlebody decomposition of Z

$$Z_{m+1} = Z_m \cup_{h_m} H_{m+1}$$

where

$$h_m \colon \partial_- Z_m \xrightarrow{\simeq} \partial_- H_{m+1} = S^{k-1} \times D^{n-k}$$

Known

$$\mathbf{qsign}_H \colon \widetilde{\mathcal{N}}^{\mathrm{TOP}}(H) \xrightarrow{\simeq} \Omega \mathbf{H}_{n-1}(H; \mathbf{L}_\bullet\langle 1 \rangle) = \mathbf{H}_n(H; \mathbf{L}_\bullet\langle 1 \rangle).$$

To do

Relate $t(Z_{m+1})$ to $\mathbf{qsign}_{\partial_- H_{m+1}}(h_m)$.

Thank you!