Bifurcations of symmetric resonances

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Outline

- 2D Hamiltonian systems around symmetric resonances
- Normal forms
- Geometric reduction

H. Hassmann, A. Marchesiello, G. Pucacco, *Journal of Nonlinear Science*, **30**, pages 2513–2544 (2020)

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We consider a family of $\mathbb{Z}_2 \times \mathbb{Z}_2$–symmetric Hamiltonian systems in two degrees of freedom, i.e. invariant with respect to the reflectional symmetries

$$\begin{aligned} \varrho_1 &: (x_1, x_2, p_1, p_2) \mapsto (-x_1, x_2, -p_1, p_2) \\ \varrho_2 &: (x_1, x_2, p_1, p_2) \mapsto (x_1, -x_2, p_1, -p_2) \end{aligned}$$

where (x, p) denote the canonical coordinates. We assume the system to be close to an elliptic equilibrium at the origin and consider

$$H(x, p; \varepsilon) = \sum_{j=0}^{\infty} \varepsilon^{2j} H_{2j}(x, p).$$

Here H_{2j} are homogeneous polynomials of degree $2(j+1)$ in the coordinates (x, p), ε is a small parameter and

$$H_0(x, p) = \frac{\omega_1}{2}(x_1^2 + p_1^2) + \frac{\omega_2}{2}(x_2^2 + p_2^2)$$

so the system can be treated as a symmetric perturbed oscillator.

The Hamiltonian

$$H = \frac{\omega_1}{2}(x_1^2 + p_1^2) + \frac{\omega_2}{2}(x_2^2 + p_2^2) + \sum_{j=1}^{\infty} \varepsilon^{2j} H_{2j}$$

is in general not integrable. Let us introduce a detuning parameter δ by assuming

$$\omega_1 = \left(\frac{m}{n} + \varepsilon^2\delta\right)\omega_2, \quad m, n \in \mathbb{N}$$

and put the term with the detuning into the perturbation, so to see the system as a perturbation of a $m{:}n$ resonant oscillator invariant under the reflection symmetries.

Then we proceed to a normalization procedure w.r.t. the unperturbed $m{:}n$ resonant oscillator: we look for a (formal) coordinate transformation that brings H into the normal form K so that after scaling $t \to \frac{\omega_2}{n}t$,

$$\{K, H_0\} = 0, \quad H_0 = \frac{m}{2}(x_1^2 + p_1^2) + \frac{n}{2}(x_2^2 + p_2^2).$$

In this way the system acquires a (formal) constant of motion $H_0 = \eta$.

The normalized system is therefore integrable.

Why the detuning? Because even if the unperturbed system is non-resonant, the non-linear coupling between the degrees of freedom induced by the perturbation determines a “passage through resonance”. This in turn is responsible for the birth of new orbit families bifurcating from the normal modes or from lower-order resonances. Moreover, in this way we can avoid the presence of terms with small denominators while normalizing the system.

We aim at a general understanding of the phase space structure and the bifurcation sequences of periodic orbits in general position from the normal modes, parametrised by the “energy” E, the detuning δ and the independent coefficients characterising the nonlinear perturbation.

Typical periodic orbits associated to the 1:1 and 1 : 2 symmetric resonances

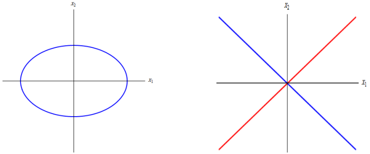


Figure: 1 : 1 symmetric resonance: loop orbits if $2(\phi_1 - \phi_2) = \pm\pi$, inclined orbits if $\phi_1 - \phi_2 = 0, \pi$.

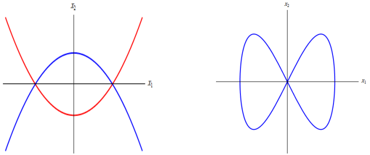


Figure: 1 : 2 symmetric resonance: anti-banana orbits if $2\phi_1 - \phi_2 = 0, \pi$, banana orbits: $4\phi_1 - 2\phi_2 = \pm\pi$.

Here action-angle like variables have been introduced:

$$p_j = \sqrt{2\tau_j}\sin\phi_j, \quad x_j = \sqrt{2\tau_j}\cos\phi_j, \quad j = 1, 2.$$

As an example, let us consider the family of systems

$$H(x,p) = \frac{1}{2}(p_1^2 + p_2^2) + V(x),$$

where

$$V(x_1, x_2) \;=\; \frac{1}{a}\left(1 + x_1^2 + \frac{x_2^2}{q^2}\right)^{a/2} \;, \quad 0 < a < 2,\; \tfrac{1}{4} < q \le 1.$$

This gravitational potential is generated by a simple but realistic matter distribution. Its astrophysical relevance is based on its ability to describe in a simple way the gross features of elliptical galaxies. In the limit $p \to 0$ we have the logarithmic potential

$$V(x_1, x_2) \;=\; \log\left(1 + x_1^2 + \frac{x_2^2}{q^2}\right).$$

After series expansion, the Hamiltonian is "prepared" for normalization by setting

$$q \;=\; \frac{\omega_1}{\omega_2} \;=\; \frac{m}{n} \;+\; \varepsilon^2\delta \;,$$

and scaling time and space variables as

$$x \;\mapsto\; \varepsilon x, \quad t \;\mapsto\; \frac{\varepsilon^2\omega_2}{n}\, t.$$

In general, let us consider

$$H = \frac{m}{2}(x_1^2 + p_1^2) + \frac{n}{2}(x_2^2 + p_2^2) + \varepsilon^2 n \frac{\delta}{2}(x_1^2 + p_1^2) + \sum_{j=1}^{\infty} \varepsilon^{2j} H_{2j}.$$

The flow $\varphi_t^{H_0}$ of the unperturbed system yields the $\mathbb{S}^1$–action φ^{H_0} on $\mathbb{R}^4 \cong \mathbb{C}^2$ given by

$$\begin{array}{rccc} \varphi^{H_0} : & \mathbb{S}^1 \times \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 \\ & (\ell, (z_1, z_2)) & \to & (e^{-\mathrm{i}m\ell} z_1, e^{-\mathrm{i}n\ell} z_2) \end{array}$$

where

$$z_j = x_j + \mathrm{i}p_j, \quad j = 1, 2.$$

or, equivalently, in action-angle like variables

$$z_j = \sqrt{2\tau_j} e^{i\phi_j}, \ \ j = 1, 2.$$

The perturbed Hamiltonian is in general not invariant under this action, however we can normalize H so that the resulting normal form K does have the oscillator symmetry, namely

$$\{K, H_0\} = 0.$$

A set of generators of the Poisson algebra of φ^{H_0}–invariant functions is given by

$$\tau_1 = \frac{z_1 \bar{z}_1}{2}, \quad \tau_2 = \frac{z_2 \bar{z}_2}{2}$$

together with

$$\sigma_1 = \frac{\mathrm{Re}\, z_1^n \bar{z}_2^m}{2}, \quad \sigma_2 = \frac{\mathrm{Im}\, z_1^n \bar{z}_2^m}{2}$$

and it is constrained by $\tau_1 \geq 0$, $\tau_2 \geq 0$ and the syzygy

$$R(\tau, \sigma) := 2^{n+m-2} \tau_1^n \tau_2^m - (\sigma_1^2 + \sigma_2^2) = 0.$$

The (truncated) normal form K is a polynomial in (τ, σ), namely

$$K = m\tau_1 + n\tau_2 + \varepsilon^2 n\, \delta \tau_1 + \sum_{j=1}^{N-1} \varepsilon^{2j} K_{2j}(\tau) + \varepsilon^{2N} K_{2j}(\tau, \sigma).$$

Without symmetries, the minimal truncation order is $m + n - 2$. With both reflection symmetries, the minimal truncation order increases to $2N = 2(m + n) - 2$ and this is why one speaks of 2m:2n resonance.

The normalization allows us to reduce the dynamics to one degree of freedom as the Poisson bracket on $\mathbb{R}^4$ induced by (τ, σ) has two Casimir elements, namely R and $H_0 = \tau_1 + 2\tau_2$.

For a fixed value $\eta \geq 0$ of H_0 we can eliminate $\tau_2 = \frac{1}{2}(\eta - \tau_1)$. The dynamics are then constrained to the reduced phase space

$$\mathcal{V}^\eta \;=\; \left\{ (\tau_1, \sigma_1, \sigma_2) \in \mathbb{R}^3 \;:\; R^\eta(\tau_1, \sigma_1, \sigma_2) = 0,\; 0 \leq \tau_1 \leq \eta \right\}$$

with Poisson structure

$$\{f, g\} \;=\; \langle \nabla f \times \nabla g, \nabla R^\eta \rangle \;,$$

where

$$R^\eta(\tau_1, \sigma_1, \sigma_2) \;=\; 2^{n-2}(\eta - \tau_1)^m \tau_1^n \;-\; (\sigma_1^2 + \sigma_2^2).$$

To understand the dynamics of the normal form we follow a geometric approach: we look at the intersections between the level sets of the normal form and the reduced phase space.

Let us focus now on the 2 : 4 resonance. The normal form, truncated at the minimal order reads

$$K(\tau,\sigma;\delta) \;=\; K_0(\tau)+\varepsilon^2 K_2(\tau;\delta)+\varepsilon^4\left[\mu\frac{\sigma_1^2-\sigma_2^2}{2}+\nu\sigma_1\sigma_2+K_4(\tau;\delta)\right]$$

with K_2, K_4 polynomials of degree 2 and 4, respectively,

$$K_0=H_0 \;=\; \tau_1+2\tau_2 \;=\; \eta.$$

We assume at least one of the coefficients μ and ν to be non-vanishing (otherwise we have to consider higher order normal form).

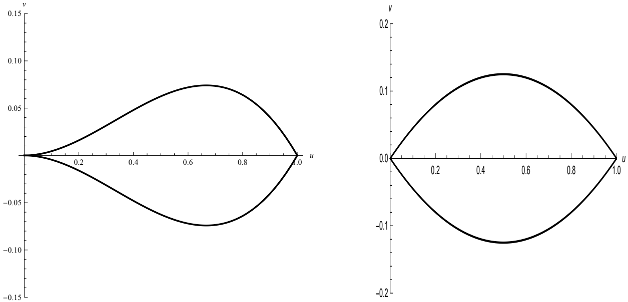
The $\mathbb{Z}_2 \times \mathbb{Z}_2$–symmetry of the original system is inherited by the normal form

Indeed, none of the invariants (τ, σ) changes under reflectional symmetry with respect to the x_1- axis. The reflectional symmetry with respect to the x_2- axis becomes the symmetry

$$(\tau, \sigma) \;\mapsto\; (\tau, -\sigma)$$

We perform a further reduction to explicitly divide out this symmetry, by introducing variables

$$\begin{aligned} u &:= \tau_1 \\ v &:= \tfrac{1}{2}(\sigma_1^2 - \sigma_2^2) \\ w &:= \sigma_1 \sigma_2 \;\; . \end{aligned}$$



Section of the (twice) reduced phase space corresponding to the 2 : 2 (right) and 2 : 4 resonance (left), for $\eta = 1$.
Singular equilibria are at the singular points of the reduced phase space

$$\mathcal{Q}_1 = (0,0,0), \quad \textit{and} \; ; \mathcal{Q}_2 = (0,0,\eta).$$

These correspond to $\tau_1 = 0$ and $\tau_1 = \eta$, i.e. the normal modes

$x_1^2 + p_1^2 = 0, \;\; x_2^2 + p_2^2 = 2\eta$ and $x_1^2 + p_1^2 = 2\eta, \;\; x_2^2 + p_2^2 = 0$ (2 : 2 resonance),

$x_1^2 + p_1^2 = 0, \; x_2^2 + p_2^2 = \eta$ and $x_1^2 + p_1^2 = 2\eta, \; x_2^2 + p_2^2 = 0$ (2 : 4 resonance)

(also called short and long axial orbits for the 2 : 4 resonance).

The type of the singularity depends on the resonance.

The normal form of the 2 : 4 resonance now reads (after neglecting constant terms and scaling one more time by ε^2)

$$K^\eta(u,v,w;\delta) \;=\; (2\delta+\alpha\eta)u + \lambda u^2 + \varepsilon^2\left[\mu v + \nu w + K_4^\eta(u;\delta)\right]$$

Note that, since the reduced phase space is a surface of revolution, by rotation we can always eliminate one of the two variables v, w from the Hamiltonian (we do not consider the case $\mu = \nu = 0$ here).

For definiteness we assume from now on $\mu > 0$ and $\nu = 0$.

We consider the level sets

$$K^\eta_{\delta,\varepsilon} \;:=\; \left\{\,(u,v,w)\in\mathbb{R}^3 \;:\; K^\eta(u,v,w;\delta) = h_0 + \varepsilon^2 h_2\,\right\}$$

which correspond to a family of third order curves when intersecting with the (u,v)–plane

$$v(u) \;=\; -\frac{1}{\varepsilon^2\mu}\left[(2\delta+\alpha\eta)u + \lambda u^2 + h_0\right] + h_2 - K_4^\eta(u;\delta).$$

The ε^2 lets the quadratic part of the curve dominate over the cubic part. Thus, we have to understand the intersections between a parabola and the reduced phase space section. Tangency points correspond to regular equilibria.

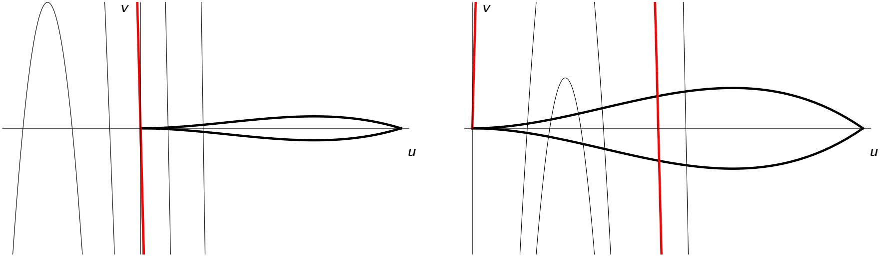


Figure: Possible configurations between the phase space section $\mathcal{P}^\eta \cap \{w = 0\}$ and a second order approximation of the level sets $\{K^\eta_\delta = h\}$ of the normal form for increasing values of η, $\delta = 0.25$ and fixed values of the coefficients in the normal form.

For values of h corresponding to the red curve we have a stable equilibrium at the origin (left) or a stable equilibrium at the origin and a periodic orbit around it (right). For values of h slightly different (gray curves) we can have periodic orbits around the origin or no dynamics; in the right figure we furthermore have periodic orbits around a regular equilibrium.

At $(0,0)$ the reduced phase space section has a cuspidal singularity.

The equilibrium $\mathcal{Q}_1 = (0,0,0)$ can be unstable only if the parabola passes through the origin $(u,v) = (0,0)$ with vanishing first derivative. This happens for

$$v'(0) = 0$$

Since we are following a perturbative approach, we look for a solution of this equation for η in the form of a power series in ε.

We find just one solution, in the form

$$\eta = \bar{\eta} := \eta_{01} + \varepsilon^2 \eta_{11}$$

acceptable for $\eta_{01} \geq 0$.

At this critical value for η two families of periodic orbits bifurcate off/from the singular equilibrium at the origin and this happens simultaneously.

This has a geometric reason related to the 2 : 4 resonance and in particular subsists through all orders of the perturbation for the normal form.

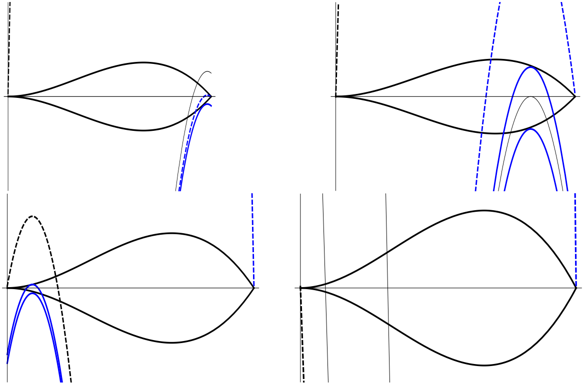


Figure: Possible tangencies between the parabola and the phase space section $\mathcal{P}^\eta \cap \{w = 0\}$ for increasing values of η, $\delta = -0.25$ and fixed values of the coefficients in the normal form. Two regular equilibria appear successively from the conical singularity and subsequently disappear simultaneously on the singular equilbrium at the origin. The equilibrium on the upper contour of the phase space is unstable while the equilibrium on the lower contour is stable.

At $\mathcal{Q}_2 = (\eta, 0, 0)$ the reduced phase space has a conical singularity.

The intersection of the reduced phase space $\mathcal{P}^\eta$ with the (u, v)–plane is given by

$$\mathcal{C}^\eta_\pm \;=\; \mathcal{P}^\eta \cap \{w = 0\} \;=\; \left\{ (u, v) \in \mathbb{R}^2 \;:\; v = \pm\frac{1}{2}(\eta - u)u^2,\; 0 \leq u \leq \eta \right\}$$

whence the slope of the two contour lines constituting the reduced phase space section at $(u, v) = (\eta, 0)$ is $\mp\frac{1}{2}\eta^2$. The corresponding equilibrium can be unstable only if the slope of the parabola at $(u, v) = (\eta, 0)$ takes values in the interval $(-\frac{1}{2}\eta^2, \frac{1}{2}\eta^2)$. Thus, to find the critical values for η which correspond to stability/instability transitions of the equilibrium, we need to solve the two equations

$$v'(\eta) \;=\; \pm\frac{\eta^2}{2}\;.$$

We arrive at the two solutions $\eta = \eta_{2,\pm} := \eta_{02} + \eta_\pm$, acceptable for $\eta_{02} \geq 0$.

In this case two families of periodic orbits can appear/ disappear, not together.

Implications for the original system: what the equilibria for the reduced system correspond to?

- The singular equilibria $\mathcal{Q}_1 = (0, 0, 0)$ and $\mathcal{Q}_2 = (0, 0, \eta)$ correspond to $\tau_1 = 0$ and $\tau_1 = \eta$, respectively.

  For the original system this are the normal modes

$$x_1^2 + p_1^2 = 0, \;\; x_2^2 + p_2^2 = \eta$$

  and

$$x_1^2 + p_1^2 = 2\eta, \;\; x_2^2 + p_2^2 = 0,$$

  also called short and long axial orbits.

- Tangencies on the lower contour of the reduced phase space are banana orbits:

$$0 = \sigma_1 = \tau_1 \sqrt{2\tau_2} \cos(2\phi_1 - \phi_2).$$

- Tangencies on the upper contour of the reduced phase space are anti-banana orbits:

$$0 = \sigma_2 = \tau_1 \sqrt{2\tau_2} \sin(2\phi_1 - \phi_2).$$

- banana and/or anti-banana orbits appear/disappear when the corresponding threshold values for η are acceptable, i.e. not negative. This is always associated with a stability/instability transition of a normal mode. This gives conditions in terms of the coefficients α, λ of the normal form and the detuning δ.
- η is not a constant for the original system; nevertheless we can use it to find threshold values for the bifurcations in terms of the (generalized) energy E (that is conserved for the original system).

On the long axial orbit ($\tau_1 = \eta$, $\tau_2 = 0$), the normal form reads as

$$K = \eta + \varepsilon^2(2\delta + \alpha_1\eta)\eta + \ldots .$$

By the scaling of time we have

$$\frac{\omega_2}{n}K + O(\varepsilon^6) = H = E$$

and we can express the (generalized) energy in terms of η as

$$E = \frac{\omega_2}{n}\left[\eta + \varepsilon^2 (2\delta + \alpha_1\eta)\,\eta + \ldots\right.$$

Substituting the threshold values for η we find the critical energy threshold values that correspond to the bifurcations off/from the long axial orbit.

Conclusion and perspectives:

- 3D problems: a geometric reduction is possible also for 3D systems that are close to resonances. However, the outcome of the normalization is in general a normal form possessing only one additional integral, besides the Hamiltonian and therefore it is not integrable. Sometimes a renormalization is possible...
- Indefinite resonances: one could consider more general systems with indefinite quadratic part, so that

$$H_0 = \frac{1}{2}(m_1\tau_1 - m_2\tau_2), \quad m_1, m_2 \in \mathbb{N}$$

These systems differ from the definite case in several features, even if their analysis can be performed almost in the same way.

K. Efstathiou, H. Hassmann, A. Marchesiello, *Bifurcations and monodromy of the axially symmetric* $1:1:-2$ *resonance*, Journal of Geometry and Physics, **146**, 103493 (2019).

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